Study of Approximated Numerical Integration Methods for the Computation of Green's Functions in Electromagnetism

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Background:

The electromagnetic modeling of antennas or radiating structures are ofter based on integral formulations for the E-or H-field involving Green's functions. Depending of the numerical method for solving the integral equation(such as Method of Moments or Finite Element Method), it is necessary to numerically evaluate the Green's functions on sampling element such as a triangle of a meshing structure. In my report, we use Gauss method in a triangle element.

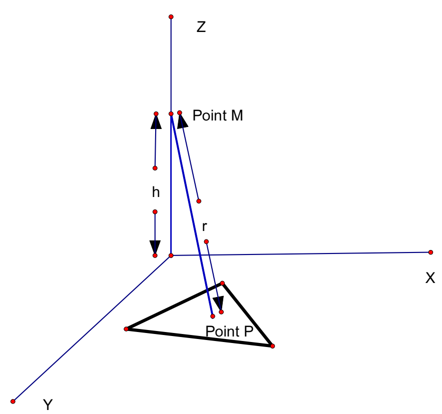
Green's functions in electromagnetism:

$$f = \frac{e^{ikr}}{r^n}, r = \sqrt{x^2 + y^2 + h^2}, n = 1, 2$$

Here k is the wavenumber. The angular wavenumber is defined as

$$k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{v_p}$$

In the formula above$\lambda$is the wave length, ν is the wave frequency and v_p is the phase velocity. In our case, we choose $v_p = c$ light speed.



My work

$\int_\Omega e^{ikr}/r^k ds$ Ω is a triangle in the mesh. This is the integration that we want to approximate.

1. For any triangle in 2D case, we have the variable h, to find a good way to divide the triangle and to find how many subdivisions do we need to make error less than what we expect, for instance 10^{-5}or 10^{-6}. It is quite usefull in electromagntic modelling.

2. Since the theory of Gauss method in 2 dimension is not well developed. We don't have the exact expression of error like the case in 1 dimension, so $err = f(...)$ in 2D in an triangle just like a black box. I want to fill this formula through analysis of experimental data. I can not get the exact expression of error in this way but some relationship. It will be useful in the deduction of in a rigorous theoretical way in the future.

Introduction of Gauss quadrature in one dimension:

In numerical analysis, a quadrature rule is an approximation of the definite integral of a function, usually stated as a weighted sum of function values at specifited values at specified points within the domain of integration. In one dimension, it usaully has a form below

$$\int_{1}^{-1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

Gauss quadrature is based on Fundamental theorem below

Let$p_n$be a nontrivial polynomial of degree n such that

$\int_a^b w(x)x^k p_n(x)dx = 0$, for all k=0, 1, ..., n-1

If we pick the nodes to be the zeros of p_n, then there exist weights w_i which make the computed integral exact for all polynomials of degree $2n-1$ or less. Furthermore, all these nodes will lie in the open interval (a, b).

The polynomial p_n is said to be an orthogonal polynomial of degree n associated to the weight function $w(x)$. It is unique up to a constant normalization factor.

We will have

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} w(x)g(x)dx \approx \sum_{i=1}^{n} w_i g(x_i)$$

Error estimate:

The error of a Gaussian quadrature rule can be stated as follows. For an integrand which has 2n continuous derivatives,

$$\int_a^b w(x)f(x)dx - \sum_{i=1}^{n} w_i f(x_i) = \frac{f^{(2n)}(\xi)}{(2n)!}(p_n, p_n)$$

for some ξin (a,b), where $p_n$is the orthogonal polynomial of order n and where

$$(f,\, g) = \int_a^b w(x)f(x)g(x)dx$$

In the important special case of $w(x) = 1$, we have the error estimate

$$\frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi),\ a < \xi < b$$

From the Gauss method in 1D case we can see:

Advantage of this method: It is esay to be implemented.

Disadvantage of this method: The error estimate is in convenient in practice, since it usually related to high order derivative of this function, and further more the actual error may be much less than a bound established by the derivative.

