High-order Time-Integration Methods for Maxwell's Equations

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Outline of the Talk

- Approximation of ODEs and PDEs
- Explicit and Implicit Schemes
- Notion of Stability
- Discontinuous Galerkin Method
- Application to Electromagnetics

Approximation of ODEs and PDEs

Differential Equation Problem

$\Downarrow \Downarrow \Downarrow$

Numerical Approximation Method

$\Downarrow \Downarrow \Downarrow$

Properties of the Method

PDE to ODE Problem Reduction by Discretization

Majority of IBVP can be reduced to the set of IVP

$$\begin{cases} \dfrac{dy}{dt} = f(t, y),\ t > 0 \\ y(0) = y_0 \end{cases}$$

and then tackled by Finite Differences approach:

$$y_n \approx y(t_n),\ t_n = n\Delta t,\ n = 0,\ \ldots,\ N.$$

Explicit and Implicit Schemes

The typical examples are

Forward Euler scheme

$$\underbrace{\frac{y_{n+1} - y_n}{\Delta t} = f(t_n, y_n)}$$

Explicit method:

y_{n+1} can be expressed at every step

Backward Euler scheme

$$\underbrace{\frac{y_{n+1} - y_n}{\Delta t} = f(t_{n+1}, y_{n+1})}$$

Implicit method:

Finding y_{n+1} requires solving system

Linear Multistep Methods

General linear s-step method: $\sum_{j=0}^{s} \alpha_j y_{n+1-j} = \Delta t \sum_{j=0}^{s} \beta_j f_{n+1-j}$

If $\beta_0 = 0$, then the method is explicit (assuming $\alpha_0 = 1$):

$$y_{n+1} = \sum_{j=1}^{s} \left(-\alpha_j y_{n+1-j} + \Delta t \beta_j f_{n+1-j}\right)$$

If $f(t, y)$ is polynomially approximated using points $t_{n+1}, t_n,\ t_{n-1},\ \ldots,\ t_{n-s+1}$ $\Longrightarrow$ Adams methods (Adams-Moulton family - if t_{n+1} is used, otherwise Adams-Bashforth family).

If $y(t)$ is polynomially approximated and $f(t, y) = f(t_{n+1}, y_{n+1})$ $\Longrightarrow$ BDF methods.

Runge-Kutta Methods

Motivated by midpoint integration formula of ODE

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(t, y(t))dt \qquad \Rightarrow$$

$$\Rightarrow \qquad \begin{cases} y_{n+1} = y_n + \Delta t f\left(t_n + \frac{\Delta t}{2}, \, y\left(t_n + \frac{\Delta t}{2}\right)\right) \\ y\left(t_n + \frac{\Delta t}{2}\right) = y_n + \frac{\Delta t}{2} f_n \end{cases}$$

we come to general s-stage Runge-Kutta Method:

$$\begin{cases} y_{n+1} = y_n + \Delta t \sum_{i=1}^{s} b_i f(t_n + c_i \Delta t, Y_i) \\ Y_i = y_n + \Delta t \sum_{j=1}^{s} a_{ij} f(t_n + c_i \Delta t, Y_j), \; 1 \le i \le s \end{cases}$$

Notion of Stability

Consider the test equation $y' = \lambda y$ providing $\Re e \lambda \leq 0$.

Numerical method application yields $y_{n+1} = R(\lambda \Delta t) y_n$.

A-stability:

$|y_{n+1}| \leq |y_n|$, i.e. $|R(\lambda \Delta t)| \leq 1$

L-stability:

$$\lim_{|\lambda| \Delta t \to \infty} |R(\lambda \Delta t)| = 0$$

Examples of Stability Analysis

Heat Equation problem:

$$\begin{cases} y_t(x,t) = a y_{xx}(x,t) \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = \phi(x) \end{cases} \Rightarrow \begin{cases} \mathbf{y}_t(t) = A\mathbf{y}(t) \\ y_0(t) = y_{N+1}(t) = 0 \end{cases} \Rightarrow \begin{cases} \mathbf{y}^{n+1} = B^{BE}\mathbf{y}^n \\ \mathbf{y}^0 = \phi(\mathbf{x}) \end{cases}$$

A is space discretization central differences matrix, B^{BE} is time discretization matrix arrising from Backward Euler scheme.

