Absorbing Boundary Conditions for Maxwell's Equations

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1. Derivation of Maxwell equations.

The equations, which are the most frequently used in electromagnetics, are Maxwell's equations. Behind microscopic Maxwell's equations one usually considers the system of 4 equations, which relate the electric and magnetic fields. This system is given by the following:

$$\begin{cases} \nabla \cdot E = \dfrac{\rho(x,t)}{\varepsilon_0} \\ \nabla \times E = -\dfrac{\partial B}{\partial t} \\ \nabla \cdot B = 0 \\ \nabla \times B = \mu_0 j(t,x) + \varepsilon_0 \mu_0 \dfrac{\partial E}{\partial t} \end{cases}$$

The first equation is a Gauss' law for electrostatics.

The second equation is a Faraday's law.

The third equation means that there are no free magnetic charges.

The fourth equation is a Maxwell's law.

Here E is an electric field,

B – magnetic field,

ε_0 is an electric permittivity in vacuum ($\approx 8.85 \times 10^{-12}$ F/m),

μ_0 is a vacuum magnetic permeability ($\approx 1.2566 \times 10^{-6}$ H/m),

j is a total electric current (which in general can be both time and space dependent),

ρ represents a total electric charge (which again can be both time and space dependent).

These equations form a system of well-posed equations and together with initial and boundary conditions which are imposed in each situation, completely determine fields E and B.

When we consider only the static fields we have equations

$$\begin{cases} \nabla \cdot E = \dfrac{\rho(x)}{\varepsilon_0} \\ \nabla \times E = 0 \\ \nabla \cdot B = 0 \\ \nabla \times B = \mu_0 j(x) \end{cases}$$

So the two fields are independent. This comes from the expressions for E^{stat} and B^{stat}. Indeed, we have

$$E^{stat}(x) = \frac{1}{4\pi\varepsilon_0} \sum_i q_i' \frac{x - x_i'}{|x - x_i'|^3} = -\frac{1}{4\pi\varepsilon_0} \int_{V'} d^3x' \, \frac{\rho(x')}{|x - x'|}$$

(comes from Coulomb's law) and

$$B^{stat}(x) = \frac{\mu_0}{4\pi}\int_{V'} d^3x' j(x') \times \frac{x - x'}{|x - x'|^3} = \frac{\mu_0}{4\pi}\nabla \times \int_{V'} d^3x' \frac{j(x')}{|x - x'|}$$

(this is Biot-Savart's law which uses Ampere's law).
But when we consider dynamic fields, we have the continuity of charge

$$\frac{\partial \rho(t,x)}{\partial t} + \nabla \cdot j(t,x) = 0,$$

and so we have Maxwell's source equation for B field

$$\nabla \times B(t,x) = \mu_0 \left(j(t,x) + \frac{\partial}{\partial t}\varepsilon_0 E(t,x) \right) = \mu_0 j(t,x) + \frac{1}{c^2}\frac{\partial}{\partial t}E(t,x)$$

and the equation which comes from applying Stokes theorem to the integral form of Faraday's law

$$\nabla \times E(t,x) = -\frac{\partial B}{\partial t}$$

2. Maxwell equations in 1D and 2D.

Firstly one write the general form of Maxwell's equations.

$$\begin{cases} \varepsilon_0 \dfrac{\partial E}{\partial t} - \nabla \times H = j(t,x) \\ \mu_0 \dfrac{\partial H}{\partial t} + \nabla \times E = 0 \end{cases}$$

We now write the system in 3D

$$\begin{cases} \varepsilon_0 \dfrac{\partial E_x}{\partial t} - \dfrac{\partial H_z}{\partial y} + \dfrac{\partial H_y}{\partial z} = j_x \\ \varepsilon_0 \dfrac{\partial E_y}{\partial t} - \dfrac{\partial H_x}{\partial z} + \dfrac{\partial H_z}{\partial x} = j_y \\ \varepsilon_0 \dfrac{\partial E_z}{\partial t} - \dfrac{\partial H_y}{\partial x} + \dfrac{\partial H_x}{\partial y} = j_z \\ \mu_0 \dfrac{\partial H_x}{\partial t} + \dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z} = 0 \\ \mu_0 \dfrac{\partial H_y}{\partial t} + \dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x} = 0 \\ \mu_0 \dfrac{\partial H_z}{\partial t} + \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = 0 \end{cases}$$

Let us now write Maxwell's equations in 2D.

