SOLVING THE SYSTEM OF LINEAR ELASTICITY BY A SCHWARZ METHOD

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Outline

Introduction.

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Linear elasticity:

- Simplified model in the case of small deformations
- Validity domain : stress states that do not produce yielding.
- Structural analysis and engineering design

Variational approach applied to the resolution of the system.

Modelisation

- Let Ω be an open set of R^N
- $f(x)$ a volumic force function from Ω to R^N
- the tensor of deformation $e(u)$:

$$e(u) = \frac{1}{2}(\nabla u + (\nabla u)^t) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \ i = j = 1 \ldots N.$$

- the tensor of constraint $\sigma(u)$:

$$\sigma = 2\mu e(u) + \lambda tr(e(u)) Id$$

Deriving the system of equations.

Using the sum of all the forces in the solid we obtain :

$$-div(\sigma) = f \quad \text{in } \Omega$$

Using the fact that $tr(u) = divu$, we can deduce the following equation :

$$-\sum_{j=1}^{N} \frac{\partial}{\partial x_j}(\mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + \lambda(divu)\delta_{ij}) = f_i \text{ in } \Omega$$

With u_i, f_i the components of f and u in the canonical basis of R^N.

Problem with mixed boundaries conditions.

Now we consider a system of linear elasticity with mixed boundaries conditions, Dirichlet and Neumann i.e

$$\begin{cases} -div(2\mu e(u)+\lambda tr(e(u))Id)=f \quad \text{in}\,\Omega \\ u=0\,\text{on}\,\partial\Omega_D \\ \sigma n = g\, on\, \partial\Omega_N \end{cases} \quad (1)$$

where $(\partial\Omega_N,\partial\Omega_D)$ is a partition of $\partial\Omega$ of non zero measure. Existence and uniqueness can be proved using Korn Inequality.

Korn Inequality interpretation

Lemma (Korn Inequality)

Let Ω be open bounded and regular set of class C^1 of R^N. There exists a constant C$>$ 0 such that for all function $v \in H^1(\Omega)^N$ we have :$||v||_{H^1(\Omega)} \leq$C $(||v||^2_{L^2(\Omega)} + ||e(v)||^2_{L^2(\Omega)})^{1/2}$.

Theorem

Let Ω an open bounded connected regular set of class C^1 of R^N.
Let $f \in L^2(\Omega)$ $g \in L^2(\partial\Omega_N)^N$ we define the space
V $= \{v \in H^1(\Omega)^N$ such that $v = 0$ on $\partial\Omega_D\}$
There exists a unique weak solution $u \in$ V of (1) which depends linearly on f and g

Minimun point of the energy.

Proposition

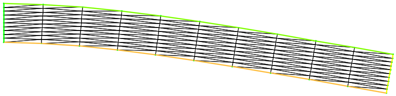
Let $j(v)$ the energy defined for all $v \in$ V by :

$$j(v) = \frac{1}{2}\int_{\Omega}(2\mu|e(v)|^2 + \lambda|divv|^2)dx - \int_{\Omega} f.vdx - \int_{\partial\Omega_N} g.vds$$

Let u be the unique solution of the variational formulation of (1), then$u$ is the unique minimum point of the above energy in V. Reciprocally if $u \in$ V is the minimum point of the energy $j(v)$ then u is the unique solution of the variational formulation.

Description of the physical model

Figure : Beam fixed on one side
Displacement under the influence of the gravity



Smith factorization

Theorem Let n be an positive integer and A an invertible $n \times n$ matrix with polynomial entries with respect to the variable λ: $A = (a_{ij}(\lambda))_{1 \leq i,j \leq n}$. Then, there exist matrices E, D and F with polynomial entries satisfying the following properties:

- $det(E)$ and $det(F)$ are constants,
- D is a diagonal matrix uniquely determined up to a multiplicative constant,
- $A = EDF$.

