MULTIOBJECTIVE OPTIMIZATION

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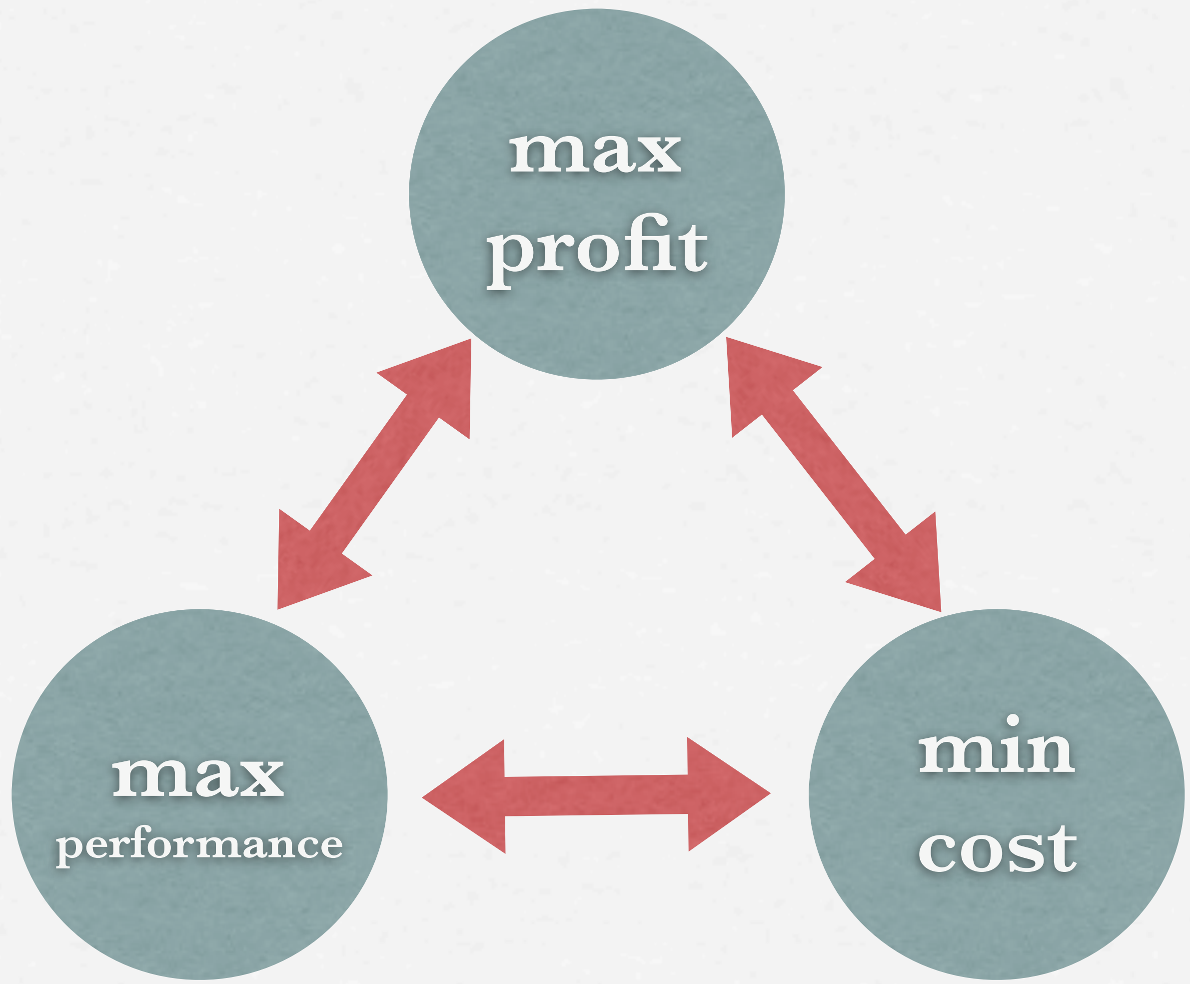