This is done by imposing the certain components of E and H to zero. One notes, that in order to preserve orthogonality one has to set to zero the components which are orthogonal. Indeed, we have $E = (E_x, E_y, E_z)$, $H = (H_x, H_y, H_z)$. Then we set

$$\begin{cases} E_y = 0 \\ E_z = 0 \\ H_x = 0 \end{cases}$$

So we obtain the system

$$\begin{cases} \varepsilon_0 \dfrac{\partial E_x}{\partial t} - \dfrac{\partial H_z}{\partial y} + \dfrac{\partial H_y}{\partial z} = j_x \\ \mu_0 \dfrac{\partial H_y}{\partial t} + \dfrac{\partial E_x}{\partial z} = 0 \\ \mu_0 \dfrac{\partial H_z}{\partial t} - \dfrac{\partial E_x}{\partial y} = 0 \end{cases}$$

To obtain the system in 1D, we need to set additionally for example $H_y = 0$.
We obtain

$$\begin{cases} \varepsilon_0 \dfrac{\partial E_x}{\partial t} - \dfrac{\partial H_z}{\partial y} = j_x \\ \mu_0 \dfrac{\partial H_z}{\partial t} - \dfrac{\partial E_x}{\partial y} = 0 \end{cases}$$

3. Domain decomposition methods for Laplace equation.

3.1. Basic aspects.

Let us consider Poisson problem of the form

$$-\Delta u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

Here $\Omega$is 2 or 3-dimensional domain with a Lipschitz boundary $\partial\Omega$.

The above Poisson problem can be reformulated in multi-domain form. Assume that domain $\Omega$is divided into two non-overlapping sub-domains Ω_1 and Ω_2 with common boundary Γ. Then one can write the following:

$$-\Delta u_1 = f \text{ in } \Omega_1$$
$$u_1 = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega$$

$u_1 = u_2$ on Γ
$\dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n}$ on Γ
$u_2 = 0$ on $\partial\Omega_2 \cap \partial\Omega$
$-\Delta u_2 = f$ in Ω_2

In other words, u_i is the restriction of u on Ω_i.

Equations 3 and 4 in the last system are called the transmission conditions for u_1 and u_2 on Γ.

3.2. Steklov-Poincare interface equation.

Let λdenote the unknown value of u on Γ. Then we consider two problems:

$-\Delta w_i = f$ in Ω
$w_i = 0$ on $\partial\Omega_i \cap \partial\Omega$
$w_i = \lambda$ on Γ

We can view w_i as the sum of solutions to corresponding two problems: one – with zero boundary conditions both on Γ and $\partial\Omega_i \cap \partial\Omega$ and right-hand side equal to f, which one denotes $G_i f$; and the other one – which is equal to λon Γ, zero on $\partial\Omega_i \cap \partial\Omega$ and is harmonic inside Ω_i, and is denoted by $H_i\lambda$.

Further, it is easy to see that $w_i = u_i$ if and only if $\dfrac{\partial w_1}{\partial n} = \dfrac{\partial w_2}{\partial n}$ on Γ.

The last condition means that λ has to satisfy Steklov-Poincare interface equation on Γ:

$$S\lambda = \Xi$$

where Ξ is defined by

$$\Xi = \frac{\partial}{\partial n} G_2 f - \frac{\partial}{\partial n} G_1 f$$

and S is Steklov-Poincare operator, defined by

$$S\eta = \frac{\partial}{\partial n} H_1\eta - \frac{\partial}{\partial n} H_2\eta$$

The Steklov-Poincare operator maps the function, defined on the interface, to the normal derivative of difference of the solutions of corresponding Laplace problems.

Remark. For the general type of equation ($Lu = f$), where L is some operator, we have to impose some transmission conditions of the form

$\Phi(u_1) = \Phi(u_2)$ on Γ
$\Psi(u_1) = \Psi(u_2)$ on Γ

These conditions should be derived in each case and they come from the fact that u should satisfy the equation not only in Ω_1 and Ω_2 but also through the interface Γ.

3.3. Domain decomposition algorithms.

Following [4], we consider two domain decomposition algorithms for wave equation.

The Dirichlet-Neumann algorithm is the following:
1. Set some initial guess – the value of the solution on the interface λ_0.
2. A) Solve the problem

$-\Delta u_1^{k+1} = f$ in Ω_1
$u_1^{k+1} = 0$ on $\partial\Omega_1 \cap \partial\Omega$
$u_1^{k+1} = \lambda^k$ on Γ

B) Solve the problem

$-\Delta u_2^{k+1} = f$ in Ω_2
$u_2^{k+1} = 0$ on $\partial\Omega_2 \cap \partial\Omega$
$\dfrac{\partial u_2^{k+1}}{\partial n} = \dfrac{\partial u_1^{k+1}}{\partial n}$ on Γ

3. Update λ: $\lambda^{k+1} = \theta u_{2|\Gamma}^{k+1} + (1-\theta)\lambda^k$
Remark. It is also possible to consider relaxation on Neumann condition.