Spectrum: $\lambda_l^{(A)} = -\frac{4a}{\Delta x^2}\sin^2\left(\frac{\pi l \Delta x}{2L}\right)$, $\lambda_l^{(B^{BE})} = \frac{1}{1+\frac{4a\Delta t}{\Delta x^2}\sin^2\left(\frac{\pi l\Delta x}{2L}\right)}$, $\rho(B^{BE}) \approx \frac{1}{\left|1+\frac{4a\Delta t}{\Delta x^2}\right|}$

$$\begin{cases} \rho(B^{BE}) < 1 \\ \lim\limits_{\Delta t \lambda^{(A)} \to -\infty} \rho(B^{BE}) = 0 \end{cases}$$ $\Rightarrow$ A- and L-stability (not depending on choice of Δx, Δt.)

Examples of Stability Analysis

Heat Equation problem:

$$\begin{cases} y_t(x,t) = a y_{xx}(x,t) \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = \phi(x) \end{cases} \Rightarrow \begin{cases} \mathbf{y}_t(t) = A\mathbf{y}(t) \\ y_0(t) = y_{N+1}(t) = 0 \end{cases} \Rightarrow \begin{cases} \mathbf{y}^{n+1} = B^{FE}\mathbf{y}^n \\ \mathbf{y}^0 = \phi(\mathbf{x}) \end{cases}$$

A is space discretization central differences matrix, B^{FE} is time discretization matrix arrising from Forward Euler scheme.

Spectrum: $\lambda_l^{(A)} = -\frac{4a}{\Delta x^2}\sin^2\left(\frac{\pi l \Delta x}{2L}\right)$, $\lambda_l^{(B^{FE})} = 1 - \frac{4a\Delta t}{\Delta x^2}\sin^2\left(\frac{\pi l \Delta x}{2L}\right)$,

$$\rho(B^{FE}) \approx \left|1 - \frac{4a\Delta t}{\Delta x^2}\right|$$

The method is absolute stable when $\rho(B^{FE}) \leq 1 \;\Rightarrow\; \Delta t \leq \frac{\Delta x^2}{2a} = O(\Delta x^2)$.

Examples of Stability Analysis

Wave Equation:

$$\Leftarrow\ y_{tt} = c^2 y_{xx}\ \Rightarrow$$

$$\frac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{\Delta t^2} = c^2 \frac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{\Delta x^2}$$

$(\ y_j^n = G^n e^{ij\zeta}\ )\ \Downarrow$ (VN analysis)

CFL stability condition:

$$\Delta t \leq \frac{\Delta x}{c} = O(\Delta x)$$

$$\Downarrow$$

Use of an explicit scheme for a hyperbolic problem is acceptable!

$$\underbrace{\begin{cases} u_t = c v_x \\ v_t = c u_x \end{cases}}_{\Downarrow} \Bigg|\ u = c y_x,\ v = y_t$$

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = c \dfrac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} \\ \dfrac{v_j^{n+1} - v_j^{n-1}}{2\Delta t} = c \dfrac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \end{cases}$$