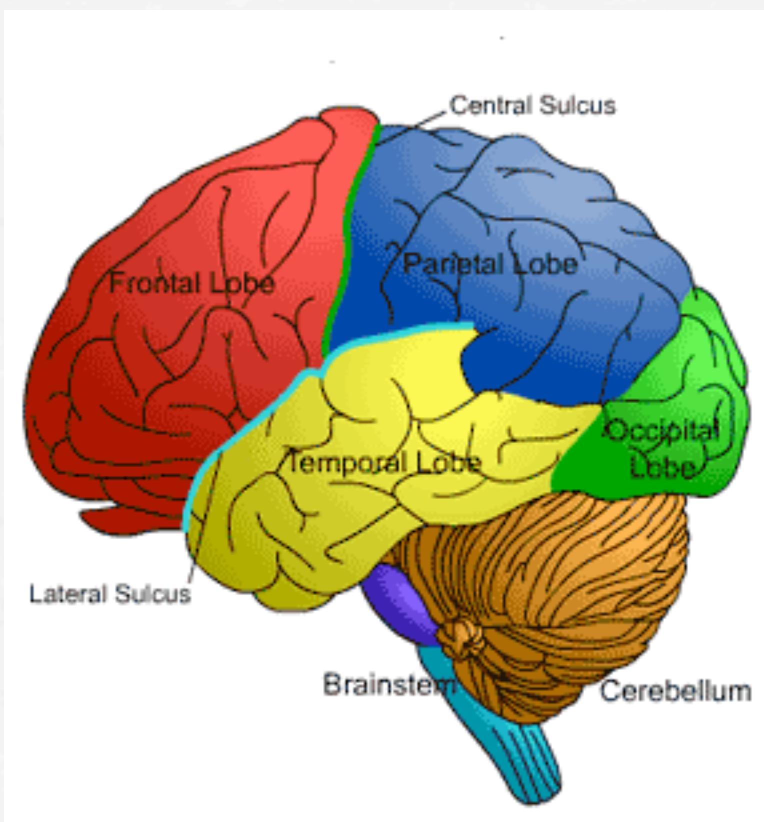
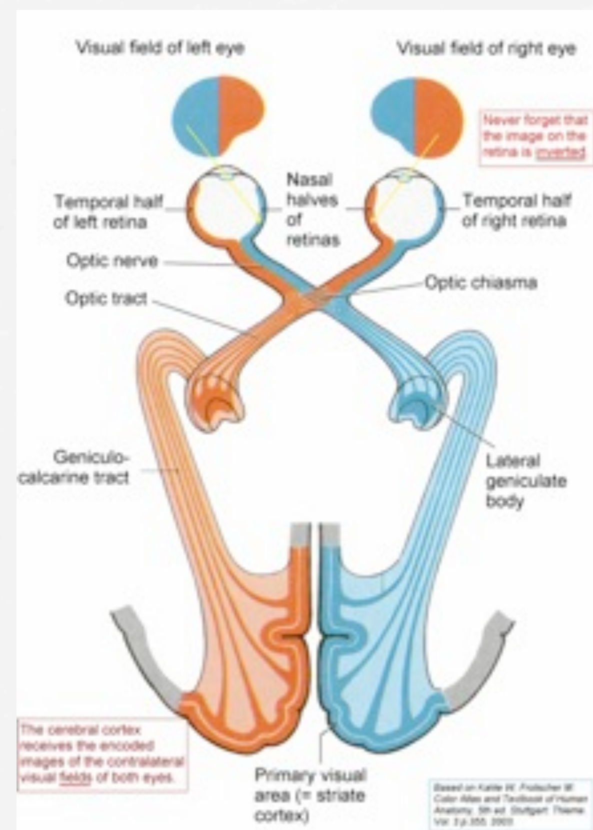
WHAT IS MULTIOBJECTIVE OPTIMIZATION?



WHY IS IT INTERESTING?