Neumann-Neumann algorithm is the following:
1. Set some initial guess – the value of the solution on the interface λ_0.
2. A) Solve the problems

$-\Delta u_i^{k+1} = f$ in Ω_i
$u_i^{k+1} = 0$ on $\partial\Omega_i \cap \partial\Omega$
$u_i^{k+1} = \lambda^k$ on Γ

B) Solve the problem

-$\Delta\psi_i^{k+1} = 0$ in Ω_i
$\psi_i^{k+1} = 0$ on $\partial\Omega_i \cap \partial\Omega$
$\psi_i = \dfrac{\partial u_1^{k+1}}{\partial n} - \dfrac{\partial u_2^{k+1}}{\partial n}$ on Γ

3. Update λ: $\lambda^{k+1} = \lambda^k - \theta(\sigma_1\psi_{1|\Gamma}^{k+1} - \sigma_2\psi_{2|\Gamma}^{k+1})$
Here θ, σ_1, σ_2 are some non-negative real parameters.

For wave equation the relaxation is needed, because non-overlapping Schwarz does not converge in this case without relaxation.

Dirichlet-Neumann algorithm cannot really be done in parallel, unlike the Neumann-Neumann algorithm.

4. Absorbing boundary conditions for wave equation.

Following [2], assume that we have to solve boundary-value problem in $\Omega \subset R^d$ with a smooth boundary Γ

$\Box_c u := \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} - \triangle u = f$ in $\Omega \times (0, T)$,
$u = g$ on $\Gamma \times (0, T)$
$u(\cdot, 0) = u_0$ in Ω
$u'(\cdot, 0) = u'_0$ in Ω

We assume also, that all conditions for existence and uniqueness of solution are satisfied.

Then we decompose Ω in two domains Ω_1 and Ω_2 that do not intersect. So we obtain parallel Schwarz method for wave equation:

$\Box_c u_1 = f$ in $\Omega_1 \times (0, T)$,
$u_1 = g$ on $\Gamma_{10} \times (0, T)$
$u_1(\cdot, 0) = u_0$ in Ω_1
$u'_1(\cdot, 0) = u'_0$ in Ω_1
$u_1 = u_2$ on Γ

and

$\Box_c u_2 = f$ in $\Omega_2 \times (0, T)$,
$u_2 = g$ on $\Gamma_{20} \times (0, T)$

$u_2(\cdot, 0) = u_0$ in Ω_2
$u_2'(\cdot, 0) = u_0'$ in Ω_2
$\dfrac{\partial u_1}{\partial n_1} + \dfrac{\partial u_2}{\partial n_2} = 0$ on Γ

We assume, that f, u_0, u_0' vanish in Ω_2.
We introduce the problem for a given $h \in L_2(0, T; H_{00}^{\frac{1}{2}})$ for some w:

$\square_c w = 0$ in $\Omega_2 \times (0, T)$,
$w = 0$ on $\Gamma_{20} \times (0, T)$
$u_2(\cdot, 0) = 0$ in Ω_2
$u_2'(\cdot, 0) = 0$ in Ω_2

Then we define the Steklov-Poincare operator by $S_{1,\Gamma}^{ex} h := \dfrac{\partial w}{\partial n_2}$. So this operator transforms the function from the boundary to the function of the dual space. As we have $\dfrac{\partial w}{\partial n_2} = -\dfrac{\partial w}{\partial n_1}$ so we have Steklov-Poincare operator of the following form:

$$\frac{\partial w}{\partial n_1} + S_{1,\Gamma}^{ex} w = 0 \text{ on } \Gamma \times (0, T)$$

The operators $B_{1,\Gamma}^{ex} := \dfrac{\partial}{\partial n_1} + S_{1,\Gamma}^{ex}$ and $B_{2,\Gamma}^{ex} := \dfrac{\partial}{\partial n_2} + S_{2,\Gamma}^{ex}$ are called the exact transparent operators.

In the case when the wave speed c is constant, it is easy to find the symbols of Steklov-Poincare operators. This is done by writing the wave equation in a Fourier space. So we do the Fourier transform of our problem in t and $d-1$ space dimensions, leaving one dimension in order to obtain ODE to find the symbol of the Steklov-Poincare operator.

So we obtain

$$-\frac{\partial^2 \tilde{w}}{\partial x^2} + \left(|k|^2 - \frac{\omega^2}{c^2} \right) \tilde{w} = 0 \text{ for } x \geq \delta$$
$$\tilde{w} = \tilde{h} \text{ at } x = \delta$$

So we have two cases:
1) $|k|^2 - \dfrac{\omega^2}{c^2} \geq 0$. Then the solution of the above IVP is $\tilde{w} = \tilde{h} e^{-(x-\delta)\sqrt{|k|^2 - \frac{\omega^2}{c^2}}}$
2) $|k|^2 - \dfrac{\omega^2}{c^2} < 0$. Then the solution of the above IVP is $\tilde{w} = \tilde{h} e^{-i(x-\delta)\sqrt{\frac{\omega^2}{c^2} - |k|^2}}$ (since the initial conditions are zero there are no waves coming from the infinity in the x direction).

