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The measure of risk aversion

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Contents

1 Introduction

Keywords: Risk aversion, Arrow-Pratt risk aversion, multivariate risk aversion, comparative risk aversion.

Behavior under uncertainty and measurement of risk aversion are interesting yet challenging topics. In this thesis, I have intended to give insights into the theory of risk aversion developed so far. In the preliminary part, some useful definitions and theorems are given. In section 3, we start with univariate risk aversion. The classical approach, the risk aversion measure $r(x) = -u''(x)/u'(x)$, corresponding to von Neumann-Morgenstern utility function u, which is a function of wealth, by Arrow [1] and Pratt [28] has made it possible to study questions involving the effect of risk aversion on economic behavior. However, when the initial wealth is random or when there is some additional noise in the risk, there are some further economic phenomena which cannot be directly explained by the Arrow-Pratt risk measure. Therefore, Ross [33] has introduced the strong measure of risk aversion. Also this problem was studied by Kihlstrom, Romer and Williams [19]. Pratt and Zeckhauser studied this from an "axiomatic point of view", and proposed "proper risk aversion", which imposes constraints on the utility functions. Moreover, "standard risk aversion"[20] was studied by Kimball.

In section 4, multivariate risk aversion is studied. When the utility function is commodity bundles, we encounter several problems to generalize the univariate case. Kihlstrom and Mirman [17] argued that a prerequisite for the comparison of attitudes towards risk is that the cardinal utilities being compared represent the same ordinal preference. Keeney [16] has defined the conditional risk premium (risk aversion) which fixes all attributes except a certain component. Extending Pratt's approximation of the univariate risk premium, Duncan [7] developed a multivariate risk aversion matrix. Also, H. Levy and A. Levy [23] studied the multivariate case in an analogous way to the Arrow's univariate derivation. Moreover, Paroush [26] has investigated the relation between KM's risk aversion and the risk premium. Using the indirect utility function, Karni [15] has proposed an approach to compare the risk averseness. Kihlstrom and Mirman [18] introduced the increasing, decreasing and constant absolute and relative risk aversion in multidimensional case. Alternative representations and interpretations of relative risk aversion using indirect utility functions and expenditure functions were given by Hanoch [11].
In section 5, other approaches to study risk aversion and also some very interesting relations are studied.

2 Preliminaries

2.1 Utility functions [9]

2.1.1 Linear utility on mixture sets

The essential features of the von Neumann-Morgenstern linear utility theory are given in this section. Here, the lower case Greek letters will always denote numbers in $[0, 1]$.

Definition 2.1 *A set M is a mixture set [13] if for any λ and any ordered pair $(x, y) \in M \times M$ there is a unique element $\lambda x \oplus (1-\lambda)y$ in M such that*

M1 $1x \oplus 0y = x$

M2 $\lambda x \oplus (1-\lambda)y = (1-\lambda)y \oplus \lambda x$

M3 $\lambda[\mu x \oplus (1-\mu)y] \oplus (1-\lambda)y = (\lambda\mu)x \oplus (1-\lambda\mu)y$

for all $x, y \in M$ *and all* λ *and* μ.

If $\mathcal{P}$ is a set of probability measures defined on an algebra $\mathcal{A}$, and if $\mathcal{P}^+$ is the set of finite convex combinations of measures in $\mathcal{P}$ (the convex hull of $\mathcal{P}$), then $\mathcal{P}^+$ is a mixture set when $\lambda p \oplus (1-\lambda)q = \lambda p + (1-\lambda)q$.
We will say that u is a linear function on a mixture set M if it is a real-valued function for which $u(\lambda x \oplus (1-\lambda)y) = \lambda u(x) + (1-\lambda)u(y)$ for all λ and $x, y \in M$. Two linear functions u and v are related by a positive affine transformation if there are real numbers $a > 0$ and b such that $v(x) = au(x) + b$ for all $x \in M$.
$\succ$ will always signify an asymmetric ($x \succ y \Rightarrow$ not $[y \succ x]$) binary relation on a designated set, which we denote for the time being as X. We define $\sim$ and $\succeq$ on X from $\succ$ by
$x \sim y$ iff not ($x \succ y$) and not ($y \succ x$), $x \succeq y$ iff $x \succ y$ or $x \sim y$.
We say that $\succ$ is an asymmetric weak order if it is $x \succ z \Rightarrow (x \succ y$ or $y \succ z)$ for all $x, y, z \in X$.

Since $\succ$ is asymmetric, $\sim$ is reflexive ($x \sim x$) and symmetric ($x \sim y \Rightarrow y \sim x$).
It can be verified that $\succ$ is an asymmetric weak order if and only if, both $\succ$ and $\sim$ are transitive. Note also that if $\succ$ is an asymmetric weak order then $\sim$ is an equivalence relation (reflexive, symmetric, transitive).

The axioms:

A1 $\succ$ on M is an asymmetric weak order.

A2 For all $x, y, z \in M$ and $0 < \lambda < 1$, if $x \succ y$ then $\lambda x \oplus (1-\lambda)z \succ \lambda y \oplus (1-\lambda)z$.

A3 For all $x, y, z \in M$, if $x \succ y$ and $y \succ z$ then there are $\alpha, \beta \in (0,1)$ such that $\alpha x \oplus (1-\alpha)z \succ y$ and $y \succ \beta x \oplus (1-\beta)z$.

From these three axioms A1, A2, A3, we can obtain the following:

J1 $(x \succ y, \lambda > \mu) \Rightarrow \lambda x \oplus (1-\lambda)y \succ \mu x \oplus (1-\mu)y$.

J2 $(x \succeq y \succeq z, x \succ z) \Rightarrow y \sim \lambda x \oplus (1-\lambda)z$ for a unique λ.