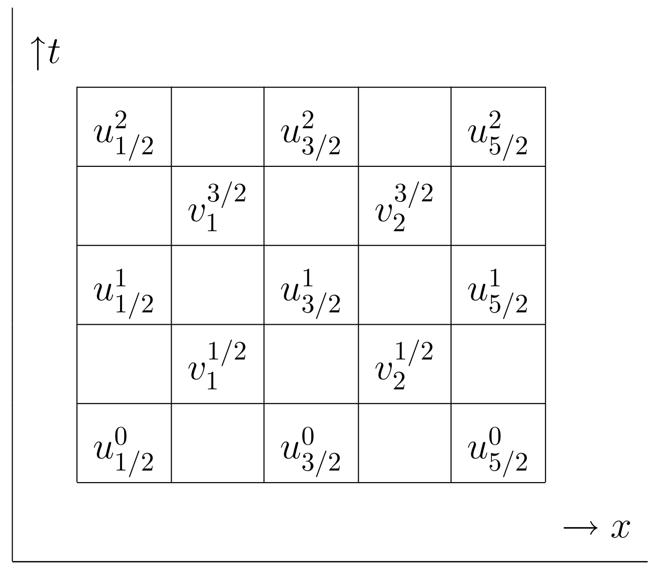
$$\downarrow$$

$$\begin{cases} \dfrac{u_{j+1/2}^{n+1} - u_{j+1/2}^n}{\Delta t} = c \dfrac{v_{j+1}^{n+1/2} - v_j^{n+1/2}}{\Delta x} \\ \dfrac{v_{j+1}^{n+3/2} - v_{j+1}^{n+1/2}}{\Delta t} = c \dfrac{u_{j+3/2}^{n+1} - u_{j+1/2}^{n+1}}{\Delta x} \end{cases}$$

Examples of Stability Analysis

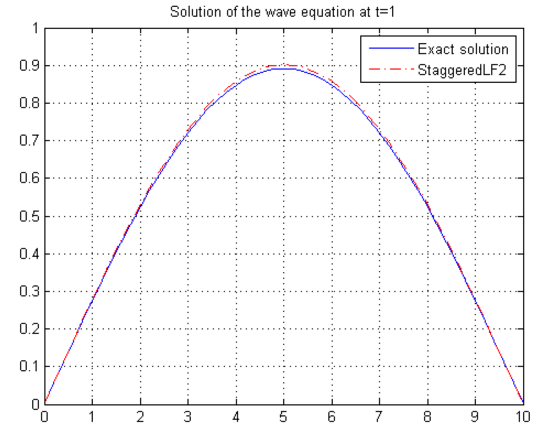
StaggeredLeapFrog2 scheme for the Wave Equation:

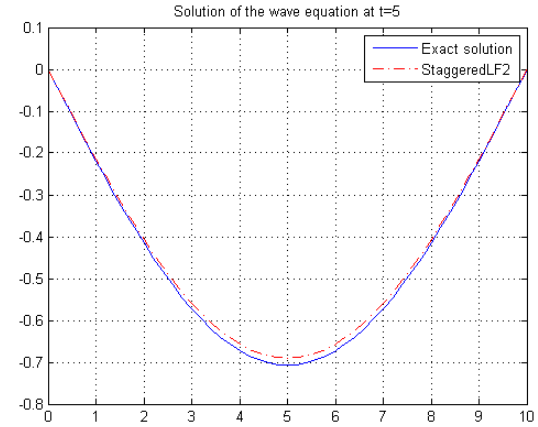
$$\begin{cases} \dfrac{u_{j+1/2}^{n+1} - u_{j+1/2}^{n}}{\Delta t} = c\dfrac{v_{j+1}^{n+1/2} - v_{j}^{n+1/2}}{\Delta x} \\ \dfrac{v_{j+1}^{n+3/2} - v_{j+1}^{n+1/2}}{\Delta t} = c\dfrac{u_{j+3/2}^{n+1} - u_{j+1/2}^{n+1}}{\Delta x} \end{cases}$$



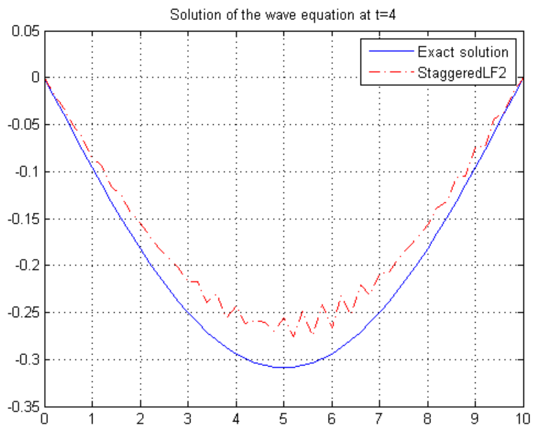
Examples of Stability Analysis

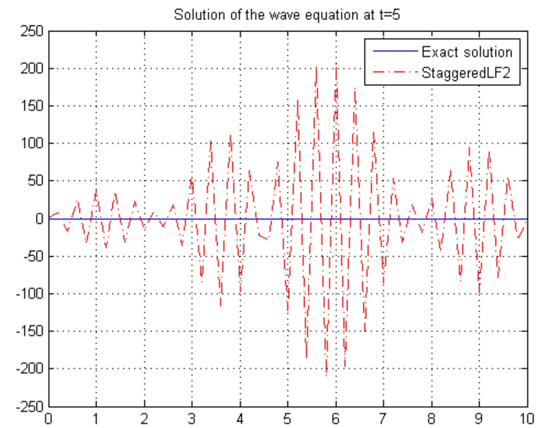
StaggeredLeapFrog2 results (CFL condition is fulfiled)