So in the case of constant wave speed we have that two operators coincide ($S_1^{ex} = S_2^{ex}$) and are independent of Γ. They are given by

$$\sigma_1^{ex} = \sigma_2^{ex} = \begin{cases} \sqrt{|k|^2 - \frac{\omega^2}{c^2}} & |k|^2 - \frac{\omega^2}{c^2} \geq 0 \\ i\sqrt{\frac{\omega^2}{c^2} - |k|^2} & |k|^2 - \frac{\omega^2}{c^2} < 0 \end{cases}$$

5. Absorbing boundary conditions for Maxwell equations.

1. 1D Calculations.

We recall Maxwell equations in 1D

$$\begin{cases} \varepsilon \frac{\partial E_x}{\partial t} - \frac{\partial H_z}{\partial y} = 0 \\ \mu \frac{\partial H_z}{\partial t} - \frac{\partial E_x}{\partial y} = 0 \end{cases}$$

Firstly, it is necessary to rewrite this system in a matrix form:

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} E \\ H \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To get the Steklov-Poincare operator, we do firstly Laplace transform in t and get

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix} = \begin{pmatrix} 0 & \mu s \\ \varepsilon s & 0 \end{pmatrix} \begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix}$$

Above system can be rewritten as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix} + \begin{pmatrix} 0 & -\mu s \\ -\varepsilon s & 0 \end{pmatrix} \begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix} = 0$$

In order to compute incoming and outgoing waves, we need to diagonalize matrix A, where

$$A = \begin{pmatrix} 0 & -\mu s \\ -\varepsilon s & 0 \end{pmatrix}$$

Define $z = \sqrt{\frac{\mu}{\varepsilon}}$, $c = \frac{1}{\sqrt{\mu\varepsilon}}$.

The eigenvalues of this matrix are $\lambda_1 = \sqrt{\mu\varepsilon}s$, $\lambda_2 = -\sqrt{\mu\varepsilon}s$. The corresponding eigenvectors are $x_1 = (-z, 1)$, $x_2 = (z, 1)$.

So we can write matrix A as $A = P\Lambda P^{-1}$, where

$P = \begin{pmatrix} -z & z \\ 1 & 1 \end{pmatrix}$,

$P^{-1} = \frac{1}{2z} \begin{pmatrix} -1 & z \\ 1 & z \end{pmatrix}$.

Then we can denote

$W = P^{-1}X$,

where

$$X = \begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix}$$

It is easy to see that $W = \dfrac{1}{2z} \begin{pmatrix} -\hat{E} + z\hat{H} \\ \hat{E} + z\hat{H} \end{pmatrix}$.

So the system of ODE's has the following form:

$$\begin{cases} \dfrac{dW_1}{dy} + \lambda_1 W_1 = 0 \\ \dfrac{dW_2}{dy} + \lambda_2 W_2 = 0 \\ \qquad W_1 = \hat{g} \end{cases}$$

The general solution of this system is

$$\begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix} = \alpha_1 e^{-\lambda_1 y} x_1 + \alpha_2 e^{-\lambda_2 y} x_2.$$

As we look for bounded solution at infinity we must impose $\alpha_2 = 0$.

So the solution of the problem has the form

$$\begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix} = \alpha_1 e^{-\lambda_1 y} \begin{pmatrix} -z \\ 1 \end{pmatrix}.$$

Inserting the general solution into a boundary condition at $y = 0$ we obtain

$$\frac{\alpha_1}{2z}(z + z) = g.$$

From this one obtains $\alpha_1 = \hat{g}$.

Finally, we conclude

$$\begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix} = e^{-\lambda_1 y} \begin{pmatrix} -z \\ 1 \end{pmatrix} \hat{g}$$

Therefore, the outgoing wave is

$$W_2 = \frac{1}{2z}(\hat{E} + z\hat{H}) = 0.$$

2. 2D Calculations.