J3 $(x \succeq y, z \succeq w) \Rightarrow \lambda x \oplus (1-\lambda)z \succ \lambda y \oplus (1-\lambda)w$.

J4 $x \sim y \Rightarrow \lambda x \oplus (1-\lambda)z \sim \lambda y \oplus (1-\lambda)z$.

Proof (We give the proof of the first two items.)
J1. We observe that using M1, M2, M3, we have $\lambda x \oplus (1-\lambda)x = x$. Using this result, we have $x \succ \mu x \oplus (1-\mu)y$, for $\mu > 0$. Hence, $\lambda x \oplus (1-\lambda)y \succ \mu x \oplus (1-\mu)y$, by M1 if $\lambda = 1$, and by M2, M3 and A2 as follows if $\lambda < 1$:

$$\begin{aligned}
&\lambda x \oplus (1-\lambda)y = (1-\lambda)y \oplus (1-(1-\lambda))x = \\
&= ((1-\lambda)/(1-\mu))[(1-\mu)y \oplus (1-(1-\mu))x] \oplus (1-((1-\lambda)/(1-\mu)))x = \\
&= ((1-\lambda)/(1-\mu))[\mu x \oplus (1-\mu)y] \oplus (1-((1-\lambda)/(1-\mu)))x = \\
&= ((\lambda-\mu)/(1-\mu))x \oplus (1-(\lambda-\mu)/(1-\mu))[\mu x \oplus (1-\mu)y] \succ \\
&\succ ((\lambda-\mu)/(1-\mu))[\mu x \oplus (1-\mu)y] \oplus (1-(\lambda-\mu)/(1-\mu))[\mu x \oplus (1-\mu)y] = [\mu x \oplus (1-\mu)y].
\end{aligned}$$

J2. Suppose first that $x \sim y$, so $y \sim x \succ z$. Then $y \sim x \oplus 0z = x$ by M1, and $x \oplus 0z \succ \mu x \oplus (1-\mu)z$ for any $\mu < 1$ by J1, so that $y \sim \lambda x \oplus (1-\lambda)z$ for a unique λ. A similar proof applies when $y \sim z$. Finally, suppose that $x \succ y \succ z$. It then follows from A1, A3 and J1 that there is a unique $\lambda \in (0,1)$ such that $\alpha x \oplus (1-\alpha)z \succ y \succ \beta x \oplus (1-\beta)z$ for all $\alpha > \lambda > \beta$.
We claim that $y \sim \lambda x \oplus (1-\lambda)z$. If $\lambda x \oplus (1-\lambda)z \succ y$ then $\lambda x \oplus (1-\lambda)z \succ y \succ z$, and by M3 and A3, $\lambda\mu x \oplus (1-\lambda\mu)z = \mu[\lambda x \oplus (1-\lambda)z] \oplus (1-\mu)z \succ y$ for some $\mu \in (0,1)$. Since $\lambda \geq \lambda\mu$, it contradicts to $y \succ \lambda\mu x \oplus (1-\lambda\mu)z$. A similar contradiction is obtained if we suppose that $y \succ \lambda x \oplus (1-\lambda)z$.Q.E.D.

Theorem 2.2 *Suppose M is a mixture set. Then the following statements are equivalent:*

a) A1, A2, A3 hold.

b) There is a linear function u on M that preserves $\succ$: for all $x, y \in M$, $x \succ y$ iff $u(x) \succ u(y)$.

In addition, a linear order-preserving u on M is unique up to a positive affine transformation.

It should be noted that non-linear order preserving utility functions exist in abundance when axioms A1-A3 hold. For if u satisfies (b), then every monotonic transformation of u also preserves $\succ$.
For the proof of this theorem, see [9, page 15].

2.1.2 Expected utility for probability measures

Let's denote $\mathcal{A}$ as a Boolean algebra (closed under complementary and finite unions) for $\mathcal{C}$ that contains the singleton $\{c\}$ for each $c \in \mathcal{C}$. $\mathcal{P}$ denotes a set of probability measures on a σ-algebra $\mathcal{A}$ that contains every one point measure: if $c \in \mathcal{C}$ and $p(\{c\}) = 1$ then $p \in \mathcal{P}$. A subset A of X is a preference interval if $z \in A$ whenever $x, y \in A$, $x \succeq z$ and $z \succeq y$. We say that $\mathcal{P}$ is closed under the formation of conditional measures $p_A(B) = p(B \cap A)/p(A), \forall B \in \mathcal{A}$, if $p_A \in \mathcal{P}$ whenever $p \in \mathcal{P}, A \in \mathcal{A}$ and $p(A) > 0$.
Axiom with finite additivity:

A0.1 $\mathcal{A}$ contains all preference intervals, and $\mathcal{P}$ is closed under countable convex combinations and under the formation of conditional measures.

Since $\mathcal{P}$ is a mixture set, axioms A1-A3 in section 2.1.1 imply the existence of a linear, order preserving utility function u on $\mathcal{P}$. When u is defined on $\mathcal{C}$ from u on $\mathcal{P}$ through one-point measures, additional axioms are needed to conclude that $u(p)$ is equal to $E(u,p)$, the expected value of u with respect to p, for each $p \in \mathcal{P}$.

A4 If $p, q \in \mathcal{P}, A \in \mathcal{A}$ and $p(A) = 1$, then $p \succeq q$ if $c \succ q$ for all $c \in A$, and $q \succeq p$ if $q \succ c$ for all $c \in A$.

Here $c \succ q$ means that $r \succ q$ when $r(\{c\}) = 1$.

Theorem 2.3 *Suppose A0.1, A1-A3, and A4 hold. Then there is a bounded real-valued function u on $\mathcal{C}$ such that, for all $p, q \in \mathcal{P}$, $p \succ q$ iff $E(u,p) > E(u,q)$, and such a u is unique up to a positive affine transformation.*

For the proof, see [9, page 26]. We can see from the proof that u is bounded. When the axiom A0.1 is weakened, u can be unbounded.
Axiom with countable additivity: When all measures in $\mathcal{P}$ are countably additive, A4 can be replaced by a dominance axiom that uses $d \in \mathcal{C}$ in place of $q \in \mathcal{P}$.

