Multi-Objective Optimization

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Presentation on Internship

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❖Prelude

❑ Internship Period

- Internship lasted from the first week of May until mid July, 2009
- Under the supervision of Mr. Désidéri

❑ Objective

- The project I was involved in was related to the Multi-objective optimization with the objective functions related to aeronautics
- The main task was to develop an algorithm in Scilab which can be used to find Pareto Optimal set and plot Pareto front

❑ Abstract

- In aeronautics, performance of an aircraft depends upon several factors such as, lift, drag, moments. Also important is the structural integrity of the aircraft. But, often optimizing one of the criteria has adverse effects on the other
- To cater this issue, simultaneous optimization of all the important factors need to be carried out

❖Optimization

❑ Optimization in aerospace

- From the beginning, engineers in this field are eager to employ the optimization methods so as to save that extra inch

❑ Multi-objective Optimization

- Multi-objective optimization has its root in late nineteenth century welfare economics, in the works of Edgeworth and Pareto
- Simply when there are two or more objective functions to be minimized
- Often these functions are contradicting in behavior
- Some examples can be found in following sectors
  - Bridge construction, Aircraft design, Chemical Plant design etc.
- Mathematically

$$\min_{x \in C} F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_{n-1}(x) \\ f_n(x) \end{bmatrix}$$

where n ≥ 2

$$C = \{x : h(x) = 0;\, g(x) \leq 0, a \leq x \leq b\}$$

❖Multi-Objective Optimization

- Multi-objective optimization can be further classified depending upon the type of objectives involved:
  - Multi-criterion Optimization
  - Multi-point Optimization
  - Multi-discipline Optimization
- In our problem, we deal with multi-discipline optimization problem
- Multi-Disciplinary Optimization
  - Optimization problem consists of objective functions from a variety of disciplines
  - Presence in a number of fields, including automobile design, naval architecture, electronics, computers and electricity distribution
  - Example: Boeing blended wing body (BWB) aircraft

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Pareto Optimality

- Important in Multi-objective optimization and a convenient representation
- Named after Vilfredo Pareto, it's a measure of efficiency
- Pareto Optimal solution is the one that cannot be further improved without hurting at least one player
- Such design vector x^* is called Pareto optimal
- The vectors x^* corresponding to the solutions included in the Pareto Optimal set are called "non-dominated"

Dominance/Non-dominance

- Y^1 is said to dominate the design point Y^2

$$Y^1 \succ Y^2$$

Iff, for all $J = J_A, J_B, \ldots$

$$J(Y^1) \leq J(Y^2)$$

and at least one of the inequalities is strict

- Otherwise, the vectors are said to be non-dominated

$$Y^1 \not\succ Y^2, Y^2 \not\succ Y^1$$

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Pareto Front

- The plot of the objective functions whose non-dominated vectors are in the Pareto optimal set is called the "Pareto Front"
- Useful in engineering
- Convenient way of considering only Pareto efficient alternatives

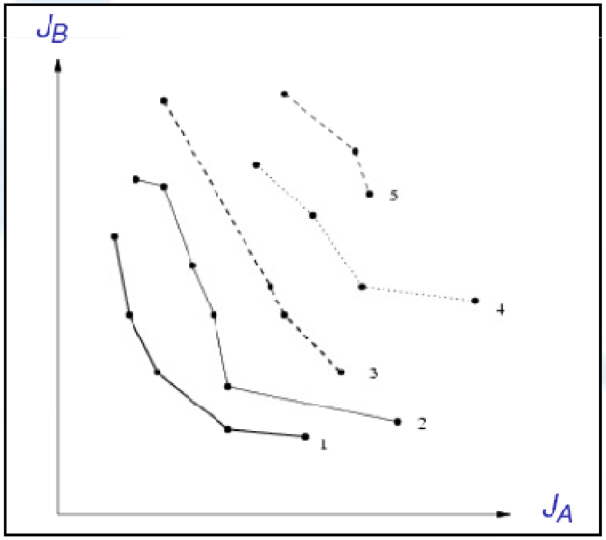

Figure 1: Pareto front

❖Problem

- Multi-objective optimization with 3 objective functions
  - J_1, J_2 and J_3
  - Most probably related to aerodynamic design, structural, thermal and acoustics etc
  - Each criterion is considered to be a smooth function of a common design vector $Y \in \mathbb{R}^4$