We write Maxwell equations in 2D

$$\begin{cases} \varepsilon\dfrac{\partial E_x}{\partial t} - \dfrac{\partial H_z}{\partial y} + \dfrac{\partial H_y}{\partial z} = 0 \\ \mu\dfrac{\partial H_y}{\partial t} + \dfrac{\partial E_x}{\partial z} = 0 \\ \mu\dfrac{\partial H_z}{\partial t} - \dfrac{\partial E_x}{\partial y} = 0 \end{cases}$$

We rewrite this system in a matrix form:

$$\begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} E \\ H_y \\ H_z \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} E \\ H_y \\ H_z \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial z} \begin{pmatrix} E \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In order to obtain Steklov-Poincare operator, we do Fourier transform in z and Laplace transform in t and get:

$$\begin{cases} s\varepsilon E_x - \dfrac{\partial H_z}{\partial y} + i\xi H_y = 0 \\ s\mu H_y + i\xi E_x = 0 \\ s\mu H_z - \dfrac{\partial E_x}{\partial y} = 0 \end{cases}$$

One rewrites the above system in a matrix form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} \hat{E}_x \\ \hat{H}_z \end{pmatrix} = \begin{pmatrix} 0 & \mu s \\ \dfrac{s^2\varepsilon\mu + \xi^2}{\mu s} & 0 \end{pmatrix} \begin{pmatrix} \hat{E}_x \\ \hat{H}_z \end{pmatrix}.$$

We can rewrite the above system in the following form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} \hat{E}_x \\ \hat{H}_z \end{pmatrix} + \begin{pmatrix} 0 & -\mu s \\ -\dfrac{s^2\varepsilon\mu + \xi^2}{\mu s} & 0 \end{pmatrix} \begin{pmatrix} \hat{E}_x \\ \hat{H}_z \end{pmatrix} = 0$$

Let us denote $B = \begin{pmatrix} 0 & -\mu s \\ -\dfrac{s^2\varepsilon\mu + \xi^2}{\mu s} & 0 \end{pmatrix}$.

The eigenvalues of B are $\lambda_1 = \sqrt{s^2\varepsilon\mu + \xi^2}$, $\lambda_2 = -\sqrt{s^2\varepsilon\mu + \xi^2}$. The corresponding eigenvectors are $x_1 = (-\frac{\mu s}{\sqrt{s^2\varepsilon\mu + \xi^2}}, 1)$, $x_2 = (\frac{\mu s}{\sqrt{s^2\varepsilon\mu + \xi^2}}, 1)$.

So we obtain the following transformation matrices

$$P = \begin{pmatrix} -\frac{\mu s}{\sqrt{s^2\varepsilon\mu + \xi^2}} & \frac{\mu s}{\sqrt{s^2\varepsilon\mu + \xi^2}} \\ 1 & 1 \end{pmatrix},$$

$$P^{-1} = -\frac{\sqrt{s^2\varepsilon\mu + \xi^2}}{2\mu s} \begin{pmatrix} \frac{\mu s}{\sqrt{s^2\varepsilon\mu + \xi^2}} & 1 \\ -\frac{\mu s}{\sqrt{\varepsilon\mu s^2 + \xi^2}} & -1 \end{pmatrix}.$$

We denote again $W = P^{-1}X$,

where

$$X = \begin{pmatrix} \hat{E}_x \\ \hat{H}_z \end{pmatrix}.$$

Let us now denote $k = \frac{\mu s}{\sqrt{\mu\varepsilon s^2 + \xi^2}}$.

It is easy to see that $W = \frac{1}{2k} \begin{pmatrix} -\hat{E}_x + k\hat{H}_z \\ \hat{E}_x + k\hat{H}_z \end{pmatrix}$.

So the system of ODE's has the following form:

$$\begin{cases} \frac{dW_1}{dy} + \lambda_1 W_1 = 0 \\ \frac{dW_2}{dy} + \lambda_2 W_2 = 0 \\ W_1 = \hat{g} \end{cases}$$

The general solution of this system is

$$\begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix} = \alpha_1 e^{-\lambda_1 y} x_1 + \alpha_2 e^{-\lambda_2 y} x_2.$$

As we look for bounded solution at infinity we must impose $\alpha_2 = 0$.

So the solution of the problem has the form

$$\begin{pmatrix} \hat{E}_x \\ \hat{H}_z \end{pmatrix} = \alpha_1 e^{-\lambda_1 y} \begin{pmatrix} -k \\ 1 \end{pmatrix}.$$

Inserting the general solution into a boundary condition at $y = 0$ we obtain

$$\frac{\alpha_1}{2k}(k + k) = g.$$

From this one obtains $\alpha_1 = \hat{g}$.

Finally, we conclude

$$\begin{pmatrix} \hat{E}_x \\ \hat{H}_z \end{pmatrix} = e^{-\lambda_1 y} \begin{pmatrix} -k \\ 1 \end{pmatrix} \hat{g}$$

Therefore, the outgoing wave is

$$W_2 = \frac{1}{2k}(\hat{E}_x + k\hat{H}_z) = 0.$$

Therefore, the outgoing wave is

$$W_2 = \frac{1}{2k}(\hat{E} + k\hat{H}) = 0,$$

where $k = \dfrac{\mu s}{\sqrt{\mu\varepsilon s^2 + \xi^2}}$.