A*4 If $p \in \mathcal{P}, A \in \mathcal{A}, p(A) = 1$ and $d \in \mathcal{C}$, then $p \succeq d^*$, where d^* is a simple measure which assigns probability 1 to consequence d, if $c \succeq d$ for all $c \in A$, and $d^* \succeq p$ if $d \succeq c$ for all $c \in A$.

Theorem 2.4 *The conclusions of theorem 2.3 remain true when its hypothesis A4 is replaced by A^*4, provided that all measures in $\mathcal{P}$ are countably additive.*

For the proof, see [9, page 29].

2.2 Other functions

Definition 2.5 *A function $u : R^n_+ \to R$ is said to be quasi-concave if the set $\{x \in R_n^+ : u(x) \geq b\}$ is a convex set for any real number b.*

Definition 2.6 *Expenditure function. Let $u : R^n_+ \to R_+$ be a continuous non-decreasing quasi-concave utility function. Let $\mathbf{P'} = (P_1, ..., P_n)$ be some positive price vector. The expenditure function $C(u_0, \mathbf{P})$ is defined as the optimal value to the problem of minimizing the cost of attaining at least a utility level u_0, given that the agent faces the price vector $\mathbf{P}$:*

$$C(u_0, \mathbf{P}) = \min_x \{\mathbf{P'} \cdot x | u(x) \geq u_0\}.$$

The expenditure function is an increasing function of the utility level. For properties of the expenditure function, see [6].

Definition 2.7 *Indirect utility function. Let u and $\mathbf{P}$ be defined as in Definition 2.6. The indirect utility function $V : R^n_{++} \times R_+ \to R$ is defined as follows:*

$$V(\mathbf{P}, I) = \max_x \{u(x) | \mathbf{P'} x \leq I\}.$$

The indirect utility is an increasing function of income.

2.3 Least concave utility function [4]

We assume that a concave representation of the preorder (a binary relation which is reflexive and transitive) exists. Let X be a convex set in a real topological vector space E, and $\succeq$ be a complete preorder on X. We say that a real-valued function u on X represents $\succeq$ if $[x \succeq y]$ is equivalent to $[u(x) \geq u(y)]$, and we denote by U the set of continuous, concave, real-valued functions on X representing $\succeq$. The set U is preordered by a relation "v is more concave than u"defined by "there is a real valued, concave function f on $u(X)$ such that $v = f \circ u$". This definition is meaningful since $u(X)$ is an interval. The function f is strictly increasing; it is also continuous since it maps the interval $u(X)$ onto the interval $v(X)$.

Theorem 2.8 *If the set U is not empty, then U has a least element.*

This result due to G. Debreu, for the proof see [4]. We observe that if u is more concave than v, and v is more concave than u, then u is derived from v by an increasing linear transformation from R to R. ($u = f \circ v$ and $v = f^{-1} \circ u$, where both f and f^{-1} are concave.) Thus, if a preference pre-order is representable by a continuous, concave, real-valued utility function, then a least concave utility representing the pre-order is another instance of a cardinal utility.

Let X be an open convex set of commodity vectors in E, and let $\mathcal{P}$ be the set of probabilities on X. We identify each element x of X with the probability having $\{x\}$ as support. Consider a risk averse agent who preorders $\mathcal{P}$ by his preferences, and who satisfies the axioms of Blackwell-Girshick [2]. This agent has a bounded von Neumann-Morgenstern utility v whose restriction v to X is a concave, real-valued function representing the restriction $\succeq$ to X of his preferences on $\mathcal{P}$. Since v is bounded, v is continuous [3, ch. 2, sect. 2.10].

3 Univariate risk aversion

3.1 Arrow measure of risk aversion [1]

3.1.1 The theory of risk aversion

It has been common to argue that the individuals tend to display aversion to the taking of risks and that risk aversion in turn is an explanation for many observed phenomena in the economic world. A risk averter is defined as one who, starting from a position of certainty, is unwilling to take a bet which is actually fair.
Let's denote Y as wealth, $U(Y)$ as total utility of wealth Y. For simplicity, we here take wealth to be a single commodity and disregard the difficulties of aggregation over many commodities. Let's assume that the utility of wealth is a twice differentiable function.
We call $U'(Y)$ the marginal utility of wealth and $U''(Y)$ the rate change of marginal utility with respect to wealth. We can always assume that wealth is desirable:

$$U'(Y) > 0. \tag{3.1}$$

Suppose $U(Y)$ is bounded:

$$\lim_{Y \to 0} U(Y) \text{ and } \lim_{Y \to \infty} U(Y) \text{ exist and are finite.} \tag{3.2}$$

Proposition 3.1 *The utility function of a risk averter is characterized by the following: $U'(Y)$ is strictly decreasing as Y increases.*

Let's illustrate the above proposition.
Consider an individual with wealth Y_0 who is offered a chance to win or lose an amount h at fair odds. His choice is then between income Y_0 with probability 1 and a random income taking on the values $Y_0 - h$ and $Y_0 + h$ with probabilities 0.5 each. A risk averter by definition prefers the certain income; by the expected utility hypothesis:

$$U(Y_0) > \frac{1}{2}U(Y_0 - h) + \frac{1}{2}U(Y_0 + h),$$

with a little rewriting:

$$U(Y_0) - U(Y_0 - h) > U(Y_0 + h) - U(Y_0).$$

The utility differences corresponding to equal chances in wealth are decreasing as wealth increases.

