CHEMOTAXIS PROBLEM

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Contents

Introduction

- The term *chemotaxis* indicates the motion of a population driven by the presence of an external stymolus, in response to gradients of such substance, called *chemoattractant*
- The most well known model of chemotaxis is the Patlak-Keller-Segel model

Introduction

- The basic unknowns in PDE models for chemotaxis are:
  - The density of individuals of the population $\rho(x,t)$
  - The concentration of the chemoattractant $c(x,t)$
- The aim of this work is to present one of the different possible study case of the Patlak-Keller-Segel model

Presentation of the problem

- We consider the following problem:

$$\begin{cases} \partial_t \rho - \mu \partial_{xx} \rho + \partial_x (\rho \chi(c) \partial_x c) - \gamma \rho = 0, \\ \epsilon \partial_t c - \nu \partial_{xx} c + \beta c - \alpha \rho = 0, \end{cases}$$

on the domain $\Omega = (a, b)$

With the boundary conditions:

$$\frac{\partial \rho}{\partial x}(a) = \frac{\partial c}{\partial x}(b) = 0$$

$$\frac{\partial \rho_0}{\partial x}(a) = \frac{\partial c_0}{\partial x}(b) = 0$$

$$\rho(0, x) = \rho_0(x) \geq 0, \text{ and } c(0, x) = c_0(x) \geq 0$$

Presentation of the problem

with $\rho(x,t) \in \Re^+, c(x,t) \in \Re^+$

and the hypotheses:

$\alpha, \beta \geq 0$

$\gamma = 0$

$\chi(c) = \chi$ constant

Weak formulation

- We consider the functional space H^1
- We seek $\rho, c \in H^1$ and $\varphi, \psi \in H^1$
- We integrate by parts and we use boundary conditions to obtain

$$\begin{cases} \langle \partial_t \rho, \varphi \rangle + \mu \langle \partial_x \rho, \partial_x \varphi \rangle + \chi \langle \rho \partial_x c, \partial_x \varphi \rangle = 0 \\ \epsilon \langle \partial_t c, \psi \rangle + \nu \langle \partial_x c, \partial_\psi \rangle + \beta \langle c, \psi \rangle - \alpha \langle \rho, \psi \rangle = 0 \end{cases}$$

Finite element method

- Let $\{V_h\}_{h\geq 0}$ be a family of approximating subspace of H^1 consisting of piecewise polynomials and let $\{T_h\}_{h\geq 0}$ be a partition of (a,b) made of intervals $(x_i, x_{i+1}), i = 1...N$ with $h = max_i(x_{i+1} - x_i)$
- We define

$$V_h = \{\varphi, \psi \in H^1; \varphi|_T, \psi|_T \in P(T), T \in T_h\}$$

Finite element method

with T generic interval and P a family of polynomials

- Finite element method consist in seeking a solution $\rho_h(t), c_h(t) \in V_h$ of the following problem:

$$\begin{cases} \langle \partial_t \rho_h, \varphi \rangle + \mu \langle \partial_x \rho_h, \partial_x \varphi \rangle - \langle \rho_h \chi \partial_x c_h, \partial_x \varphi \rangle = 0 \\ \epsilon \langle \partial_t c_h, \psi \rangle + \nu \langle \partial_x c_h, \partial_x \psi \rangle + \beta \langle c_h, \psi \rangle - \alpha \langle \rho_h, \psi \rangle = 0 \end{cases}$$

for all $\varphi, \psi \in V_h$

Energy estimate

- We put $\varphi = \rho_h$ and $c = c_h$ respectively in the two equation of the finite element formulation
- We sum the two equations to get an estimation
- We use Cauchy-Schwarz inequalities and Young inequalities to estimate the non linear term

Energy estimate

- We obtain the following energy estimate

$$\frac{1}{2}\frac{d}{dt}(\|\rho_h\|^2_{L^2} + \epsilon\|c_h\|^2_{L^2})$$
$$+\left(\mu - \frac{\chi\delta_2}{2}\right)\|\partial_x\rho_h\|^2_{L^2} + \left(\nu - \frac{\chi}{2\delta_2}\|\rho_h\|^2_{L^\infty}\right)\|\partial_x c_h\|^2_{L^2}$$
$$+\left(\beta - \frac{\alpha\delta_1}{2}\right)\|c_h\|^2_{L^2} \leq \frac{\alpha}{2\delta_1}\|\rho_h\|^2_{L^2} \quad (1)$$

which holds only under the condition

$$\|\rho_h\|^2_{L^\infty} \leq \frac{\nu 2\delta_2}{\chi}$$

Energy estimate

and considering small initial data

$\rho(0), c(0)$ and short time.