6. Numerical algorithm.

6.1. Finite Volumes discretization for one domain.

Before doing Finite Volumes discretization it is useful to rewrite Maxwell equations in a quasi-nonconservative form

$$B\frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} = 0 \text{ on }]a,b[,\ t > 0,$$

where

$$W = \begin{pmatrix} E \\ H \end{pmatrix},\ B = B(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix},\ F(W) = AW,\ A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The idea of Finite Volumes discretization is the following: we divide the interval $]a,b[$ into N cells with $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$. After that we define the mean-value of W on each cell and denote it by W_j.

In other words,

$$W_j = \frac{1}{\Delta x_j}\int_{C_j} W(x,t)dx.$$

Multiplying the equations in quasi-nonconservative form by some test function $\psi(x)$ and intergrating over the domain yields into

$$\int_C \left(B\frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} \right) \psi(x) = 0$$

Now as test function we take the characterization function on each cell. So we get

$$B_j \frac{dW_j}{dt} + \phi_{j,j-1} + \phi_{j,j+1} = 0$$

The question is how to approximate the fluxes $\phi_{j,j-1}$ and $\phi_{j,j+1}$. The simplest way is to take the centered approximation. So in this case we get

$$\phi_{j,j\pm1} = \frac{1}{2}\left(F_j + F_{j\pm1}\right) = \mp\frac{1}{2}\begin{pmatrix} H_j + H_{j\pm1} \\ E_j + E_{j\pm1} \end{pmatrix}$$

6.2. Choice of schemes.

For the discretization of Maxwell equations we use forward difference in time and Crank-Nicolson scheme in space

This gives the following discretized system

$$\begin{cases} \varepsilon_j \dfrac{E_j^{n+1} - E_j^n}{\Delta t} = \dfrac{1}{2\Delta x_j}\left[\left(\dfrac{H_{j+1}^n + H_{j+1}^{n+1}}{2}\right) - \left(\dfrac{H_{j-1}^n + H_{j-1}^{n+1}}{2}\right)\right] \\ \mu_j \dfrac{H_j^{n+1} - H_j^n}{\Delta t} = \dfrac{1}{2\Delta x_j}\left[\left(\dfrac{E_{j+1}^n + E_{j+1}^{n+1}}{2}\right) - \left(\dfrac{E_{j-1}^n + E_{j-1}^{n+1}}{2}\right)\right] \end{cases}$$

Reorganizing the above system we get

$$\begin{cases} \sigma_j \varepsilon_j E_j^{n+1} + H_{j-1}^{n+1} - H_{j+1}^{n+1} = \sigma_j \varepsilon_j E_j^n + H_{j+1}^n - H_{j-1}^n \\ \sigma_j \mu_j H_j^{n+1} + E_{j-1}^{n+1} - E_{j+1}^{n+1} = \sigma_j \mu_j H_j^n + E_{j+1}^n - E_{j-1}^n \end{cases}$$

Here $\sigma_j = \dfrac{4\Delta x_j}{\Delta t}$.

6.3. Satisfying absorbing boundary conditions.

Following [1], we have That the total flux for $j = 1$ is

$$\phi_{1.2} + \phi_{1,0} = \frac{1}{2}\left(AW_2 + AW_1\right) + (-Z_1)^+ B_1 W_1$$

It is easy to see that

$$\phi_{j,j\pm1} = \mp\frac{1}{2}\left(\begin{array}{c} H_j + H_{j\pm1} \\ E_j + E_{j\pm1} \end{array}\right)$$

(if the centered approximation of flux is considered).

From the above formula for $\phi_{1,2}$ and $\phi_{1,0}$ it is obvious that $\phi_{1,2} + \phi_{1,0} = -\frac{1}{2}\left(\begin{array}{c} H_0 - H_2 \\ E_0 - E_2 \end{array}\right)$

From the definition of Z and B we get $(-Z)_1^+ \, BW = -\frac{1}{2}\left(\begin{array}{c} c\varepsilon E_1 - H_1 \\ -E_1 + c\mu H_1 \end{array}\right)$

From the definition of A we have

$$\frac{1}{2}(AW_2 + AW_1) = -\frac{1}{2}\left(\begin{array}{c} H_1 + H_2 \\ E_1 + E_2 \end{array}\right)$$

The discretization is

$$\left\{\begin{array}{l} \sigma_j \varepsilon_j E_j^{n+1} + H_{j-1}^{n+1} - H_{j+1}^{n+1} = \sigma_j \varepsilon_j E_j^n + H_{j+1}^n - H_{j-1}^n \\ \sigma_j \mu_j H_j^{n+1} + E_{j-1}^{n+1} - E_{j+1}^{n+1} = \sigma_j \mu_j H_j^n + E_{j+1}^n - E_{j-1}^n \end{array}\right.$$