Let's try to justify the predominance of risk aversion over risk preference.
Suppose that, for some positive number ϵ, the total length of all the intervals on which $U'(Y) \geq \epsilon$ is

infinite. Since $U(Y)$ is in any case increasing even on the remaining intervals, $U(Y)$ would have to tend to infinity as Y approaches infinity. This is a contradiction to (3.2). Therefore, for any positive number ϵ, we must have $U'(Y) < \epsilon$ for all but a set of intervals whose total length is finite. Hence, with little exceptions, $U'(Y)$ must be decreasing.

From Proposition 3.1, which is a necessary and sufficient condition for risk aversion, it is tempting to use the rate of change of $U'(Y)$ as a measure. However, the utility function is defined only up to positive linear transformations. Therefore, we seek our measure to remain invariant under positive linear transformations of the utility function.

The following are these type of measures:

Definition 3.2

$$R_A(Y) = -\frac{U''(Y)}{U'(Y)} \text{ \textit{absolute risk aversion}} \tag{3.3}$$

$$R_R(Y) = -\frac{YU''(Y)}{U'(Y)} \text{ \textit{relative risk aversion.}} \tag{3.4}$$

Let's see the simple behavioral interpretation of these two measures. Consider an individual with wealth Y who is offered a bet which involves winning or losing an amount h with probabilities p and $1-p$ respectively. The individual will be willing to accept the bet for values of p sufficiently large (certainly for $p = 1$) and will refuse if p is small (certainly for $p = 0$; a risk averter will refuse the bet if $p = \frac{1}{2}$ or less). The willingness to accept or reject a given bet will in general also depend on his present wealth Y.
Given the amount of the bet h and the wealth Y, by continuity, there will be a probability $p(Y, h)$ such that the individual is just indifferent between accepting and rejecting the bet. The absolute risk aversion directly measures the insistence of an individual for more than fair odds, when bets are small.

Proposition 3.3 *For the small values of h and for fixed Y, the function $p(Y, h)$ can be approximated by a linear function of h:*

$$p(Y, h) = \frac{1}{2} + \frac{R_A(Y)}{4}h + o(h^2) \tag{3.5}$$

where $p(Y, h)$ is the probability which makes indifferent between the choices.

Proof. Since the individual is indifferent between the certainty of Y and the gamble of winning h with probability $p(Y, h)$ and losing h with probability $1 - p(Y, h)$, the expected utility theorem implies,

$$U(Y) = p(Y, h)U(Y + h) + [1 - p(Y, h)]U(Y - h).$$

Expanding $U(Y + h)$, we obtain $U(Y + h) = U(Y) + hU'(Y) + (h^2/2)U''(Y) + R_1$, where R_1/h^2 approaches zero with h. Similarly, $U(Y - h) = U(Y) - hU'(Y) + (h^2/2)U''(Y) + R_2$, where R_2/h^2 approaches zero with h. By substituting and simplifying we have $U(Y) = U(Y) + (2p - 1)hU'(Y) + (h^2/2)U''(Y) + R$, where $R = pR_1 + (1 - p)R_2$, and therefore R/h^2 approaches zero with h. If we solve for p, we find $p(Y, h) = 1/2 + (h/4)[-U''(Y)/U'(Y)] - [R/2hU'(Y)]$. Q.E.D.

If we measure the bets not in absolute terms but in proportion to Y, the absolute risk aversion is replaced by the relative risk aversion. Denote the amount of bet by nY, where n is the fraction of wealth at stake. If we let $h = nY$ in (3.5) and use the definitions (3.3) and (3.4), we have,

$$p(Y, nY) = \frac{1}{2} + \frac{R_R(Y)}{4} n + o(n^2).$$

The behavior of these measures as Y changes is of interest. Two hypotheses are needed:

Hypothesis 3.4 *The relative risk aversion $R_R(Y)$ is an increasing function of Y.*
The absolute risk aversion $R_A(Y)$ is a decreasing function of Y.

If absolute risk aversion increased with wealth, it would follow that as an individual became wealthier, he would decrease the amount of risky assets held. The hypothesis of increasing relative risk aversion is saying that if both wealth and the size of the bet are increased in the same proportion, the willingness to accept the bet should decrease.

Note that the variation of the relative risk aversion with changing wealth is connected with the boundedness of the utility function.

Proposition 3.5 *If the utility function is to remain bounded as wealth becomes infinite, then the relative risk aversion cannot tend to a limit below one; similarly, for the utility function to be bounded from below as wealth approaches zero, the relative risk aversion cannot approach a limit above one as wealth tends to zero.*

Proof. Let R be a number such that $R_R(Y) \leq R$ for all $Y \geq Y_0$, for some Y_0. $U''(Y)/U'(Y) \geq -R/Y$. Integrating from Y_0 to Y yields, $\log U'(Y) - \log U'(Y_0) \geq -R(\log Y - \log Y_0)$
or $U'(Y) \geq U'(Y_0) Y_0^R Y^{-R}$ for $Y \geq Y_0$.
Let $C = U'(Y_0) Y_0^R > 0$, and integrate both sides from Y_0 to Y:
$U(Y) \geq U(Y_0) + [C/(1-R)](Y^{1-R} - Y_0^{1-R})$ if $R \neq 1$, and $U(Y) \geq U(Y_0) + C(\log Y - \log Y_0)$ if $R = 1$. If $R \leq 1$, it is a contradiction to the boundedness of the utility function from above. Hence, $R_R(Y) > 1$ for arbitrarily large Y values. In particular $R_R(Y)$ can not converge to a limit less than 1. Similarly, we can see that $R_R(Y)$ must be less than 1 for values of Y arbitrarily close to 0. Q.E.D.

Therefore, it is broadly permissible to assume that the relative risk aversion increases with wealth, though the proposition does not exclude fluctuations.