Application of the Smith factorization to the system of linear elasticity

The two dimensional system of linear elasticity is given by :

$$S_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (2\mu+\lambda)\partial_{xx} + \mu\partial_{yy} & \lambda\partial_{xy} + \mu\partial_{yx} \\ \mu\partial_{xy} + \lambda\partial_{yx} & \mu\partial_{xx} + (2\mu+\lambda)\partial_{yy} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Application of the Smith factorization to the system of linear elasticity

We transform this equations as follows :
we perform Fourier transform in the y-direction with the dual variable k,
we perform Laplace transform in the x-direction with dual variable Λ,
we obtain the following equation :

$$A\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} := \begin{pmatrix} (2\mu+\lambda)\Lambda^2 - k^2\mu & (\lambda+\mu)ik\Lambda \\ (\mu+\lambda)ik\Lambda & \mu\lambda^2 - (2\mu+\lambda)k^2 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \hat{f}$$

Result

$$A = EDF$$

$$E=\begin{pmatrix} -\mu k^2 & 0 \\ \frac{i\mu\Lambda((\lambda+2\mu)\Lambda^2-(2\lambda+3\mu)k^2)}{(\lambda+\mu)k} & 1 \end{pmatrix}$$

$$D=\begin{pmatrix} 1 & 0 \\ 0 & -(\Lambda^2-k^2)^2 \end{pmatrix}$$

$$F=\begin{pmatrix} \frac{-(\lambda+2\mu)\Lambda^2}{\mu k^2}+1 & \frac{-i(\lambda+\mu)\Lambda}{\mu k} \\ \frac{i(\lambda+2\mu)^2\Lambda}{(\lambda+\mu)k^3} & \frac{\lambda+2\mu}{k^2} \end{pmatrix}$$

An efficient (optimal) algorithm for the system of linear

We consider the following problem : Find ϕ : $R^2 \rightarrow R$ such that
-$\triangle^2\phi = f$ in R^2, $|\phi(\overrightarrow{x})| \rightarrow 0$ for $|\mathbf{x}| \rightarrow \infty$
where f is given right hand side. The domain Ω is decomposed into two halfplanes $\Omega_1 = R^- \times R$ and $\Omega_2 = R^+ \times R$. Let the interface $\{0\} \times R$ be denoted by Γ and $(\mathbf{n}\ _i)_{i=1,2}$ be the outward normal of $(\Omega_i)_{i=1,2}$. The algorithm, we propose, is given as follows:
Algorithm3.1. We choose the initial values ϕ_1^0 and ϕ_2^0 such that $\phi_1^0 = \phi_2^0$ and $\triangle\phi_1^0 = \triangle\phi_2^0$ on Γ. We obtain $(\phi_i^{n+1})_{i=1,2}$ from $(\phi_i^n)_{i=1,2}$ by the following
iterative procedure:

Iterative procedure

Correction step. We compute the corrections $(\tilde{\phi}_i^{n+1})_{i=1,2}$ as the solution of the homogeneous local problems

$$\begin{cases} -\triangle^2 \tilde{\phi}_i^{n+1} = 0 \, in \, \Omega_i, \\ lim_{|\mathbf{x}| \to 0} |\tilde{\phi}_i^{n+1}| = 0, \\ \frac{\partial \tilde{\phi}_i^{n+1}}{\partial \mathbf{n}_i} = \gamma_1^n \, on \, \Gamma, \\ \frac{\partial \triangle \tilde{\phi}_i^{n+1}}{\partial \mathbf{n}_i} = \gamma_2^n \, on \, \Gamma, \end{cases}$$

where $\gamma_1^n = -\frac{1}{2}(\frac{\partial \phi_1^n}{\partial \mathbf{n}_1} + \frac{\partial \phi_2^n}{\partial \mathbf{n}_2})$ and γ_2^n=-$\frac{1}{2}(\frac{\partial \triangle \phi_1^n}{\partial \mathbf{n}_1} + \frac{\partial \triangle \phi_2^n}{\partial \mathbf{n}_2})$.

Iterative procedure

Udapting step. We update $(\phi_i^{n+1})_{i=1,2}$ by solving the local problems

$$\begin{cases} -\triangle\phi_i^{n+1} = f \; in \, \Omega_i \\ lim_{|\mathbf{x}|\to 0}|\phi_i^{n+1}| = 0, \\ \phi_i^{n+1} = \phi_i^n + \delta_1^{n+1} \; on \, \Gamma \\ \triangle\phi_i^{n+1} = \triangle\phi_i^n + \delta_2^{n+1} \; on \, \Gamma \end{cases}$$

where $\delta_1^{n+1} = \frac{1}{2}(\tilde{\phi}_1^{n+1} + \tilde{\phi}_2^{n+1})$ and $\delta_2^{n+1} = \frac{1}{2}(\triangle\tilde{\phi}_1^{n+1} + \triangle\tilde{\phi}_2^{n+1})$.