- Problem Statement
  - Design vector

$$Y \in \mathbb{R}^N$$

  - Minimize the given criteria

$$J_i(Y) \quad i = 1, 2, \dots, n$$

  - Where

$$N \geq n$$

  - We consider N=4 and n=3
  - The space of design vector can be a Hilbert Space usually equal to $\mathbb{R}^N$, but it can also be a subspace of L^2

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❖Pareto Concepts

- Some important concepts
  - Y ∈ $\mathcal{H}$; $\mathcal{H}$: working space, a Hilbert space equal to $\mathbb{R}^N$, can also be a subspace of L^2
  - The objective functions are assumed to be class C^2 in some working open ball of the design space $\mathcal{H}$

- Lemma 1
  - Let be a Pareto optimal point of the smooth criteria $J_i(Y)\ (1 \le i \le n \le N)$ and define the gradient vectors $u_i^0 = \nabla J_i(Y^0)$ in which denotes the gradient operator. There exists a convex combination of the gradient vectors that is equal to zero:

$$\sum_{i=1}^{n} \alpha_i u_i^0 = 0, \quad \alpha_i \ge 0, \quad \sum_{i=1}^{n} \alpha_i = 1$$

Pareto Stationary

- The smooth criteria $J_i(Y)$ are said to be Pareto stationary if they satisfy lemma 1, i.e.

$$\sum_{i=1}^{n} \alpha_i u_i^0 = 0, \quad \alpha_i \geq 0, \quad \sum_{i=1}^{n} \alpha_i = 1$$

- For smooth unconstrained criteria, Pareto stationarity is a necessary condition for Pareto optimality
- If the smooth criteria $J_i(Y)$ are not Pareto stationary at a given design point then descent directions common to all criteria exist

Lemma 2

- Let $\mathcal{H}$ be a Hilbert space of finite or infinite dimension N, and $\{u_i\}(1 \leq i \leq n \leq N)$ a family of n vectors in $\mathcal{H}$. Let $\mathcal{U}$ be the set of strict convex combinations of these vectors

$$\mathcal{U} = \left\{ \omega \in \mathcal{H} \,/\, \omega = \sum_{i=1}^{n} \alpha_i u_i \,;\, \alpha_i > 0 \,;\, \sum_{i=1}^{n} \alpha_i = 1 \right\}$$

- and $\overline{\mathcal{U}}$ its closure (the convex hull of the family). Then, there exists a unique element of minimum norm, and:

$$\forall \bar{u} \in \overline{\mathcal{U}} : (\bar{u}, \omega) \geq (\omega, \omega) = \|\omega\|^2 := C_\omega$$

Theorem 1

- Let H be a Hilbert space of infinite or finite dimension N. Let $J_i(Y)\ (1 \leq i \leq n \leq N)$ be n smooth functions of the vector Y Є H, and Y^0 a particular admissible design point, at which the gradient vectors are denoted by $u_i^0 = \nabla J_i(Y^0)$ and,

$$\mathcal{U} = \left\{ \omega \in \mathcal{H} \,/\, \omega = \sum_{i=1}^{n} \alpha_i u_i \,;\, \alpha_i > 0 \,;\, \sum_{i=1}^{n} \alpha_i = 1 \right\}$$

- Let ω be the minimal norm element of the convex hull $\overline{\mathcal{U}}$, closure of U. Then:
  - Either ω = 0, and the criteria are Pareto stationary at $Y = Y^0$
  - Or ω ≠ 0 and – ω is a descent direction common to all the criteria; additionally, if ωЄU, the inner product $(\overline{u}, \omega)$ is equal to $\|\omega\|^2$ for all $\overline{u} \in \overline{\mathcal{U}}$

- Hence,
  - We identify the following vector

$$\omega = \sum_{i=1}^{n} \alpha_i u_i^0$$

by solving the following constrained minimization problem

$$\min_{\alpha \in \mathbb{R}^n} \left\| \sum_{i=1}^{n} \alpha_i u_i^0 \right\|^2$$