3.1.2 Model

Now, we would like to apply these concepts to a specific model of choice between risky and secure assets. It is assumed that the distribution of the rate of return is independent of the amount invested (stochastic constant returns to scale). An individual with given initial wealth invests part of it in the risky asset and the rest in the secure asset. The wealth at the end of the period is then a random variable. Let's denote:
X rate of return on the risky asset (a random variable)
A initial wealth
a amount invested in the risky asset
m amount invested in the secure asset $A - a$

Y final wealth
It follows from the definition that

$$Y = A + aX. \tag{3.6}$$

The decision of the individual is to choose a, the amount invested in the risky asset, so as to maximize:

$$E[U(Y)] = E[U(A + aX)] = W(a) \tag{3.7}$$

where $0 \leq a \leq A$. (If the secure asset has a positive rate of return ρ, the model is essentially the same, except that X is now interpreted as the difference between the rates of return on the risky and secure assets and, in (3.6), A is replaced by $A' = A(1 + \rho)$.)
The first two derivatives of (3.7) with respect to a are:

$$W'(a) = E[U'(Y)X], \quad W''(a) = E[U''(Y)X^2].$$

It is risk averse, $U''(Y) < 0$ for all Y, so $W''(a) < 0$ for all a. Therefore $W'(a)$ is a decreasing function, $W(a)$ must have one of the three following shapes.

- $W(a)$ has its maximum at $a = 0$ and it is a decreasing function. A necessary and sufficient condition is that $W'(0) \leq 0$. But if $a = 0$ then $Y = A$ and $U'(Y) = U'(A)$, which is a positive constant. Therefore, $W'(0) = U'(A)E(X) \Rightarrow a = 0$ if and only if $E(X) \leq 0$. In equivalent form, $a > 0$ if and only if $E(X) > 0$. It means he always takes some part of a favorable gamble.

- This is the case where the individual invests all his wealth in the risky asset. The condition is $W'(A) \geq 0$ or $E[U'(A + AX)X] \geq 0$.

- When the first two cases do not hold, there is an interior maximum at which

  $$W'(a) = E[U'(Y)X] = 0, \tag{3.8}$$

  implying the individual invests something but not all.

In the third case, the variation of the optimal solution, a, is of interest.
Let's see the effects of shifts in A on a. Let's assume a quadratic utility function, $U(Y) = a + bY + cY^2$, then with some simplification $a = \frac{(d-A)E(X)}{E(X^2)}$, where d is a constant.
We see clearly that investment in the risky asset would decrease as initial wealth, A, increases. Note that for the quadratic utility function, risk aversion requires that $c < 0$. However, the absolute risk aversion , $R_A(Y) = 1/[(-b/2c) - Y]$, is not decreasing.
More generally, for any utility function to see the dependence of a on A, we can differentiate (3.8) with respect to A and derive

$$\frac{da}{dA} = -\frac{E[U''(Y)X]}{E[U''(Y)X^2]}. \tag{3.9}$$

Since $U''(y) < 0$, the sign of da/dA is the same as the one in the numerator. It can be shown that decreasing absolute risk aversion implies that the numerator is positive, hence the amount of risky investment increases with final wealth.

Next, let's consider shifts in the distribution of X. We can think of them as a family of transformations of the original random variable X, characterized by the value of the shift parameter, h. Then the equation (3.8) becomes $E\{U'[Y(h)]X(h)\} = 0$ and $dY/dh = a(dX/dh) + X(h)(da/dh)$. By differentiating (3.8) with respect to h, we have $E\{[aU''(Y)X(h) + U'(Y)](dX/dh)\} + (da/dh)E\{U''(Y)[X(h)]^2\} = 0$. Therefore, da/dh has the same sign as $E\{[aU''(Y)X(h) + U'(Y)](dX/dh)\}$.

Let's take an additive shift, $X(h) = X + h$. Then from $dX/dh = 1$ and (3.9), da/dh has the same sign as da/dA. Therefore, for an additive shift in the probability distribution of rates of return, the demand for the risky asset increases with the shift parameter if the demand for the risky asset increases with wealth.

Let's take $X(h) = (1 + h)X$. Then $dX/dh = X$ and da/dh has the same sign as $E[aU''(Y)X^2] + E[U'(Y)X]$. Since a is the optimal solution, $E[U'(Y)X] = 0$. Therefore, da/dh is negative.
But a much stronger statement is made by Tobin, see [36].

Proposition 3.6 *If a is the demand for investment goods when the return is a random variable X, then $a/(1 + h)$ is the demand when the return is the variable $(1 + h)X$.*

Proof. Let $a(h)$ be the optimum investment in risky assets when the return is $X(h)$, and let $a = a(0)$. Then $E\{U'[Y(h)](1 + h)X\} = 0, E[U'(Y)X] = 0$. Let $a' = a(h)(1 + h)$ and $Y' = A + a'X$. Then $Y' = A + a(h)X(h) = Y(h)$. Therefore, $E[U'(Y')X] = 0$, which implies that a' is optimal when the return variable is X. That is, $a' = a \Rightarrow a(h) = a/(1 + h).Q.E.D.$

Finally, we consider a multiplicative shift about an arbitrary center $\tilde{X}$. Then $X(h) = (\tilde{X}) + (1 + h)(X - \tilde{X}) = (1 + h)X - h\tilde{X}$. Therefore, it can be regarded as a multiplicative shift about the origin, followed by a downward additive shift $h\tilde{X}$.
Using the previous results, we can see that a multiplicative shift about a non-negative center diminishes the demand for risky assets in even greater proportion than the shift itself.
On the other hand, for $\tilde{X}$, the demand for risky assets decreases in smaller proportion and might even increase.