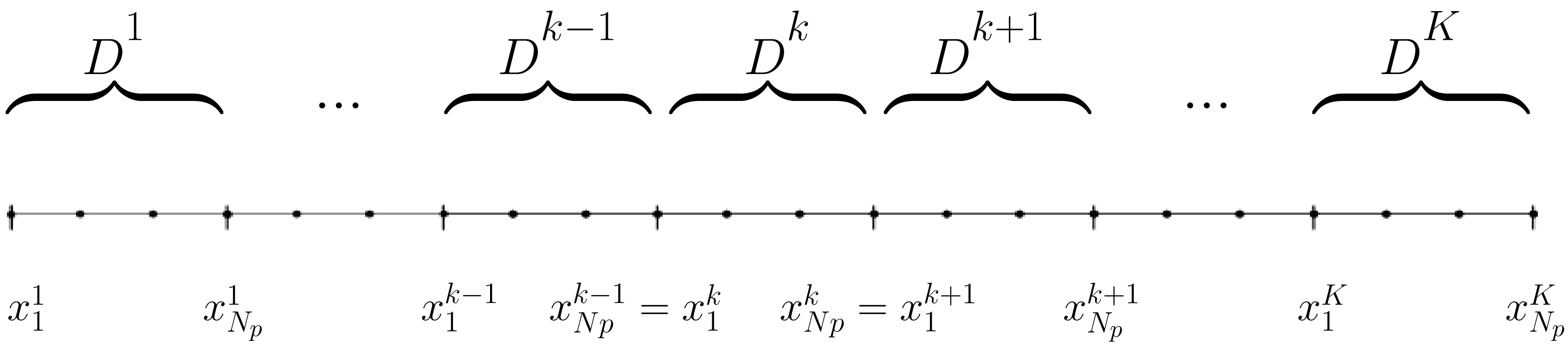


StaggeredLeapFrog2 results (CFL condition is exceeded)





Discontinuous Galerkin Method



Approximated solution is sought inside an element: $u_h^k(x,t) = \sum_{n=1}^{N_p} \hat{u}_n^k(t)\psi_n^k(x)$

Global solution is directly recovered: $u(x,t) \approx u_h(x,t) = \bigoplus_{k=1}^{K} u_h^k(x,t)$

Discontinuous Galerkin Method

Advection equation: $$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad f(u) = cu, \qquad c > 0.$$

According to Galerkin method:

$$\int_{D^k} \left(\frac{\partial u_h}{\partial t} + \frac{\partial f(u_h)}{\partial x} \right) \psi_n^k(x) dx = 0 \;\Rightarrow\; \int_{D^k} \left(\frac{\partial u_h}{\partial t} \psi_n^k - c u_h \frac{\partial \psi_n^k}{\partial x} \right) dx = - \left[f^\star \psi_n^k \right] \Big|_{x_1^k}^{x_{N_p}^k}$$

More generally, strong formulation reads

$$\int_{D^k} \left(\frac{\partial u_h^k}{\partial t} + \frac{\partial f(u_h^k)}{\partial x} \right) \psi_n^k(x) dx = \left[\left(c u_h^k - f^\star \right) \psi_n^k \right] \Big|_{x_1^k}^{x_{N_p}^k}$$

Numerical flux choice (e.g. on the left boundary):