- In brain, there are several regions with several functions. They work together to control the body.





pictures from : http://scienceblogs.com/purepedantry/2007/10/ocular_dominance_columns_and_t.php

MULTIOBJECTIVE OPTIMIZATION IS VERY USEFUL IN MANY FIELDS.

MOPs

- ★ Product and process design
- ★Finance
- ★Aircraft design
- ★Oil and gas industry
- ★Automobile design

How about NEUROSCIENCE?

PLAN

- Study multiobjective optimization in general.
- Try to apply MOPs to make some case studies in neuroscience.

HOW DO WE WORK?

Main references:

NONLINEAR MULTIOBJECTIVE OPTIMIZATION
(Kaisa M. Miettinen, 1999)

MATHEMATICAL PROBLEMS IN IMAGE PROCESSING
(G.Aubert and P.Kornprobst, 2002)

Along the way

- Calculus of variations
- Differential geometry
- Multiobjective optimization with equilibrium constraints
- Bilevel optimization
- Evolutionary mutiobjective optimization

Case study: Ambrosio-Tortorelli image segmentation

MULTIOBJECTIVE OPTIMIZATION

$$\text{minimize } f(x) = (f_1(x), \ldots, f_k(x))$$
$$\text{subject to } x \in S = \{x \in \mathbb{R}^n | g(x) = (g_1(x), \ldots, g_m(x)) \leq 0)\}$$

$$f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, k$$

$$g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$$

We call S the "feasible region"
and $Z = f(S)$ the "feasible objective region".