With following constraints

$$\alpha_i \geq 0 \, (\forall i) \, ; \, \sum_{i=1}^{n} \alpha_i = 1$$

❖Scilab Routines

❑ Optim

- Solve non-linear optimization problem
- Can define the prefered algorithm to solve the problem
- But, allows constraints in the form of lower and upper bounds on design variable **only**
- Can only define lower bound not the upper bound

❑ Quapro/Qpsolve

- Older version of Scilab has quapro, now replaced by qpsolve in recent versions
- Used for objective function consisting of a quadratic form plus a linear combination of the design variables
- Constraint functions can be defined in addition to bounds on design variable
- The constraints functions should be linear
- But the matrix describing the quadratic part of objective function should be only positive definite symmetric matrix
- We cannot guarantee the positive definitivity

❖Algorithm

❑ Minimization problem

$$\omega = \sum_{i=1}^{n} \alpha_i u_i^0$$

- Minimize

$$\min_{\alpha \in \mathbb{R}^n} \left\| \sum_{i=1}^{n} \alpha_i u_i^0 \right\|^2$$

- Subject to constraints

$$\alpha_i \geq 0 \ (\forall i) \ ; \ \sum_{i=1}^{n} \alpha_i = 1$$

❑ Initial step

- Choose a design vector Y^0 and make an initial guess on α

$$\alpha^0 = \left[\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right]$$

- Our case, n = 3 and N = 4

$$\alpha^0 = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$$

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- Criteria and derivatives at Y^0

$$J_i(Y^0)\,(1 \le i \le n \le N)$$
$$u_i^0 = \nabla J_i(Y^0)$$

- Design of Experiment
  - Proposed by Ronald A. Fisher, in his innovative book The Design of Experiments (1935)
  - Design of all information-gathering exercises where variation is present
  - Purpose of it is to study the effect of some processes or intervention on some objects

- Check condition on ω, i.e.
  - If ω = 0, stop, that is it is already a Pareto optimal point
  - If ω ≠ 0, ω is the descent direction

- Redefine, ω as

$$\omega = \sum_{i=1}^{n-1} \alpha_i u_i^0$$

$$q = \min_{\alpha \in \mathbb{R}^{n-1}} \|\omega\|^2 = \min_{\alpha \in \mathbb{R}^{n-1}} \left\| \sum_{i=1}^{n-1} \alpha_i u_i^0 \right\|^2$$

- Constraints take the following new form

$$\alpha_i \geq 0 \; (\forall i) \; ; \; \alpha_n = 1 - \sum_{i=1}^{n-1} \alpha_i$$

- Define Partial derivatives of q w.r.t. A

$$\frac{\partial q}{\partial \alpha_i} = 2(\omega, u_i^0 - u_n^0) \qquad \forall\, i = 1, 2, \ldots n-1$$

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- For i = 1,…, n-1
  - Define α′ as

$$\alpha_i' = \alpha_i^0 - \rho \left(\frac{\partial q}{\partial \alpha_i}\right)_{\alpha_i^0} \qquad \forall\, i = 1,2,\dots n-1$$

  - α′ can be set to value either 0 or 1 depending on the condition and then ρ can be defined as

$$\frac{\partial q}{\partial \alpha_i} \geq 0 \quad \Rightarrow \rho_i \leq \alpha_i^0 \Big/ \frac{\partial q}{\partial \alpha_i}$$

$$\frac{\partial q}{\partial \alpha_i} < 0 \quad \Rightarrow \rho_i \leq 1 - \alpha_i^0 \Big/ \left|\frac{\partial q}{\partial \alpha_i}\right|$$

  - Summarizing

$$\rho_{i,max} := \begin{cases} \alpha_i^0 \Big/ \frac{\partial q}{\partial \alpha_i} & \frac{\partial q}{\partial \alpha_i} > 0 \\ 1 - \alpha_i^0 \Big/ \left|\frac{\partial q}{\partial \alpha_i}\right| & \frac{\partial q}{\partial \alpha_i} < 0 \end{cases}$$