3.2 Pratt measure of risk aversion [28]

Definition 3.7 *Consider a decision maker with assets x and utility function U. Risk premium π is the real number such that receiving a risk $\tilde{z}$ or receiving a non-random amount $E(\tilde{z}) - \pi$ is indifferent. As it depends on x and on the distribution of $\tilde{z}$, it will be denoted $\pi(x, \tilde{z})$.*

By the properties of the utility,

$$U(x + E(\tilde{z}) - \pi(x, \tilde{z})) = E\{U(x + \tilde{z})\}. \tag{3.10}$$

We will consider the situations where $E\{U(x + \tilde{z})\}$ exists and is finite. Then $\pi(x, \tilde{z})$ exists and is uniquely defined by (3.10). It follows immediately from (3.10) that, for any constant μ,

$$\pi(x, \tilde{z}) = \pi(x + \mu, \tilde{z} - \mu).$$

Definition 3.8 *The cash equivalent, $\pi_a(x, \tilde{z}) = E(\tilde{z}) - \pi(x, \tilde{z})$, is the smallest amount for which the decision maker would not choose $\tilde{z}$, if he had it. It is given by $U(x + \pi_a(x, \tilde{z})) = E\{U(x + \tilde{z})\}$.*

Definition 3.9 *The bid price, $\pi_b(x, \tilde{z})$, is the largest amount the decision maker will pay to choose $\tilde{z}$. It is given by $U(x) = E\{U(x + \tilde{z} - \pi_b(x, \tilde{z}))\}$.*

For an unfavorable risk $\tilde{z}$, it is natural to consider the insurance premium $\pi_I(x, \tilde{z})$ such that the decision maker is indifferent between facing the risk $\tilde{z}$ and paying the non-random amount $\pi_I(x, \tilde{z})$. Since paying π_I is equivalent to receiving $-\pi_I$, we have $\pi_I(x, \tilde{z}) = -\pi_a(x, \tilde{z})$.

Proposition 3.10 *Let's denote the risk by $\tilde{z}$ and its variance by σ_z^2. We assume that the third absolute central moment of $\tilde{z}$ is of smaller order than σ_z^2.*

$$\pi(x, \tilde{z}) = \frac{1}{2}\sigma_z^2 r(x + E(\tilde{z})) + o(\sigma_z^2),$$

where $r(Y) = -\frac{U''(Y)}{U'(Y)}$ is the absolute risk aversion.

Proof. At first let's consider the risk neutral case, where $E(\tilde{z}) = 0$. Expanding the equation (3.10) around x on both sides gives $U(x - \pi) = U(x) - \pi U'(x) + O(\pi^2)$,
$E\{U(x + \tilde{z})\} = E\{U(x) + \tilde{z}U'(x) + \frac{1}{2}\tilde{z}^2 U''(x) + O(\tilde{z}^3)\}$.
By simplifying the equation, $\pi(x, \tilde{z}) = \frac{1}{2}\sigma_z^2 r(x) + o(\sigma_z^2)$. Q.E.D.

A sufficient regularity condition for the above equation is that U has a third derivative which is continuous and bounded over the range of all $\tilde{z}$.
Since $\pi(x, \tilde{z}) = \pi(x + \mu, \tilde{z} - \mu)$, for $\mu = E(\tilde{z})$: $\pi(x, \tilde{z}) = \frac{1}{2}\sigma_z^2 r(x + E(\tilde{z})) + o(\sigma_z^2)$.
Thus the risk premium for a risk $\tilde{z}$ with mean $E(\tilde{z})$ and small variance is approximately $r(x + E(\tilde{z}))$ times half the variance of $\tilde{z}$.

Proposition 3.11 *The utility function, $U(x)$, is equivalent to $\int e^{-\int r}$ by a positive linear transformation, where $r(x) = -\frac{U''(x)}{U'(x)}$*

Proof. $-\int r(x) = \log U'(x) + c$. Exponentiating and integrating again, we have $e^c U(x) + d$, which is the positive linear transformation of $U(x)$. Therefore, $U(x) \sim \int e^{-\int r}$.Q.E.D.

We observe that the absolute risk aversion function r associated with any utility function U contains the essential information about U, while eliminating some information of secondary importance.

3.2.1 Comparative risk aversion

Let U_1 and U_2 be utility functions with absolute risk aversion functions r_1 and r_2 respectively. If, at a point x, $r_1(x) > r_2(x)$, then U_1 is locally more risk averse than U_2 at the point x; that is, the corresponding risk premia satisfy $\pi_1(x, \tilde{z}) > \pi_2(x, \tilde{z})$ for sufficiently small risks $\tilde{z}$. The following theorem says that the corresponding global properties also hold.

Definition 3.12 *In the specific case where the risk is to gain or lose a fixed amount $h > 0$ with corresponding probabilities $P(\tilde{z} = h)$ and $P(\tilde{z} = -h)$, let $U(x) = E\{U(x + \tilde{z})\}$ and $\tilde{z} = \pm h$. Let's denote by $p(x, h)$ the probability premium, $p(x, h) = P(\tilde{z} = h) - P(\tilde{z} = -h)$.*

Such risk is neutral if h and $-h$ are equally probable, so $P(\tilde{z}=h)-P(\tilde{z}=-h)$ measures the probability premium of $\tilde{z}$. Note that $P(\tilde{z}=h)$ is the same as $P(Y,h)$ in section 3.1, equation (3.5).

Proposition 3.13 *For the case $\tilde{z}=\pm h$ and $h>0$, the probability premium can be approximated $p(x,h)=\frac{1}{2}hr(x)+O(h^2)$ where $r(x)$ is the absolute risk aversion.*

Proof. Since $p(x,h)=P(\tilde{z}=h)-P(\tilde{z}=-h)$, we have $P(\tilde{z}=h)=\frac{1}{2}[1+p(x,h)]$, $P(\tilde{z}=-h)=\frac{1}{2}[1-p(x,h)]$. Moreover, $U(x)=E\{U(x+\tilde{z})\}=\frac{1}{2}[1+p(x,h)]U(x+h)+\frac{1}{2}[1-p(x,h)]U(x-h)$.
When U is expanded around x, the above equation becomes $U(x)=U(x)+hp(x,h)U'(x)+\frac{1}{2}h^2U''(x)+O(h^3)$. Therefore, solving for $p(x,h)$, we find $p(x,h)=\frac{1}{2}hr(x)+O(h^2)$.