Proposition

Algorithm 3.1. converges in two iterations

optimal algorithm for the system of linear elasticity

After having found an optimal algorithm which convergers in two steps for the fourth order operator -$\triangle^2$ problem, we focus on the linear elasticity system .

Algorithm3.2. We choose the initial values $(u_1^0, v_1^0)$and (u_2^0, v_2^0) such that
$(\mathrm{F}(u_1^0, v_1^0)^T)_2 = (\mathrm{F}(u_2^0, v_2^0)^T)_2$ and $\triangle(\mathrm{F}(u_1^0, v_1^0)^T)_2 = \triangle(\mathrm{F}(u_2^0, v_2^0)^T)_2$ on Γ. We compute $((u_i^{n+1}, v_i^{n+1}))_{i=1,2}$ from $((u_i^n, v_i^n))_{i=1,2}$ by the following iterative procedure :

optimal algorithm

Corrrection step. We compute the corrections $((\tilde{u}_i^{n+1}, \tilde{v}_i^{n+1}))_{i=1,2}$ as the solution of the homogeneous local problems

$$\begin{cases} S_2(\tilde{u}_i^{n+1}, \tilde{v}_i^{n+1}) = 0 \, in \, \Omega_i \\ Lim_{|\mathbf{x}| \to \infty} |\tilde{\mathbf{u}}_i^{\ n+1}| = 0, \\ \frac{\partial (F(\tilde{u}_i^{n+1}, \tilde{v}_i^{n+1})^T)_2}{\partial \mathbf{n}_i} = \gamma_1^n \, on \, \Gamma, \\ \frac{\partial \triangle (F(\tilde{u}_i^{n+1}, \tilde{v}_i^{n+1})^T)_2}{\partial \mathbf{n}_i} = \gamma_2^n \, on \, \Gamma, \end{cases}$$

where

$$\gamma_1^n = -\frac{1}{2}\left(\frac{\partial (F(u_1^n, v_1^n)^T)_2}{\partial \mathbf{n}_1} + \frac{\partial (F(u_2^n, v_2^n)^T)_2}{\partial \mathbf{n}2}\right)$$

$$\gamma_2^n = -\frac{1}{2}\left(\frac{\partial \triangle (F(u_1^n, v_1^n)^T)_2}{\partial \mathbf{n}_1} + \frac{\partial \triangle (F(u_2^n, v_2^n)^T)_2}{\partial \mathbf{n}_2}\right)$$

optimal algorithm

Updating step. We update $((u_i^{n+1}, v_i^{n+1}))_{i=1,2}$ by solving the local problems:

$$\begin{cases} S_2(u_i^{n+1}, v_i^{n+1}) = \mathbf{f} \; in \, \Omega_{i,} \\ Lim_{|\mathbf{x}|\to\infty} |\mathbf{u}\;_i^{n+1}| = 0, \\ (F(u_i^{n+1}, v_i^{n+1})^T)_2 = (F(u_i^n, v_i^n)^T)_2 + \delta_1^{n+1} \; on \, \Gamma \\ \triangle(F(u_i^{n+1}, v_i^{n+1})^T)_2 = \triangle(F(u_i^n, v_i^n)^T)_2 + \delta_2^{n+1} \; on \, \Gamma \end{cases}$$

where

$$\delta_1^{n+1} = \frac{1}{2}[(F(\tilde{u}_1^{n+1}, \tilde{v}_1^{n+1})^T)_2 + (F(\tilde{u}_2^{n+1}, \tilde{v}_2^{n+1})^T)_2]$$

$$\delta_2^{n+1} = \frac{1}{2}[\triangle(F(\tilde{u}_1^{n+1}, \tilde{v}_1^{n+1})^T)_2 + \triangle(F(\tilde{u}_2^{n+1}, \tilde{v}_2^{n+1})^T)_2]$$

An efficient algorithm for the system of linear elasticity

Algorithm3.3. We choose the initial values (u_1^0, v_1^0) and (u_2^0, v_2^0) such that $v_1^0 = v_2^0$ and $\frac{\partial u_1^0}{\partial \overrightarrow{n_1}} = \frac{\partial u_2^0}{\partial \overrightarrow{n_2}}$ on Γ. We compute $((u_i^{n+1}, v_i^{n+1}))_{i=1,2}$ from $((u_i^n, v_i^n))_{i=1,2}$ by the following iterative procedure :

efficient algorithm

Correction step. We compute the corrections $((\tilde{u}_i^{n+1}, \tilde{v}_i^{n+1}))_{i=1,2}$ as the solution of the homogeneous local problems :