- For α_n
  - We make use of the condition that we defined earlier

$$\alpha_n = 1 - \sum_{i=1}^{n-1} \alpha_i$$

$$\sum_{i=1}^{n-1} \alpha_i' = \sum_{i=1}^{n-1} \alpha_i^0 - \rho \sum_{i=1}^{n-1} \frac{\partial q}{\partial \alpha_i}$$

$$\alpha_n' = \alpha_n^0 + \rho Q' \qquad Q' := \sum_{i=1}^{n-1} \frac{\partial q}{\partial \alpha_i}$$

  - We again have similar conditions

$$\rho_{n,max} := \begin{cases} {1 - \alpha_n^0}/{Q'} & Q' > 0 \\ {\alpha_n^0}/{|Q'|} & Q' < 0 \end{cases}$$

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Maximum ρ

- Combined, these two gives us a condition on the choice of ρ. We will now consider the minimum of ρ in the set

$$\rho_{max} = \min_{i=1,\dots,n} \rho_{i,max}$$

- Discretize this ρ_{max} into a number of intervals, say 5 or 10

$$\rho_k = \frac{\rho_{max}}{k} \qquad k = 1, \dots, m$$

- Then recalculate α′ and

$$\alpha_i' = \alpha_i^0 - \rho_k \left(\frac{\partial q}{\partial \alpha_i}\right)_{\alpha_i^0} \qquad \forall\, i = 1,2, \dots n-1$$

$$\alpha_n' = \alpha_n^0 + \rho_k Q' \qquad \forall\, i = n$$

- Recalculate ω
  - This new α takes place of α^0 and we again calculate ω for every ρ_k until it reaches ρ_{max} or the value of ω starts to increase
  - As soon as one of the criteria increase, we stop

- Convergence on ω
  - We take this new α as α^0 and repeat the steps until we get convergence on ω

- Compute new values of criteria
  - Once convergence on ω is achieved we get out of loop and compute an interval using

$$\Delta t_{i,max} = \frac{l_i}{u_i.\omega}$$

  - Take maximum among the 3 and discretize the interval $[0, t_{max}]$

$$t_{max} = \max_{i=1,\dots,n} \Delta t_{i,max}$$

- Set design variable as
  - For every point in the interval [0, tmax],

$$Y^0 = Y^0 - t\omega$$

  - And then evaluate the criteria at each new vector

$$j_i(t) = J_i(Y^0 - t\omega) \ (1 \le i \le n)$$

  - The step size should be such that it is the largest strictly positive real number for which all the functions are monotone-decreasing over the interval $[0,t_{max}]$
  - Continue until increase in any one of the criteria is encountered
- This is our updated design vector
  - Set this new design vector as Y^0 and start anew the whole process from the beginning
  - Continue until the condition on ω is satisfied