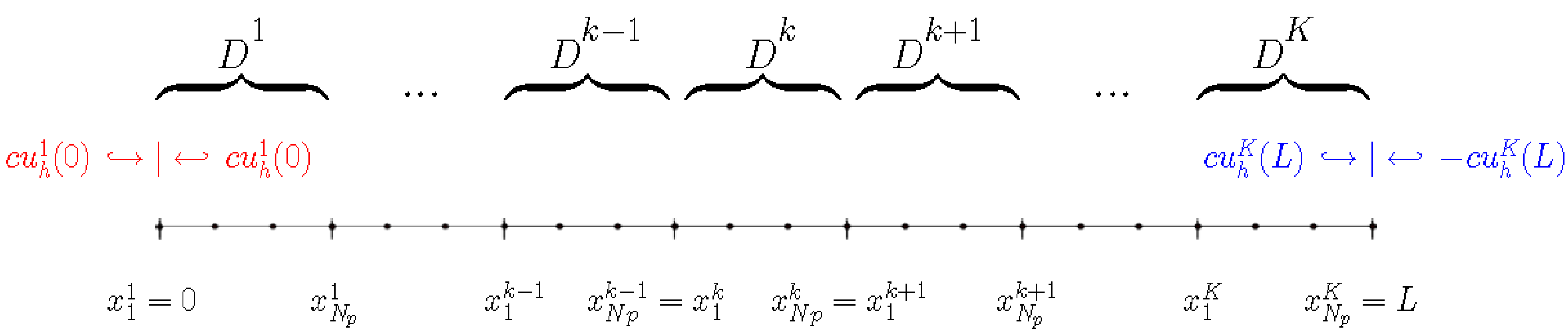
$$f^\star(x_1^k, x_{N_p}^{k-1}) = \begin{cases} c u_h^{k-1}(x_{N_p}^{k-1}) & \text{- purely upwind flux} \\ \frac{1}{2} \left[c u_h^{k-1}(x_{N_p}^{k-1}) + c u_h^k(x_1^k) \right] & \text{- central flux} \end{cases}$$

Discontinuous Galerkin Method

Imposing boundary conditions (general case)

Neumann boundary condition: $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$

Dirichlet boundary condition: $u|_{x=L} = 0$



Application to Electromagnetics

Maxwell's set of equations, 1D case reduction:

$$\begin{cases} \nabla \cdot \mathbf{E} = 4\pi\rho \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \end{cases} \quad \underset{(\,\mathbf{J}=\mathbf{0}\,)}{\overset{(\,\rho=0\,)}{\Rightarrow}} \quad \begin{cases} \dfrac{\partial E}{\partial t} = -c\dfrac{\partial B}{\partial x} \\ \dfrac{\partial B}{\partial t} = -c\dfrac{\partial E}{\partial x} \end{cases} \quad \left| \begin{array}{l} \mathbf{E} = (0,\ 0,\ E(x,t)) \\ \mathbf{B} = (0,\ B(x,t),\ 0) \end{array} \right.$$

➤ Electrostatic CGS Units system is used.

➤ Waves in free space are considered, i.e. $\epsilon = \mu = 1$

Application to Electromagnetics

Sample problem formulation and its DG spatial discretization:

$$\begin{cases} \dfrac{\partial E}{\partial t} = -c\dfrac{\partial B}{\partial x} \\ \dfrac{\partial B}{\partial t} = -c\dfrac{\partial E}{\partial x} \\ E(0,t) = E(L,t) = 0 \\ B_x(0,t) = B_x(L,t) = 0 \\ E(x,0) = \sin\left(\dfrac{\pi x}{L}\right) \\ B(x,0) = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{d\mathbf{E}}{dt} = cA_B\mathbf{B} \\ \dfrac{d\mathbf{B}}{dt} = cA_E\mathbf{E} \\ \mathbf{E}(0) = \sin\left(\dfrac{\pi \mathbf{x}}{L}\right) \\ \mathbf{B}(0) = \mathbf{0} \end{cases}$$

Application to Electromagnetics

StaggeredLeapFrog4 scheme *(Verwer, 2007)*:

$$\begin{cases} \dfrac{\mathbf{E}^{n+1}-\mathbf{E}^{n}}{\Delta t} = \left(A_B + \dfrac{1}{24}c^2\Delta t^2 A_B A_E A_B\right) c\mathbf{B}^{n+1/2} \\ \dfrac{\mathbf{B}^{n+3/2}-\mathbf{B}^{n+1/2}}{\Delta t} = \left(A_E + \dfrac{1}{24}c^2\Delta t^2 A_E A_B A_E\right) c\mathbf{E}^{n+1} \end{cases}$$