Theorem 3.14 *Let $r_i(x)$, $\pi_i(x,\tilde{z})$ and $p_i(x)$ be the absolute risk aversion, the risk premium and the probability premium corresponding to the utility function U_i, $i=1,2$. Then the following conditions are equivalent, in either the strong form (indicated in brackets), or the weak form (with the bracketed part omitted):*

a) $r_1(x)\geq r_2(x)$ for all x [and $>$ for at least one x in every interval].

b) $\pi_1(x,\tilde{z})\geq[>]\pi_2(x,\tilde{z})$ for all x and $\tilde{z}$.

c) $p_1(x,h)\geq[>]p_2(x,h)$ for all x and all $h>0$.

d) $U_1(U_2^{-1}(t))$ is a [strictly] concave function of t.

e) $\frac{U_1(y)-U_1(x)}{U_1(w)-U_1(v)}\leq[<]\frac{U_2(y)-U_2(x)}{U_2(w)-U_2(v)}$ for all v,w,x,y with $v<w\leq x<y$.
The same equivalence holds for the interval, if $x, x+\tilde{z}, x+h, x-h, U_2^{-1}(t), v, w, y$ all lie in the specified interval.

Proof To show that (b) follows from (d), using equation (3.10), we have $\pi_i(x,\tilde{z})=x+E(\tilde{z})-U_i^{-1}(E\{U_i(x+\tilde{z})\})$. Then

$$\pi_1(x,\tilde{z})-\pi_2(x,\tilde{z})=U_2^{-1}(E\{\tilde{t}\})-U_1^{-1}(E\{U_1(U_2^{-1}(\tilde{t}))\}), \tag{3.11}$$

where $\tilde{t}=U_2(x+\tilde{z})$.
If $U_1(U_2^{-1}(t))$ is [strictly] concave, then by Jensen's inequality $E\{U_1(U_2^{-1}(\tilde{t}))\}\leq[<]U_1(U_2^{-1}(E\{\tilde{t}\}))$. Substituting equation (3.11) to the above inequality, we obtain (b).
To show that (d) follows from (a), note that $\frac{d}{dt}U_1(U_2^{-1}(t))=\frac{U_1'(U_2^{-1}(t))}{U_2'(U_2^{-1}(t))}$ which is [strictly] decreasing if and only if $\log U_1'(x)/U_2'(x)$ is.
Moreover, $\frac{d}{dt}\log\frac{U_1'(x)}{U_2'(x)}=r_2(x)-r_1(x)$. After some simplification, it immediately follows that $a\Rightarrow d$.
To show that (a) implies (e), integrate (a) from w to x, obtaining $-\log\frac{U_1'(x)}{U_1'(w)}\geq[>]-\log\frac{U_2'(x)}{U_2'(w)}$ for $w<x$, which is equivalent to $\frac{U_1'(x)}{U_1'(w)}\leq[<]\frac{U_2'(x)}{U_2'(w)}$ for $w<x$. This implies $\frac{U_1(y)-U_1(x)}{U_1'(w)}\leq[<]\frac{U_2(y)-U_2(x)}{U_2'(w)}$ for $w\leq x<y$, as may be seen by applying the Mean Value Theorem to the difference of the two sides of the inequality regarded as a function of y.
Condition (e) follows by using the Mean Value Theorem taking now w as variable.
We have proved that $a\Rightarrow d\Rightarrow b$ and $a\Rightarrow e\Rightarrow c$. Therefore, it is sufficient to show that $b\Rightarrow a$ and $c\Rightarrow a$, or equivalently that not (a) implies not (b) and (c). But this follows from what has already been proved, for if the weak [strong] form of (a) does not hold, then the strong [weak] form of (a) holds on some interval with U_1 and U_2 interchanged, so the weak [strong] forms of (b) and (c) do not hold. Q.E.D.

3.2.2 Increasing and decreasing risk aversion

Consider a decision maker who (*i*) attaches a positive risk premium to any risk, but (*ii*) attaches a smaller risk premium to any given risk the greater his assets x.

(i) $\pi(x, \tilde{z}) > 0$ for all x and $\tilde{z}$;

(ii) $\pi(x, \tilde{z})$ is strictly decreasing function of x for all $\tilde{z}$.

We will call a utility function to be risk averse if the weak form of (*i*) holds, that is, if $\pi(x, \tilde{z}) \geq 0$ for all x and $\tilde{z}$; it is well known that this is equivalent to concavity of U, and hence $U'' \leq 0$ and to $r \geq 0$. A utility function is strictly risk averse if it is strictly concave.

Theorem 3.15 *The following conditions are equivalent.*

a') The absolute risk aversion function r(x) is [strictly] decreasing.

b') The risk premium $\pi(x, \tilde{z})$ is a [strictly] decreasing function of x for all $\tilde{z}$.

c') The probability premium $p(x, h)$ is a [strictly] decreasing function of x for all $h > 0$.

For the proof, see [28, page 130]. The same equivalence holds if "increasing"is substituted for "decreasing". Also, for a given interval, the theorem holds, if $x, x + \tilde{z}, x + h, x - h$ all lie in the specified interval.

3.2.3 Operations which preserve decreasing risk aversion

Definition 3.16 *A utility function is called [strictly] decreasingly risk averse if its local risk aversion function r is [strictly] decreasing and nonnegative.*

By theorem 3.15, conditions (*i*) and (*ii*) are equivalent to the utility being strictly decreasingly risk averse.