Gauss method on a triangle in 2 dimension:

Numerical Integration over Simplexes and Cones

THEOREM 1 . If

$$\sum_j a_j f(\xi) - \int_R f(\xi) = E(f)$$

then

$$\sum_j W a_j g(\eta_j) - \int_{TR} g(\eta) dV = W E(f)$$

Where T is an affine transformation, $\eta = A\xi + \eta_0$, of $E_n$onto itself; $g(\eta) = f(\xi)$; W is the absolute value of the determinant of A; R is an n-dimensional region included in the domain of f and $\xi_{1,}...\xi_k, ...,$ are points in the domain of f.

This theorem allowed us to some specific shape regions to develop formulas for the class of all affine transforms. All triangles in 2 dimension are equivalent under affine transformations.

How to get extend this Gauss method to high dimension like a triangle

Let an n-dimensional region R be embedded in the hyperplane $x = 1$ in the E_{n+1} where we represent the points in $E_{n+1}$by (ξ, x), where ξ is a point in E_n.Then the set of all points xR, where $0 \leq x \leq 1$, is a cone C with base R and vertex at the origin in E_{n+1}.

Let f(ξ, x) be a function defined over C and suppose that a suitable numerical integration formula is given over the base R of C. If,

$$\int_R f(\xi, 1) dV_n = \sum_j a_j f(\xi_j, 1)$$

then

$$\int_C f(\xi, x) dV = \int_0^1 dx \int_{xR} f(\xi, x) dV_n = \int_0^1 x^n \sum_j a_j f(x\xi_j, x) dx$$

since the Jacobian of the affine transformtion from R to $xR$is x^{-n}. Define a function

$$g(x) = \sum_j a_j f(x\xi_j, x)$$

and then we have

$$\int_C f dv = \int_0^1 x^n g(x) dx$$

Now we ask for numerical integration formulas of the form

$$\int_0^1 x^n g(x) dx = \sum_i b_i g(x_i)$$

Since such formulas may certainly be found we then have

$$\int_C f dv = \sum_i \sum_j b_i a_i f(x_i \xi_j, x_i)$$

Use the method above we can decrease the dimension to get a Gauss approximation of integral in higher dimension.

The quintic polynomial is integrated precisely with seven points in the triangle using $rVi + (1-r)C$ $C = \frac{1}{3}\sum_1^3 V_i$ weight a; $sV_i + (1-s)C$, weight b; and C, weight c. We find $r = \frac{1+\sqrt{15}}{7}$, $s = \frac{1-\sqrt{15}}{7}$, and $a = (\frac{155-\sqrt{15}}{1200})\Delta$, $b == (\frac{155+\sqrt{15}}{1200})\Delta$, $c = \left(\frac{9}{40}\right)\Delta$.Since the general quintic polynomial in two variables has 21 terms this formula appears to be a type we can efficient nothing that noting that one might not hope to accomplish a formula with fewer than 7=21/3 points. Here the "3" is the number of degrees of freedom for each point due to coordinates and weight. However, there are known hyperefficient formulas. which use fewer points than indicated by this argument. While we will not reproduce the argument here, we used a triangle with vertices (0,0), (1,-1), (1,1). Then the requirements of the affine symmetry of the formula with the form of the region assured that all monomials with odd powers of y could be omitted. This left 12 equations. We chose five of these and solved them for a, b, c, r, and s, and verified that the remaining 7 were satisfied.

From paper of P.C.HAMMER, O.J.MARLOWE and A.H.STROUD

Error Control of Numerical integration in 2D

We already have

$$\int\limits_{\Omega} f(x)ds = \sum_{i=1}^{7} w_i f(x_i)$$

Ω is a triangle region, w_i, x_i are chosen according the result we have got. This formula is exact for $f(x)$ is polynomial with the degree ≤ 5.

If $f(x)$ is any function, we have

$$\int\limits_{\Omega} f(x)ds = \sum_{i=1}^{7} w_i f(x_i) + R$$

Then we divide the triangle into small triangle regions

$$\sum_{j=1}^{m} \int\limits_{\Omega_j} f(x)ds = \sum_{j=1}^{m}\sum_{i=1}^{7} w_{ij} f(x_{ij}) + R$$

$$|R| \leq \sum_{j}^{m} |R_j| \leq m\,|R_{max}| < \varepsilon$$

ε we usually choose 10^{-5}.