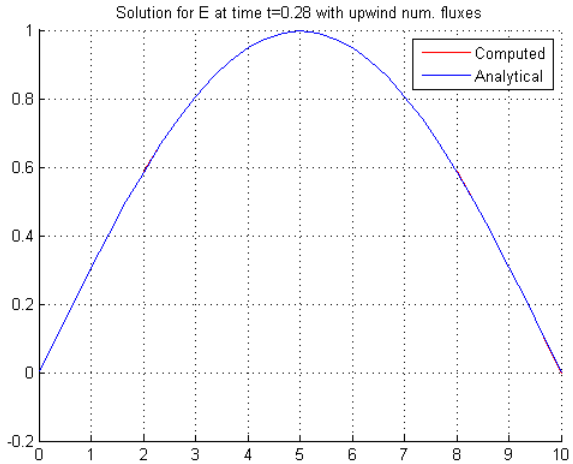
In matrix form:

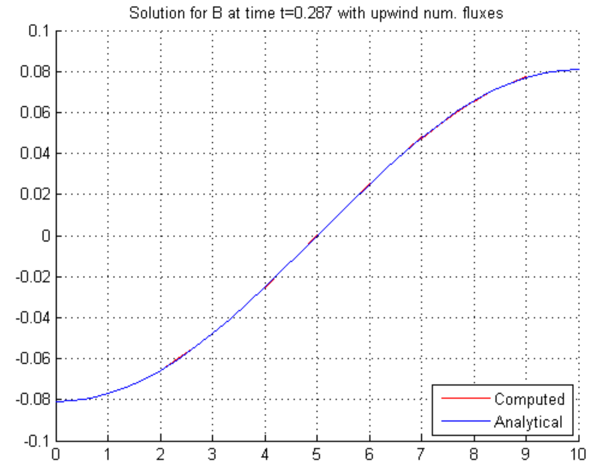
$$\begin{pmatrix} \mathbf{E}^{n+1} \\ \mathbf{B}^{n+3/2} \end{pmatrix} = \begin{pmatrix} I & S_1 \\ S_2 & I + S_2 S_1 \end{pmatrix} \begin{pmatrix} \mathbf{E}^{n} \\ \mathbf{B}^{n+1/2} \end{pmatrix}$$

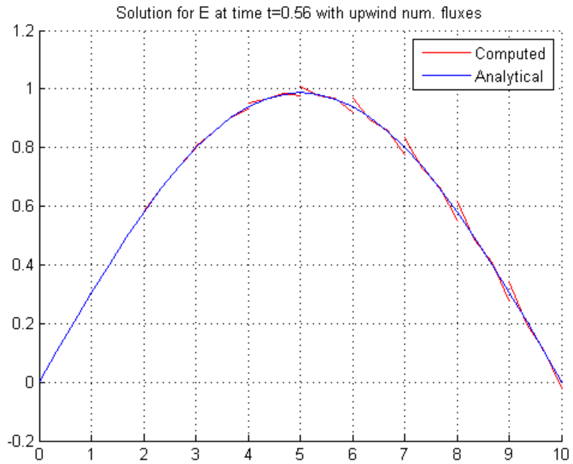
$$S_1 = kc\left(A_B + \frac{1}{24}c^2\Delta t^2 A_B A_E A_B\right), \qquad S_2 = kc\left(A_E + \tfrac{1}{24}c^2\Delta t^2 A_E A_B A_E\right)$$

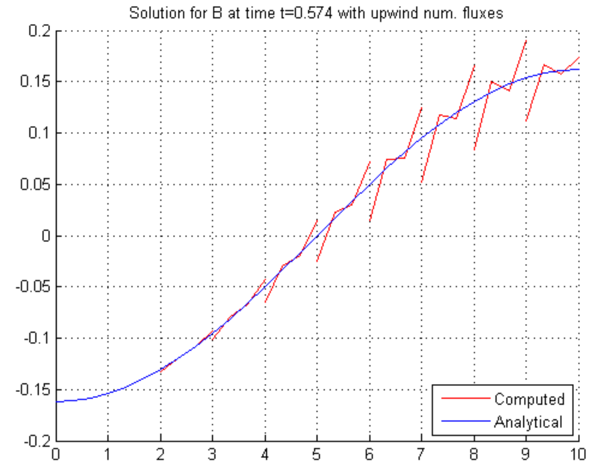
Application to Electromagnetics

Results with purely upwind numerical fluxes
(for reasonable choice of Δt and on stability region boundary)