- In particular for the time we have

$$t < \ln\left(\frac{2\nu\delta_2}{\chi} h \frac{1}{\|\rho(0)\|_{L^2}^2 + \epsilon\|c(0)\|_{L^2}^2}\right)\frac{2\delta_1}{\alpha}$$

Error estimate

- We define

$$E_\rho = \rho - \rho_h = (\rho - \Pi_h \rho) + (\Pi_h \rho - \rho_h) = \eta_\rho + \theta_\rho$$

$$E_c = c - c_h = (c - \Pi_h c) + (\Pi_h c - c_h) = \eta_c + \theta_c$$

where Π_h is the elliptic projection operator.

- Using the property of Π_h and the inverse Sobolev inequality we can have:

Error estimate

$$\|(c - c_h)(t)\| \leq \|c_0 - c_h(0)\| + C^{st} h^r (\|c_0\| + \|c(t)\|)$$

$$+ C_1 C^{st} h^r \int_0^{t^*} (\|\partial_t c(t)\| + \|c(t)\|) dt + C_1 \int_0^{t^*} \|(\rho - \rho_h)(t)\| dt$$

where C_1, C are some constants depending on the differents parameters $\alpha, \beta, \epsilon \ldots$

- Similar arguments can be use for the estimate of E_ρ

Numerical implementation

- Simplified model
- We consider:
  - The constant $\chi = 1$
  - The Implicit Euler for time discretisation
  - The finite element method in space
  - The domain $(a, b) = (0, 1)$

Numerical implementation

- First we use the Implicit Euler scheme to obtain

$$\begin{cases} \eta\rho^{n+1} - \mu\partial_{xx}\rho^{n+1} + \partial_x(\rho^{n+1}\partial_x c^{n+1}) = \eta\rho^n + f^{n+1} \\ \epsilon\eta c^{n+1} - \nu\partial_{xx}c^{n+1} + \beta c^{n+1} - \alpha\rho^{n+1} = \eta\epsilon c^n, \end{cases} \quad (1)$$

- We consider the weak formulation of the problem
- We expand ρ and c in respect to the basis φ_i

Numerical implementation

- We can rewrite the system in the following form

$$\begin{cases} \eta \sum_{j=1}^{N} M_{ij} u_j^{n+1} + \mu \sum_{j=1}^{N} K_{ij} u_j^{n+1} \\ - \sum_{k=1}^{N} \sum_{j=1}^{N} c_j^{n+1} T_{ijk} u_k^{n+1} u_{j+N}^{n+1} \\ = \eta \sum_{j=1}^{N} M_{ij} u_j^{n} + b_i^n \\ (\epsilon\eta + \beta) \sum_{j=1}^{N} M_{ij} u_{j+N}^{n+1} + \nu \sum_{j=1}^{N} K_{ij} u_{j+N}^{n+1} \\ -\alpha \sum_{j=1}^{N} M_{ij} u_j^{n+1} = \eta\epsilon \sum_{j=1}^{N} M_{ij} u_{j+N}^{n} \end{cases} \quad (1)$$

Numerical implementation

- M mass matrix $M = \langle \varphi_j, \varphi_i \rangle$
- K stiffness matrix $K = \langle \varphi'_j, \varphi'_i \rangle$
- T triple matrix related to the non linear term $T = \langle \varphi_k \varphi'_j, \varphi'_i \rangle$
- b right hand side
- $\eta = \dfrac{1}{\delta t}$

➢ How deal with the non linear term?