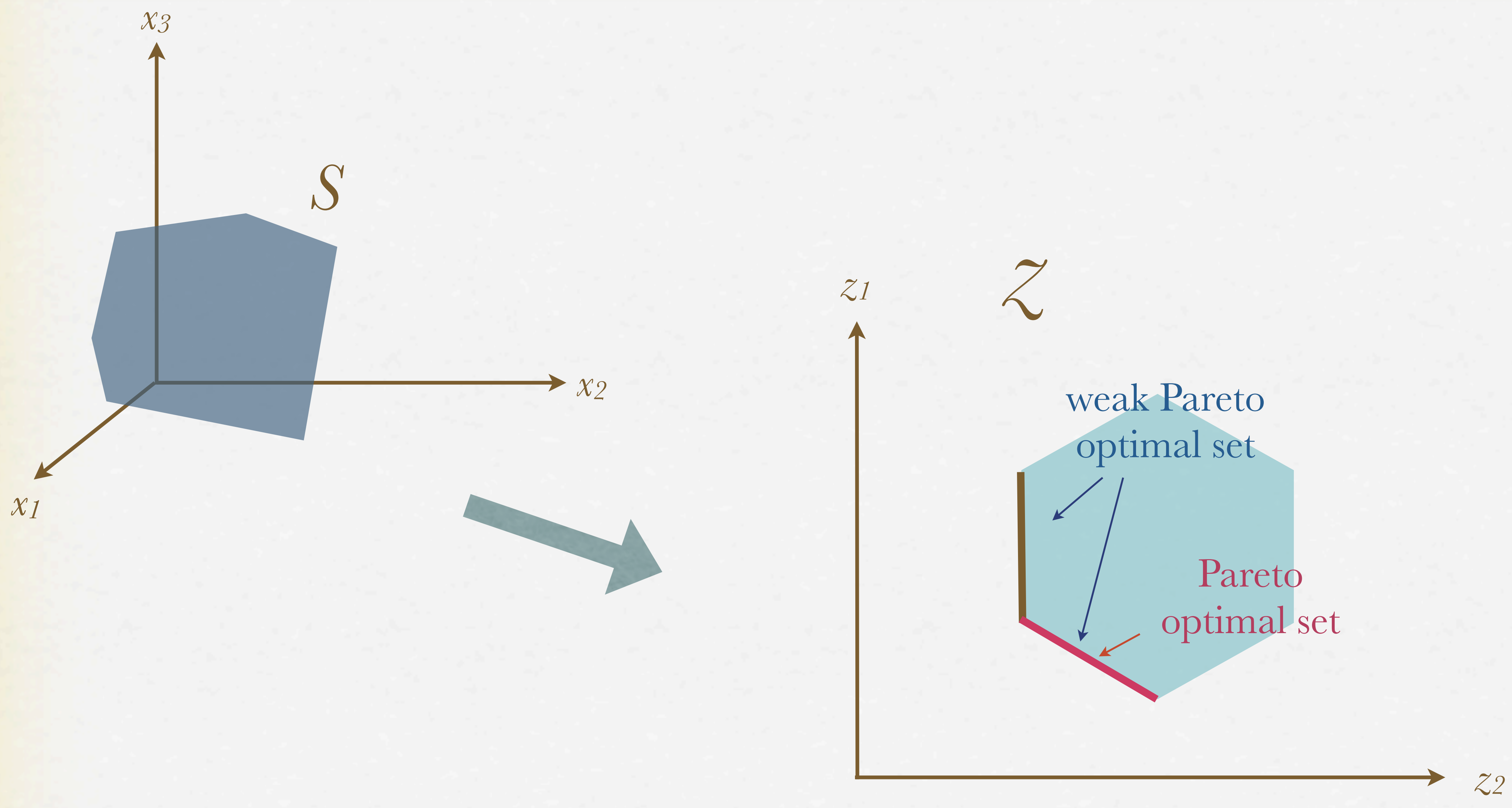
PARETO OPTIMAL

Pareto optimality : An decision vector $x^* \in S$ is Pareto optimal if there does not exist another decision vector $x \in S$ such that $f_i(x) \leq f_i(x^*)$ for all $i = 1, \ldots, k$ and $f_j(x) < f_j(x^*)$ for at least on index j.

Weak Pareto optimality : An objective vector z^* is weakly Pareto optimal if there does not exist another decision vector $z \in S$ such that $f_i(y) < f_i(x)$ for all $i = 1, \ldots, k$.

An objective vector z^* is Pareto optimal(weakly Pareto optimal) if the decision vector corresponding to it is Pareto optimal.

EXAMPLE



GENERAL PROCEDURE

Pareto optimal set

The decision maker select the final solution from the Pareto optimal set.