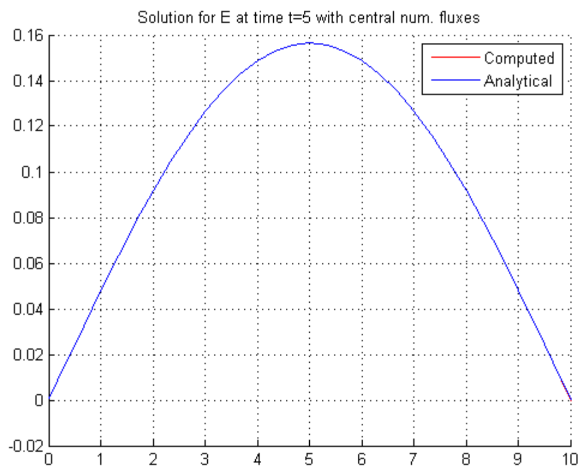


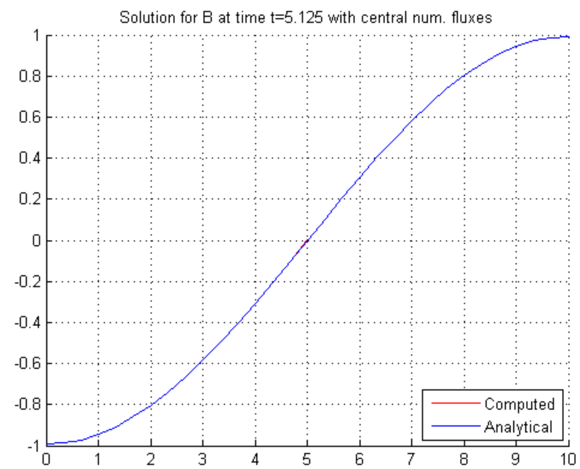


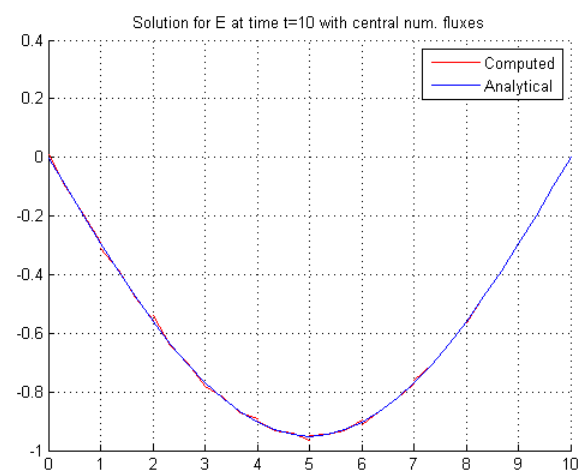


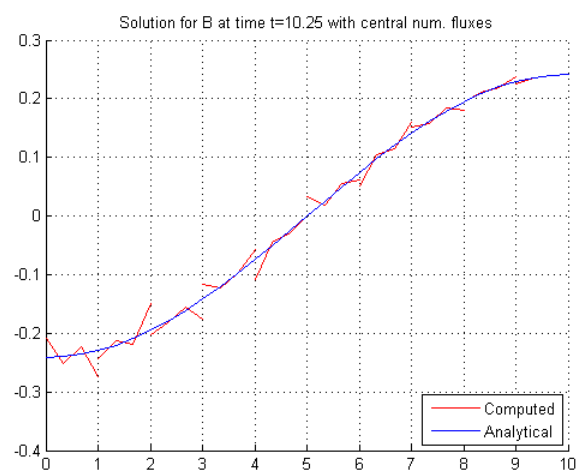
Application to Electromagnetics

Results with central numerical fluxes
(for reasonable choice of Δt and on stability region boundary)









Conclusions

- DG + StaggeredLF4 is good combination for solving a hyperbolic problem.
- With Legendre polynomials basis functions, choice of numerical flux heavily affects amplification matrix.
- Central numerical fluxes are more preferrable due to better stability characteristics (use of purely upwind numerical fluxes results in more than 15 times stricter condition on Δt).
- On-going study: check the same tendency with Lagrange polynomials basis functions.

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