Numerical implementation

- We consider the matricial form of the system $AU = B$
- First case: linearitation around u_k

Matrix A

$$\mathcal{A} = \begin{pmatrix} \eta M + \mu K & \tilde{T} \\ -\alpha M & (\epsilon\eta + \beta)M + \nu K \end{pmatrix}$$

where

$$\tilde{T} = -\sum_{k=1}^{N} T_{ijk} u_k^n$$

Numerical implementation

- Second case : linearization around u_{j+N}

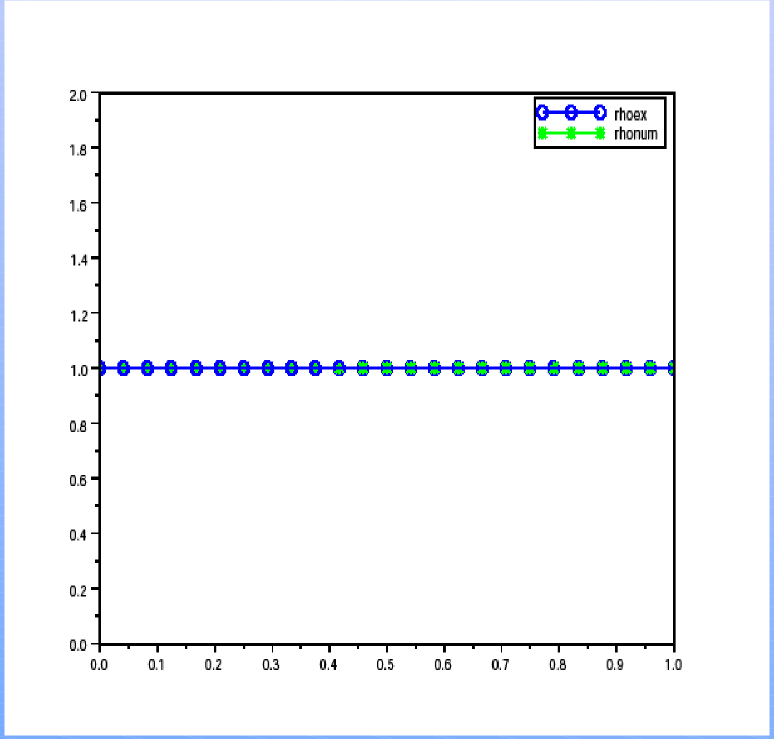
Matrix A

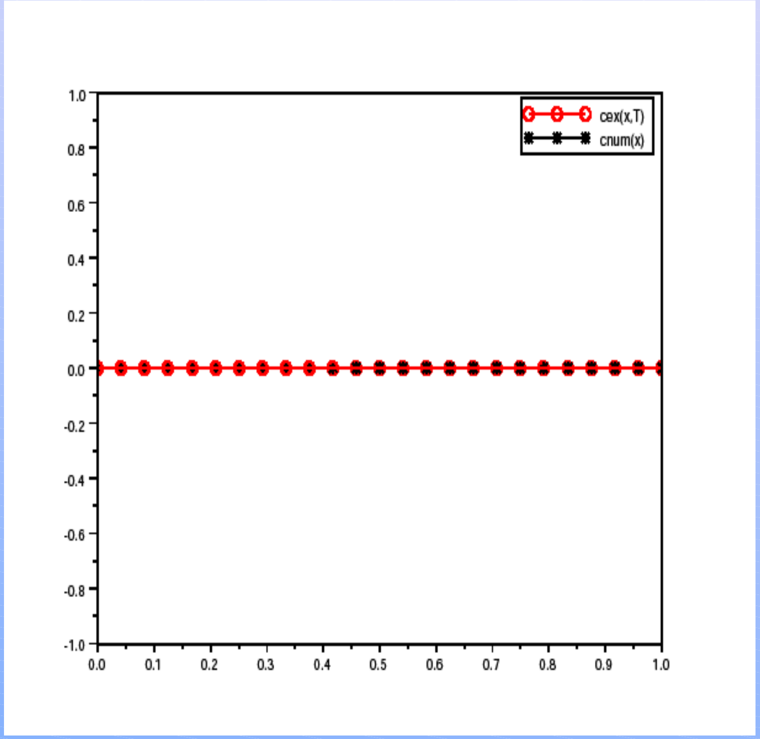
$$\mathcal{A} = \begin{pmatrix} \eta M + \mu K + \tilde{T} & 0 \\ -\alpha M & (\epsilon\eta + \beta)M + \nu K \end{pmatrix}$$

where

$$\tilde{T} = -\sum_{j=1}^{N} T_{ijk} u^n_{j+N}$$

Numerical results





Conclusions

- This work is manly an analytical study of a particolar case of the Keller-Segel model for chemotaxis
- We focus our attention in a 1D case to prove the convergence of the solution
- The same arguments can be used in more general case and in higher dimensions
- It's possible to prove the convergence of non constant solutions

Thanks for your attention