METHODS

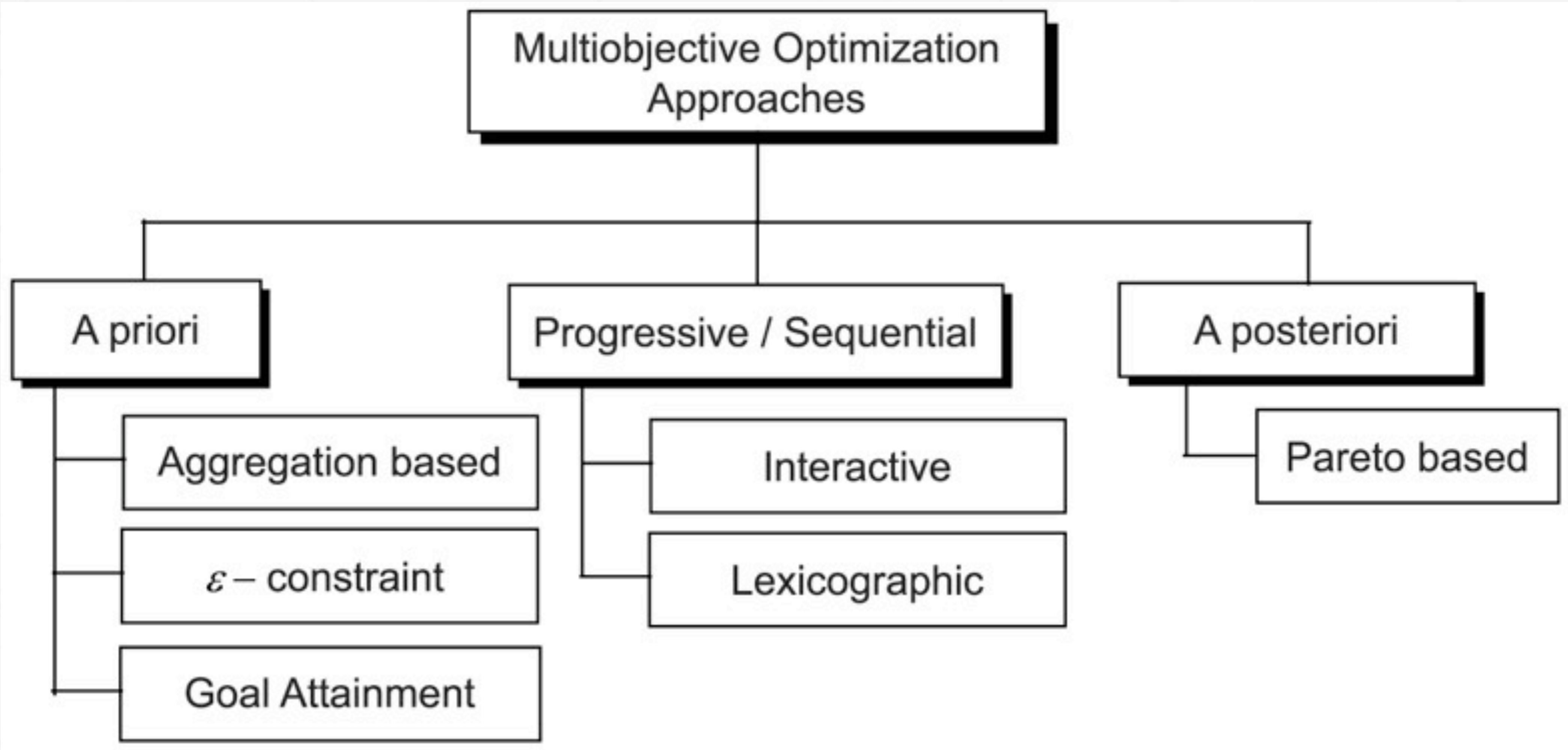


chart from : http://www.emeraldinsight.com/fig/1740240307004.png

EPSILON-CONSTRAINT METHOD

$$\begin{aligned} &\text{minimize } (f_1(x), \ldots, f_k(x)) \\ &\text{subject to } (g_1(x), \ldots, g_k(x)) \leq 0 \end{aligned}$$

ϵ-Constraint method:

$$\begin{aligned} &\text{minimize } f_l(x) \\ &\text{subject to } (g_1(x), \ldots, g_m(x)) \leq 0 \\ &\text{and } f_j(x) \leq \epsilon_j \text{ for all } j \neq l \end{aligned}$$

KARUSH-KUNH-TUCKER CONDITIONS APPLIED TO THE EPSILON-CONSTRAINT PROBLEM

Let the objective and the constraint functions be continuously differentiable at x^* which is regular point of the constraint problem of the ϵ-constraint problem.A necessary condition for x^* to be a solution of the ϵ-constraint problem is that there exist vector $0 \leq \lambda \in \mathbb{R}^{k-1}$ and $0 \leq \mu \in \mathbb{R}$ such that

(1) $\nabla f_l(x^*) + \sum_{j=1, j \neq l}^{n} \lambda_j \nabla (f_j(x^*) - \epsilon_j) + \sum_{i=1}^{m} \mu \nabla g(x^*) = 0$

(2) $\lambda_j (f_j(x^*) - \epsilon_j) = 0$ for all $j \neq l$, $\mu_i g_i(x^*) = 0$ for all $i = 1, \ldots, m$

CASE STUDY

- Given $\Omega \in \mathbb{R}^2$
- Image as a function $u : \Omega \to \mathbb{R}$
- $u(x, y) :=$ Intensity