$$\left\{\begin{array}{c} S_2(\tilde{u}_1^{n+1}, \tilde{v}_1^{n+1}) = 0 \, in \, \Omega_1, \\ \frac{\partial \tilde{u}_1^{n+1}}{\partial x} = -\frac{1}{2}(\frac{\partial u_1^n}{\partial x} - \frac{\partial u_2^n}{\partial x}) \, on \, \Gamma, \\ \frac{\partial \tilde{u}_1^{n+1}}{\partial x} + \frac{\partial \tilde{v}_1^{n+1}}{\partial y} = \gamma_{2,1}^n \, on \, \Gamma \end{array}\right. \quad \text{and}$$

$$\left\{\begin{array}{c} S_2(\tilde{u}_2^{n+1}, \tilde{v}_2^{n+1}) = 0 \, in \, \Omega_2, \\ \frac{\partial \tilde{u}_2^{n+1}}{\partial x} = \frac{1}{2}(\frac{\partial u_1^n}{\partial x} - \frac{\partial u_2^n}{\partial x}) \, on \, \Gamma, \\ -\frac{\partial \tilde{u}_2^{n+1}}{\partial x} - \frac{\partial \tilde{v}_2^{n+1}}{\partial y} = \gamma_{2,1}^n \, on \, \Gamma \end{array}\right.$$

where

$$\gamma_{2,1}^n = -\frac{1}{2}(\frac{\partial u_1^n}{\partial x} + \frac{\partial v_1^n}{\partial y} - \frac{\partial u_2^n}{\partial x} - \frac{\partial v_2^n}{\partial y})$$

An efficient algorithm

Updating step. We update $((u_i^{n+1}, v_i^{n+1}))_{i=1,2}$ by solving the local problems

$$\left\{\begin{array}{c} S_2(u_i^{n+1}, v_i^{n+1}) = \overrightarrow{f} \, in \, \Omega_{i,} \\ u_i^{n+1} = u_i^n + \frac{1}{2}(\tilde{u}_1^{n+1} + \tilde{u}_2^{n+1}) \, on \, \Gamma \\ \frac{\partial u_i^{n+1}}{\partial y} - \frac{\partial v_i^{n+1}}{\partial x} = \frac{\partial u_i^n}{\partial y} - \frac{\partial v_i^n}{\partial x} + \delta_{2,1}^n \, on \, \Gamma \end{array}\right.$$

where

$$\delta_{2,1}^n = \frac{1}{2}\left(\frac{\partial \tilde{u}_1^{n+1}}{\partial y} - \frac{\partial \tilde{v}_1^{n+1}}{\partial x} + \frac{\partial \tilde{u}_2^{n+1}}{\partial y} - \frac{\partial \tilde{v}_2^{n+1}}{\partial x}\right)$$

Schwarz overlap scheme applied to the system of linear elasticity

We want to solve

$$S_2(\mathbf{w}) = \mathbf{f}\, in\, \Omega_1 \bigcup \Omega_2$$

where $\mathbf{w} = (u, v)$.
The schwarz algorithm runs like this :
Start from (u_1^0, v_1^0), (u_2^0, v_2^0) we compute $\mathbf{w}_1^{n+1}$, $\mathbf{w}_2^{n+1}$ from $\mathbf{w}_1^n$, $\mathbf{w}_2^n$as follows :

$$\begin{cases} S_2(\mathbf{w}_1^{n+1}) = \mathbf{f}\, in\, \Omega_1 \\ \mathbf{w}_1^{n+1} = \mathbf{w}_2^n\, on\, \partial\Omega_1 \bigcap \Omega_2 \end{cases}$$

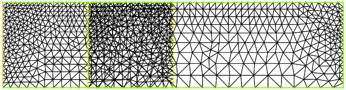
and

$$\begin{cases} S_2(\mathbf{w}_2^{n+1}) = \mathbf{f}\, in\, \Omega_2 \\ \mathbf{w}_2^{n+1} = \mathbf{w}_1^n, on\, \partial\Omega_2 \bigcap \Omega_1 \end{cases}$$

Application

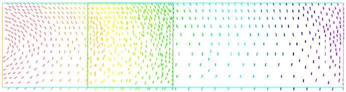
Here we take 1 and 2 to be rectangle, we apply the algorithm starting from zero.

Figure : The 2 overlapping mesh TH and th



Solution

Figure :Displacement fields during the iterations



Solution

Figure : Final configuration of the bean after convergence