The convergence of this approximation method. If we assume

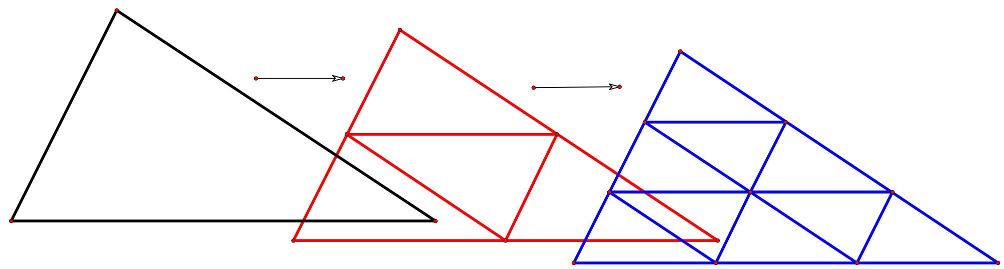
$$R \sim \triangle^K \; k \geq 2$$

$\triangle$ is the area of the whole triangle. Then after divided

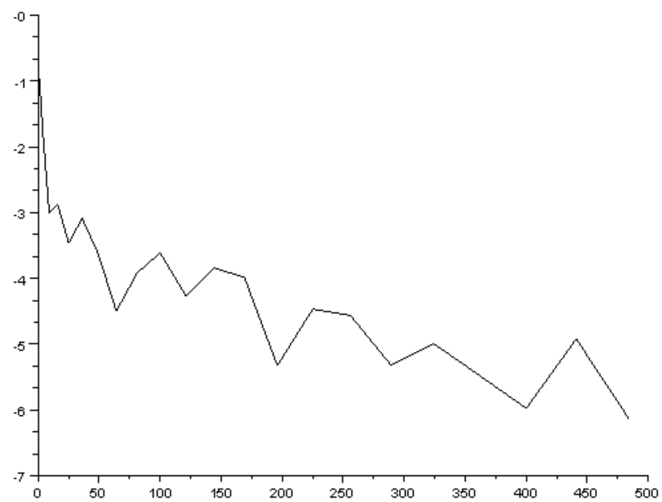
$$|R| \leq m\,|R_{max}| \sim m(c_0 \frac{\triangle}{m})^k, \; k \geq 2$$

When $m \to \infty$, error$\to 0$.

We use the lines which parallel the edges to divide the whole triangle to n^2 subdivisions, here each edge is divided to n equal line segments.



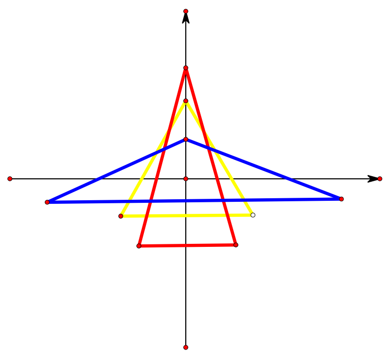
I tested the convergence of this way in many triangles.

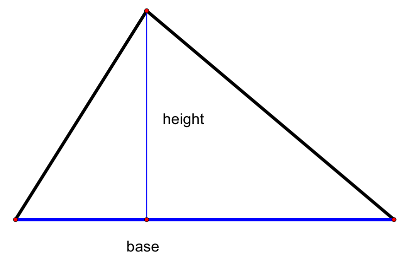


With the experimental calculations in FORTRAN program, I mainly consider how many subdivisions we need according to three aspects.

1. The parameter h, the range of h we consider $\frac{L}{100} < h < L$, the L is the longest edge of triangles.

2. The shape of triangles





In my FORTRAN program, I mainly change the ratio of height and base to control the shape of triangle. Firstly, I choose the triangle with the origin in the interior of triangle.

3. The positon of triangles in the coordinates.

From the Green function

$$f = \frac{e^{ikr}}{r^n}, r = \sqrt{x^2 + y^2 + h^2}, n = 1, 2$$

For any triangle, we do mapping of this triangle according to x axis, y axis and the origin, we will not change the value of integral of Green function. Also, we rotate this triangle with the center of the origin, the value of the integral will not change. So we can just move the triangles in one direction.