$$\|\omega\| < tol \qquad stop$$

  - This results in one point on the Pareto Optimal set

❖Example and Solution

- Criteria J1, J2 and J3

$$J_1(y) = 2\left((2+\sqrt{2})y_1^2 + \sqrt{2}y_3^2 + y_4^2\right)$$

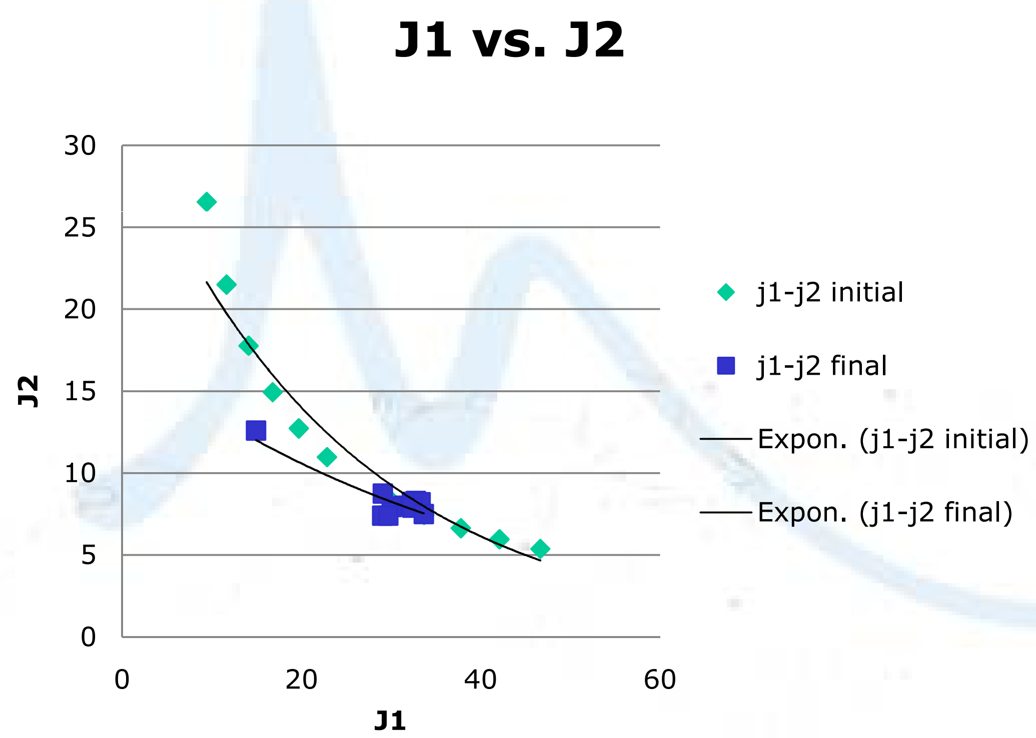
$$J_2(y) = 3\left(\frac{5}{3y_1^2} + \frac{3}{2y_2^2} + \frac{2}{y_3^2} + \frac{2}{y_4^2}\right)$$

$$J_3(y) = \frac{1}{y_1^2} + \frac{2}{y_2^2} + \frac{2}{y_3^2} + \frac{1}{4y_4^2}$$

- Set of design variables

$$y = \left\{ \begin{bmatrix}0.9\\0.9\\0.9\\0.9\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1.1\\1.1\\1.1\\1.1\end{bmatrix}, \begin{bmatrix}1.2\\1.2\\1.2\\1.2\end{bmatrix}, \begin{bmatrix}1.3\\1.3\\1.3\\1.3\end{bmatrix}, \begin{bmatrix}1.4\\1.4\\1.4\\1.4\end{bmatrix}, \begin{bmatrix}1.6\\1.6\\1.6\\1.6\end{bmatrix}, \begin{bmatrix}1.7\\1.7\\1.7\\1.7\end{bmatrix}, \begin{bmatrix}1.8\\1.8\\1.8\\1.8\end{bmatrix}, \begin{bmatrix}1.9\\1.9\\1.9\\1.9\end{bmatrix}, \begin{bmatrix}2.0\\2.0\\2.0\\2.0\end{bmatrix} \right\}$$