IMAGE SEGMENTATION

Input | Outputs

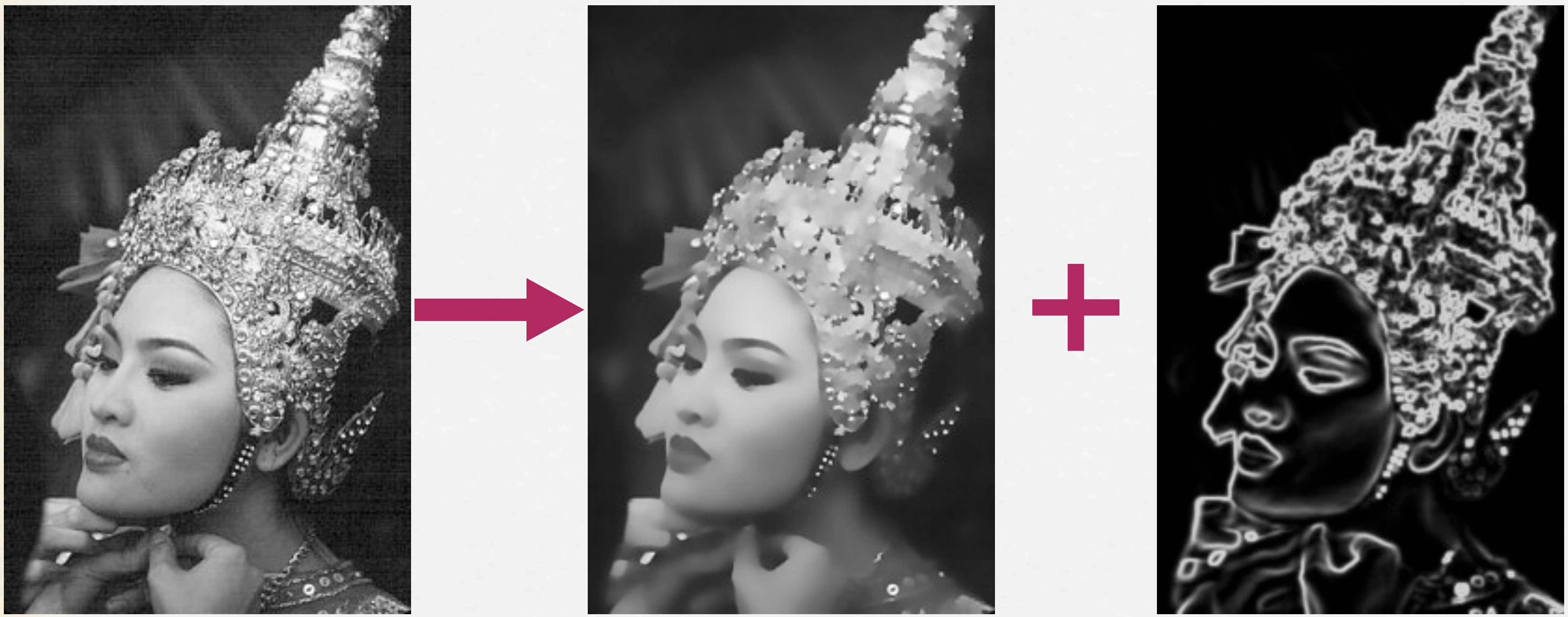


Image segmentation is typically used to locate objects and boundaries (lines, curves, etc.) in images.

MUMFORD-SHAH SEGMENTATION FUNCTIONAL

$$E_{MS}(u,B) = \alpha \int\int_{R \setminus B} \|\nabla u\|^2 dxdy + \beta \int\int_R (u-g)^2 dxdy + |B|$$

where
R is connected, bounded, open set of $\mathbb{R}^2$
B is a curve segmenting R
$|B|$ is the length of B
g is the feature intensity
u is the smoothed image $\subset \mathbb{R}^2 \setminus B$
α, β are the weights.

The minimizer u of this functional is a smooth approximation of g in each sub-domain segmented by B.