For $j = 1$ the discretization is

$$\left\{\begin{array}{l} \sigma_1 \varepsilon_1 E_1^{n+1} + H_0^{n+1} - H_2^{n+1} = \sigma_1 \varepsilon_1 E_1^n + H_2^n - H_0^n \\ \sigma_1 \mu_1 H_1^{n+1} + E_0^{n+1} - E_2^{n+1} = \sigma_1 \mu_1 H_1^n + E_2^n - E_0^n \end{array}\right.$$

Substituting the expressions for H_0 and E_0 into the above discretization results in

$$\left\{\begin{array}{l} (\sigma_1 + c)\,\varepsilon_1 E_1^{n+1} - H_2^{n+1} = (\sigma_1 - c)\,\varepsilon_1 E_1^n + H_2^n \\ (\sigma_1 + c)\,\mu_1 H_1^{n+1} - E_2^{n+1} = (\sigma_1 - c)\,\mu_1 H_1^n + E_2^n \end{array}\right.$$

Performing very similar calculations for the total flux for $j = N$, it is easy to get

$$\left\{\begin{array}{l} (\sigma_N + c)\,\varepsilon_N E_N^{n+1} - H_{N-1}^{n+1} = (\sigma_N - c)\,\varepsilon_N E_N^n + H_{N-1}^n \\ (\sigma_N + c)\,\mu_N H_N^{n+1} - E_{N-1}^{n+1} = (\sigma_N - c)\,\mu_N H_N^n + E_{N-1}^n \end{array}\right.$$

6.4. Numerical results for one domain.

We consider a test problem in the open space of the form

$$\left\{\begin{array}{l} \dfrac{\partial E}{\partial t} - \dfrac{\partial H}{\partial y} = 0 \\ \dfrac{\partial H}{\partial t} - \dfrac{\partial E}{\partial y} = 0 \end{array}\right.$$

$E(0, y) = cosy + siny$
$H(0, y) = cosy - siny$

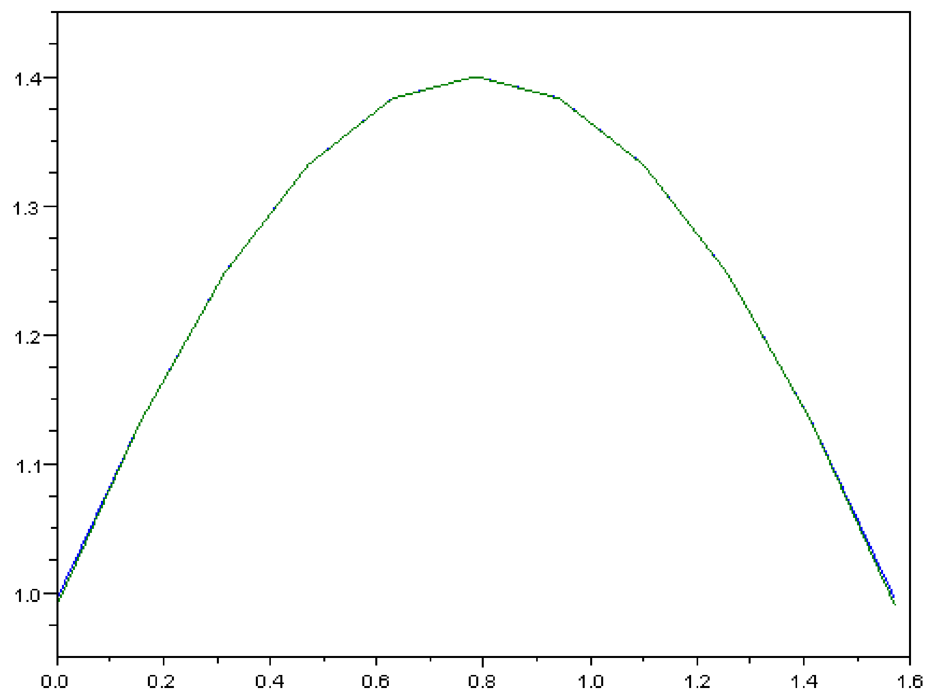
Assume that we solve it on $[0, \frac{\pi}{2}]$.
Then the solution to this problem, which satisfies absorbing boundary conditions