We have two Green functions

$$f = \frac{e^{ikr}}{r} \; and \; f = \frac{e^{ikr}}{r^2}$$

Parameter $k = 6\pi$, since we choose frequency $\nu = 900MHz$, then $k = \frac{2\pi\nu}{C} = 6\pi$

The size of triangle we choose

The longest edge of triangle $L < \frac{\lambda}{5}$

The results

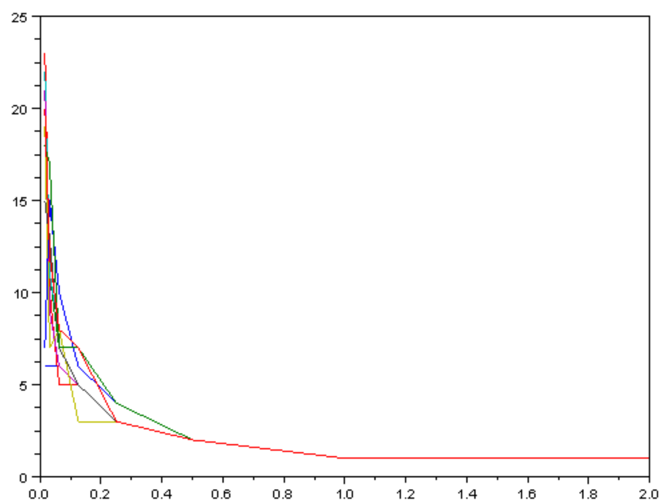
The first Green function $f = e^{ikr}/r$

1. The relation beweeen the parameter h and the subdivision we need n^2

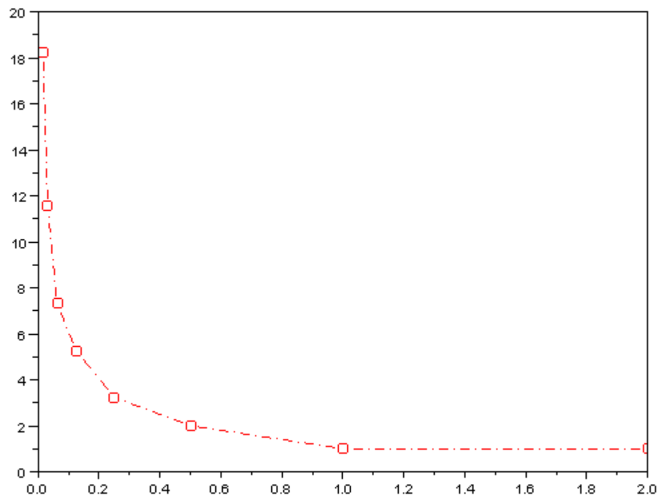
Firstly, we see the table , in the table L is the longest edge of the triangle.

Ratio	Tri1 0.120	Tri2 0.244	Tri3 0.400	Tri4 0.599	Tri5 0.866	Tri6 1.239	Tri7 1.799	Tri8 2.732	Tri9 4.598	Tri10 10.20
2*L	1	1	1	1	1	1	1	1	1	1
1*L	1	1	1	1	1	1	1	1	1	1
0.5*L	4	4	4	4	4	4	4	4	4	4
1/4*L	9	9	9	9	9	9	9	16	16	9
1/8*L	25	25	25	9	25	9	25	36	49	49
1/16*	36	49	25	64	36	64	49	100	49	64
1/32*L	36	121	100	121	81	49	169	225	289	144
1/64*L	36	484	529	484	441	361	225	49	324	400

The graph of this table, x axis is the h, y axis in n not n^2



The graph of the relation in average



2. The relation beween the position of triangle and the subdivision n^2we need

Here, I choose the equalateral triangle. Since for any triangle, we do mapping of this triangle according to x axis, y axis and the origin, we will not change the value of integral of Green function. Also, we rotate this triangle with the center of the origin, the value of the integral will not change. We only need to move the triangle in one direction to compare the difference.