AMBROSIO-TORTORELLI EDGE-STRENGTH FUNCTIONAL

$$E_{AT}(u,v) = \int\int_R \{\alpha(1-v)^2\|\nabla u\|^2 + \beta(u-g)^2 + \frac{\rho}{2}\|\nabla v\|^2 + \frac{v^2}{2\rho}\}dxdy$$

where
R is connected, bounded, open set of $\mathbb{R}^2$
g is the feature intensity
u is the smoothed image
v is a continuous variable
α, β, ρ are the weights.

AMBROSIO-TORTORELLI FUNCTIONAL

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where
R is connected, bounded, open set of $\mathbb{R}^2$
g is the feature intensity
u is the smoothed image
v is a continuous variable
α, β, ρ are the weights.

GOAL

Find (u^*, v^*) which minimizes $E_{AT}(u, v)$ that is

$$\begin{aligned} &\text{minimize } E_{AT}(u,v) \\ &\text{subject to } (u,v) \in W^{1,2}(R) \end{aligned}$$

which is "Variational optimization problem".

STANDARD APPROACH

$$\text{minimize} \int_R F(x, y(x), y'(x))dx$$

Necessary condition for y to be the minimum : the Euler-Lagrange equation

$$\frac{d}{dx}\left[\frac{\partial F(x,y,y')}{\partial y'}\right] = \frac{\partial F(x,y,y')}{\partial y}$$

GRADIENT DESCENT EQUATION

Fix v,
$$\frac{\partial u}{\partial t} = -2\nabla v \cdot \nabla u + (1 - v)\nabla^2 u - \frac{\beta(u - g)}{\alpha(1 - v)}$$

Fix u,
$$\frac{\partial v}{\partial t} = \nabla^2 v - \frac{v}{\rho^2} + \frac{2\alpha}{\rho}(1 - v)\|\nabla u\|^2$$

$$\frac{\partial u}{\partial n}|_{\partial R} = 0, \ \frac{\partial v}{\partial n}|_{\partial R} = 0$$

ALTERNATIVE IDEA

$$E_{AT}(u,v) = \int\int_R \{\alpha(1-v)^2\|\nabla u\|^2 + \beta(u-g)^2 + \frac{\rho}{2}\|\nabla v\|^2 + \frac{v^2}{2\rho}\} dxdy$$

Split and solve

$$E_1(u,v) = \int\int_R \alpha(1-v)^2\|\nabla u\|^2 dxdy$$

$$E_2(u,v) = \int\int_R \beta(u-g)^2 + \frac{\rho}{2}\|\nabla v\|^2 + \frac{v^2}{2\rho} dxdy$$

$$\text{minimize } \{E_1(u,v), E_2(u,v)\}$$
$$\text{subject to } (u,v) \in (W^{1,2}(R))^2$$

APPLY TO THE AMBROSIO-TORRORELLI FUNCTIONAL

ϵ-Constraint problem

$$\text{minimize } E_1(u, v)$$
$$\text{subject to } E_2(u, v) \leq \epsilon$$

Remark : The Ambrosio-Tortorelli functional is defined on continuous domains but we are going to apply the epsilon-constraint method which is for vector domains. Now I am considering the discretized images as vectors.

KARUSH-KUHN-TUCKER CONDITION

$$
\begin{aligned}
L(u, v, \lambda) &= E_1(u, v) + \lambda(E_2(u, v) - \epsilon) \\
L_u(u, v, \lambda) &= \frac{\partial}{\partial u}(E_1(u, v) + \lambda E_2(u, v) - \lambda\epsilon) = 0 \\
L_v(u, v, \lambda) &= \frac{\partial}{\partial v}(E_1(u, v) + \lambda E_2(u, v) - \lambda\epsilon) = 0 \\
& \lambda(E_2(u, v) - \epsilon) = 0 \text{ and } \lambda \geq 0
\end{aligned}
$$