$(E - H)(t, 0) = 0$
$(E + H)(t, \frac{\pi}{2}) = 0$

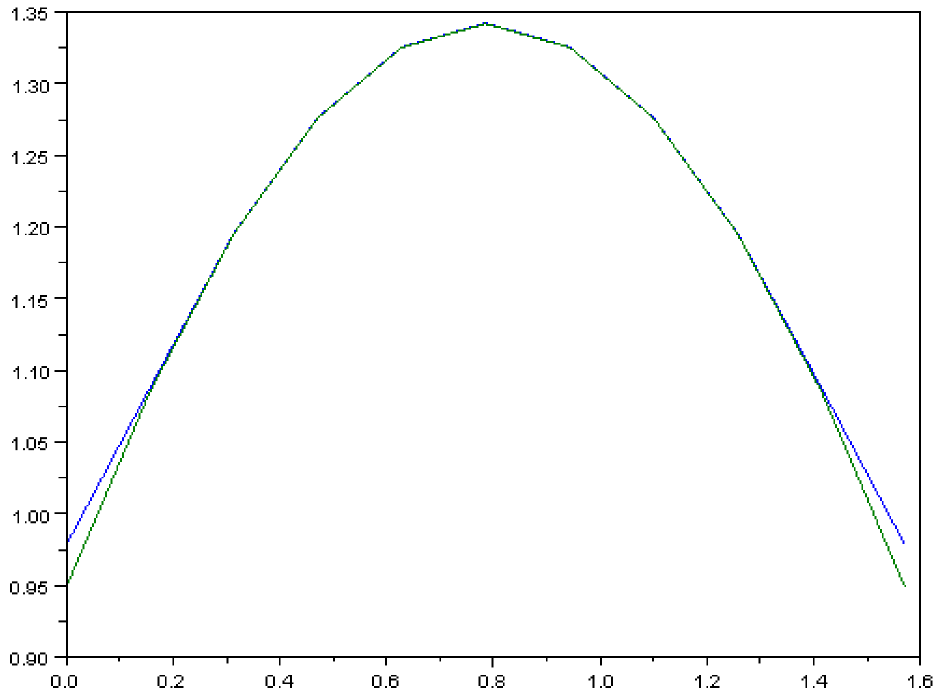
is the following:

$E(t, y) = (cosy + siny)\,(cost - sint)$
$H(t, y) = (cosy - siny)\,(cost + sint)$

Below on the picture the comparison between approximate and exact solution is given. The approximate solution is given in the first case using 11 nodes for the space discretization(9 internal). The solution is calculated at the time $T = 0.01$.

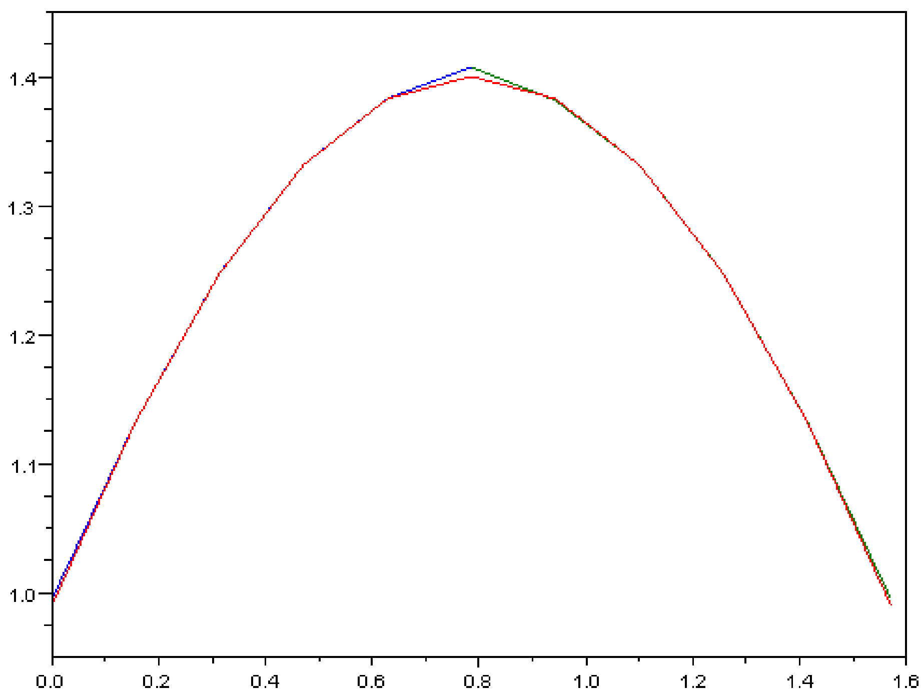


In the second example the time $T = 0.05$. We can see that the approximation gets worse as the time interval gets larger.



7. Numerical results for domain decomposition.

Below the domain is divided into two equal domains. For the space discretization in each domain 6 nodes are taken. The approximate solution is calculated for the time $T = 0.01$.



8. Conclusions.

– It is possible in some cases to write down the conditions to restrict the problem on the whole space to the problem on some interval.

– In 1D the Steklov-Poincare operator in the case of Maxwell's equations is local, unlike in 2D where it is non-local (because it is not a polynomial in Fourier space). Therefore in 2D, one needs to approximate the exact operator in order to implement boundary conditions on the computer.

– An alternative way for numerically solving PDE's using domain decomposition is to implememnt finite element method which requires the weak formulation.

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