- Graph of the Pareto front involving



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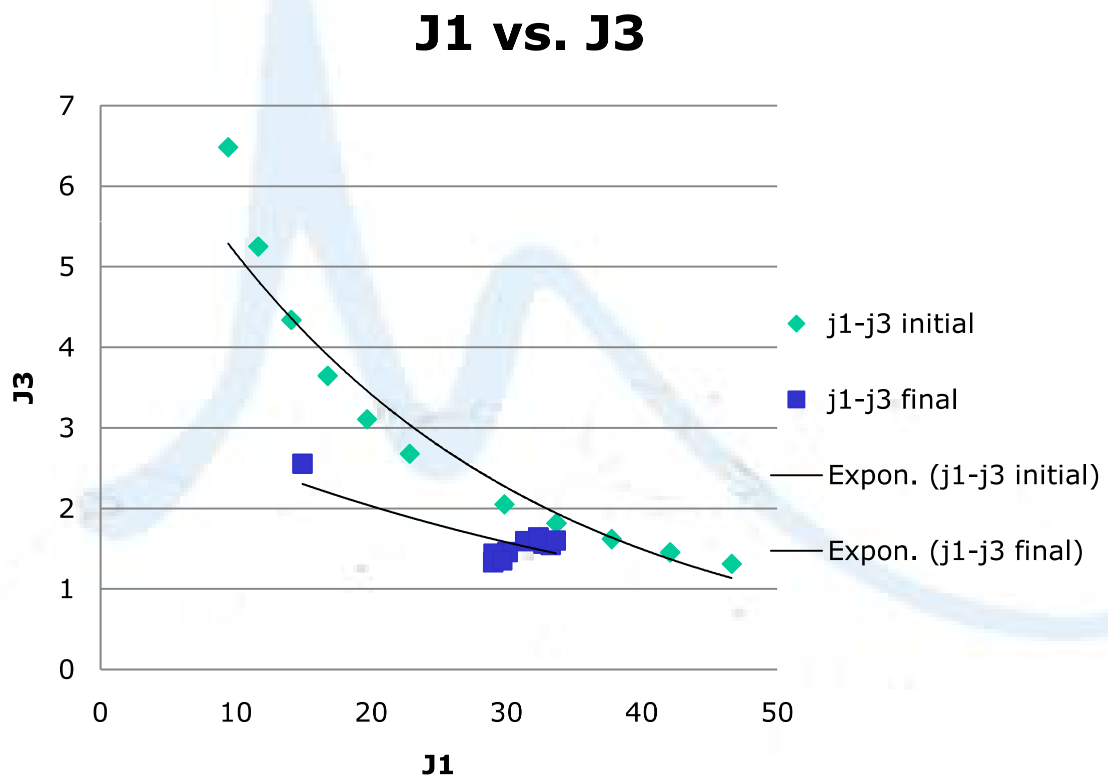
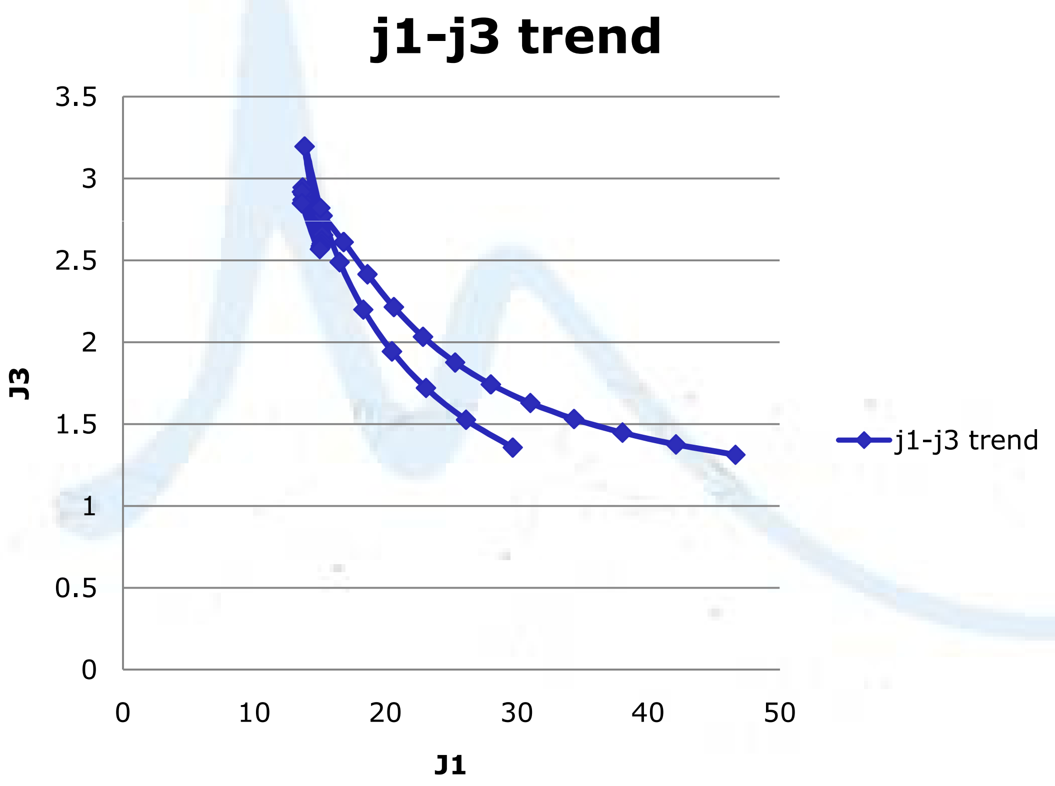
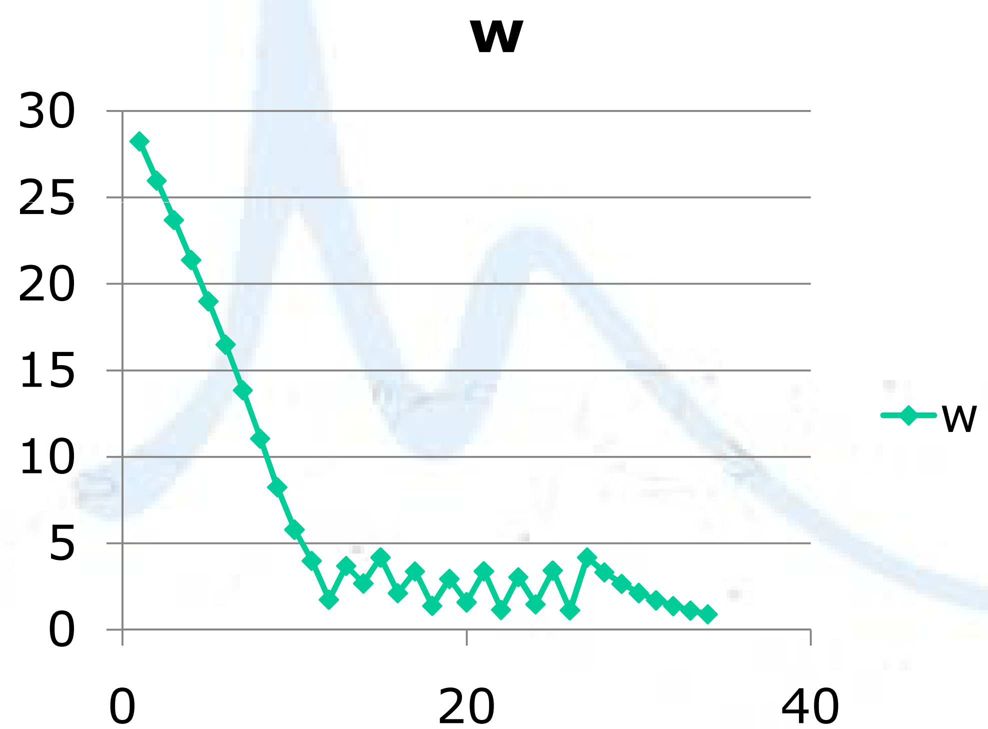