GRADIENT DESCENT EQUATIONS

$$\frac{\partial u}{\partial t} = div(\alpha(1-v)^2 \nabla u) - \beta(u-g)$$

$$\frac{\partial v}{\partial t} = \nabla^2 v + \frac{2\alpha}{\rho}(1-v)\|\nabla u\|^2 - \frac{v}{\rho^2}$$

$$\lambda = \frac{\int\int_R \alpha(u-g) div((1-v)^2 \nabla u) dxdy}{\epsilon - \int\int_R \frac{\rho}{2}\|\nabla v\|^2 + \frac{v^2}{2\rho} dxdy}$$

We implement into computer by finite difference schemes.

THE EXPERIMENTS

- We do experiments on two images, one is an image with simple detail and another is more complicating. The parameters are fixed except for epsilon.

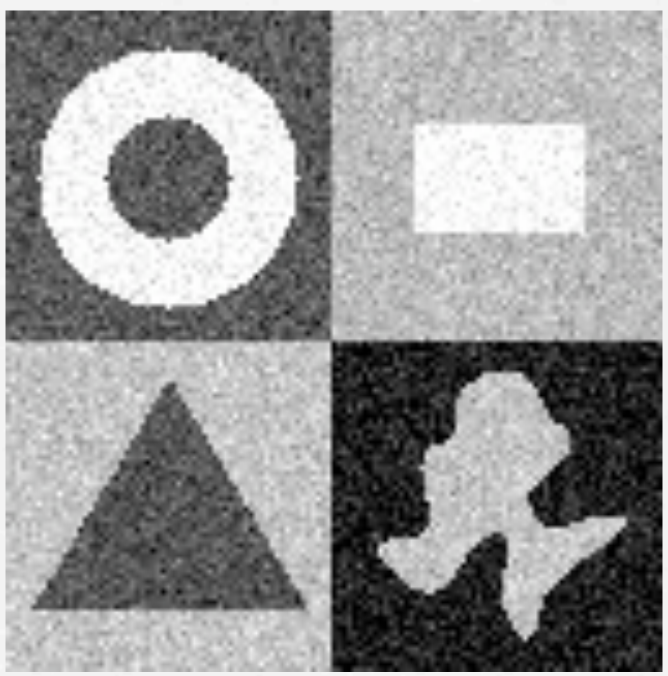


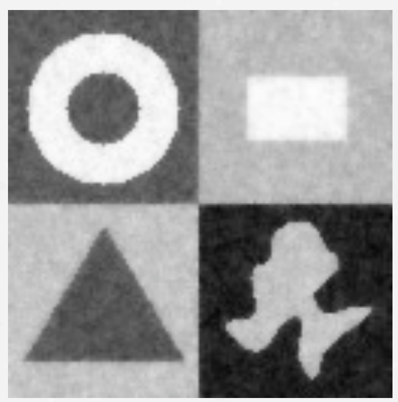
image 1

image 2

RESULTS

Parameters : $\alpha = 0.10, \beta = 0.001, \rho = 1.0,$ number of iterations $= 10$

standard



alternative epsilon=0.05

alternative epsilon=0.10

alternative epsilon=0.15

alternative epsilon=0.20

RESULTS

Parameters : $\alpha = 0.10, \beta = 0.001, \rho = 1.0$, number of iterations = 300

standard

alternative epsilon = 0.05

alternative epsilon = 0.10

alternative epsilon = 0.15

alternative epsilon = 0.20

DISCUSSION

- We haven't designed a good measurement to compare the efficiency between the standard approach and the alternative approach yet.
- The results can be observed by horizontal section which the solutions from standard approach seem to be smoother.
- We have some reformations from continuous domains to vector domains which have not been checked the availability well. The results are calculated approximately.

FUTURE WORK

- Try to apply other methods and problems.
- We have to study multiobjective optimization in infinite dimensional domains. Where most neuroscience problems locate in.
- The evolutionary algorithm for multiobjective optimization is also promising to be apply in this field.

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