In the table below, D is the shortest distance of the triangle from the origin

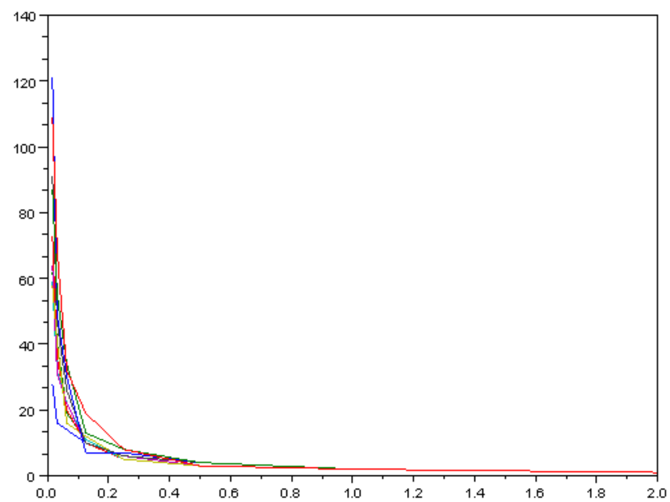
D	0.001*L	0.01*L	0.05*L	0.1*L	0.5*L	1*L	2*L	5*L
2*L	1	1	1	1	1	1	1	1
1*L	1	1	1	1	1	1	1	1
0.5*L	4	4	4	4	1	1	1	1
1/4*L	4	4	4	4	4	1	1	1
1/8*L	16	16	9	9	4	1	1	1
1/16*L	36	36	16	16	4	1	1	1
1/32*L	81	36	16	16	4	1	1	1
1/64*L	144	196	16	16	4	1	1	1

From the table, let's fix parameter h. When we increase D, the subdivison what we need decrease rapidly like we increase parameter h in the case D=0.

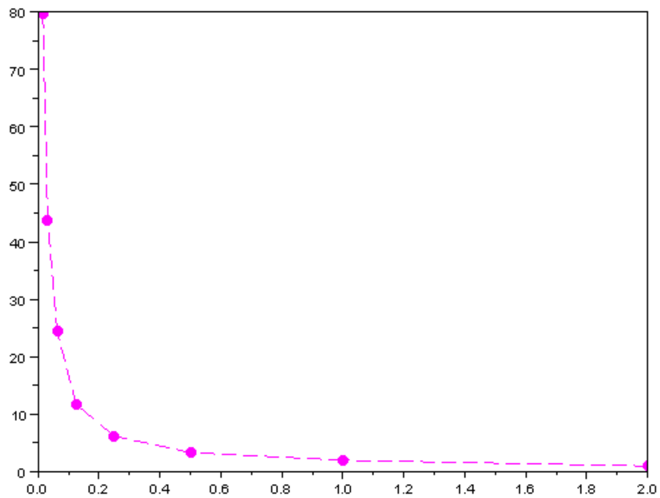
The second Green function $f = e^{ikr}/r^2$

Ratio	0.120	0.244	0.400	0.599	0.866	1.239	1.799	2.732	4.598	10.20
2*L	1	1	1	1	1	1	1	1	1	1
1*L	4	4	4	4	4	4	4	4	4	4
0.5*L	9	9	9	9	9	9	16	16	16	9
1/4*L	36	25	25	25	36	25	36	49	64	64
1/8*L	100	121	100	121	100	144	100	49	169	361
1/16*L	196	361	400	484	484	256	676	900	1156	1024
1/32*L	256	1681	1296	1024	961	1681	2209	2401	3025	4624
1/64*L	784	3844	5329	3481	4096	3364	8281	14641	7569	11881

The graph of this table, x axis is the h, y axis in n not n^2



The graph of the relation in average



Summary

1. If h $\geq$1L, we don't need to divide the triangle.

2. There is no big influence of the shape of the triangle to the error.

3. If the shortest distance from the traingle to the origin $\geq$1L, we don't need to divide the triangle.

4. A better way to divide the triangle.

Further work

1. To implement the new way to divide the triangle;

2. Use experimental calculation to test the probable relation of error and other variables.

3. The most difficult one: doing theorectical study to give an expression of error in general case.