- For one set of iterations, the trend in the relative values



- Trend in the variation of ω in one set of iterations



- Results for first iteration for the complete set of design vectors

	initial			
Y	j1	j2	j3	w
0.9	9.442052	26.54321	6.481481	8.258974
1	11.65685	21.5	5.25	6.020793
1.1	14.10479	17.7686	4.338843	4.523512
1.2	16.78587	14.93056	3.645833	3.484256
1.3	19.70008	12.72189	3.106509	2.740463
1.4	22.84743	10.96939	2.678571	2.194167
1.6	29.84155	8.398438	2.050781	20.14365
1.7	33.68831	7.439446	1.816609	22.29636
1.8	37.76821	6.635802	1.62037	24.35648
1.9	42.08124	5.955679	1.454294	26.33168
2	46.62742	5.375	1.3125	28.23428

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- Results for last iteration for the complete set of design vectors

	final			
Y	j1	j2	j3	w
0.9	32.73655	8.294122	1.562837	0.906045
1	33.25493	8.20128	1.548888	0.894161
1.1	32.3329	8.233263	1.603136	0.945527
1.2	31.38221	7.985843	1.595362	0.944998
1.3	32.34522	7.881773	1.641581	0.993237
1.4	33.62845	7.48323	1.603102	0.96563
1.6	29.00605	7.39153	1.333886	0.918005
1.7	30.07409	7.913961	1.460697	0.88562
1.8	29.05121	8.738566	1.440301	0.905873
1.9	14.92747	12.59104	2.556135	0.813994
2	29.67797	7.391229	1.356703	0.892021

Thank You