Theorem 3.17 *Suppose $a > 0$. $U_1(x) = U(ax+b)$ is [strictly] decreasingly risk averse for $x_0 \leq x \leq x_1$ if and only if $U(x)$ is [strictly] decreasingly risk averse for $ax_0 + b \leq x \leq ax_1 + b$.*

Proof. This follows from the formula $r_1(x) = ar(ax + b)$. Q.E.D.

Theorem 3.18 *If $U_1(x)$ is decreasingly risk averse for $x_0 \leq x \leq x_1$, and $U_2(x)$ is decreasingly risk averse for $U_1(x_0) \leq x \leq U_1(x_1)$, then $U(x) = U_2(U_1(x))$ is decreasingly risk averse for $x_0 \leq x \leq x_1$, and strictly so unless one of U_1 and U_2 is linear from some x on and the other has constant risk aversion in some interval.*

Proof. We have $\log U'(x) = \log U_2'(U_1(x)) + \log U_1'(x)$. Therefore $r(x) = r_2(U_1(x))U_1'(x) + r_1(x)$. The functions $r_2(U_1(x)), U_1'(x)$ and $r_1(x)$ are positive and decreasing, therefore so is $r(x)$. Furthermore, $U_1'(x)$ is strictly decreasing as long as $r_1(x) > 0$, so $r(x)$ is strictly decreasing as long as $r_1(x)$ and $r_2(U_1(x))$ are both positive. If one of them is 0 for some x, then it is 0 for all larger x, but if the other is strictly decreasing, then so is r. Q.E.D.

Theorem 3.19 *If $U_1, ..., U_n$ are decreasingly risk averse on an interval $[x_0, x_1]$, and $c_1, ..., c_n$ are positive constants, then $U = \sum_1^n c_i U_i$ is decreasingly risk averse on $[x_0, x_1]$, and strictly so except on subintervals (if any) where all U_i have equal and constant risk aversion.*

Proof. The general statement follows from the case $U = U_1 + U_2$.
For this case $r = -\frac{U_1'' + U_2''}{U_1' + U_2'} = \frac{U_1'}{U_1' + U_2'} r_1 + \frac{U_2'}{U_1' + U_2'} r_2$;
$r' = \frac{U_1'}{U_1'+U_2'} r_1' + \frac{U_2'}{U_1'+U_2'} r_2' + \frac{U_1'' U_2' - U_1' U_2''}{(U_1'+U_2')^2}(r_1 - r_2) = \frac{U_1' r_1' + U_2' r_2'}{U_1' + U_2'} - \frac{U_1' U_2'}{(U_1'+U_2')^2}(r_1 - r_2)^2$.
We have $U_1' > 0, U_2' > 0, r_1 \leq 0$ and $r_2 \leq 0$. Therefore $r' \leq 0$, and $r' < 0$ unless $r_1 = r_2$ and $r_1' = r_2'$.
Q.E.D.

3.2.4 Relative risk aversion

So far, we have been concerned with risks that remained fixed while assets varied. Let us now view everything as a proportion of assets.

Definition 3.20 *Let $\pi^*(x, \tilde{z})$ be the proportional risk premium corresponding to a proportional risk $\tilde{z}$; that is, a decision maker with assets x and utility function U would be indifferent between receiving a risk $x\tilde{z}$ and receiving the non-random amount $E(x\tilde{z}) - x\pi^*(x, \tilde{z})$.*

From definition, $\pi^*(x, \tilde{z}) = \frac{1}{x}\pi(x, x\tilde{z})$.
Using the risk premium properties, we have $\pi^*(x, \tilde{z}) = \frac{1}{2}\sigma_z^2 r^*(x + xE(\tilde{z})) + o(\sigma_z^2)$, where $r^*(x) = xr(x)$ is the relative risk aversion.

Similarly, we can define the proportional probability premium $p^*(x, h) = p(x, xh)$, corresponding to a risk of gaining or losing a proportional amount h.
Moreover, a utility function is [strictly] increasingly or decreasingly proportionally risk averse if it has a [strictly] increasing or decreasing local proportional risk aversion function.

Theorem 3.21 *The following conditions are equivalent.*

a") The relative risk aversion function $r^*(x)$ is [strictly] decreasing.

b") The proportional risk premium $\pi^*(x, \tilde{z})$ is a [strictly] decreasing function of x for all $\tilde{z}$.

c") The proportional probability premium $p^*(x, h)$ is a [strictly] decreasing function of x for all $h > 0$.

The same equivalence holds if "increasing"is substituted for "decreasing". Also, for a given interval, the theorem holds, if $x, x + \tilde{z}, x + h, x - h$ all lie in the specified interval.

3.3 Some stronger measures of risk aversion [33]

For simplicity, we will assume that all utility functions are strictly monotone and concave in C^3. The statement "A is more risk averse than B in the Arrow-Pratt sense"is denoted by $A \supseteq_{AP} B$.

Definition 3.22 *The relationship $A \supseteq_{AP} B$ holds if and only if* $(\forall x)$

$$-\frac{A''(x)}{A'(x)} \geq -\frac{B''(x)}{B'(x)}.$$

Unfortunately, in situations where only incomplete insurance is available, the Arrow-Pratt measure is not strong enough to support the economic intuition. The example below illustrates a case where $A \supseteq_{AP} B$, but B's premium exceeds A's.

Example 3.23 *Suppose that wealth is distributed by the lottery*

$$\widetilde{w} = \begin{cases} w_1 & \textit{with probability } p \\ w_2 & \textit{with probability 1-p} \end{cases}$$

and suppose that the lottery for decision $\widetilde{\epsilon}$ $(E\{\widetilde{z}\} = 0)$ *is*

$$\widetilde{\epsilon} = \begin{cases} 0, & \textit{if } \widetilde{w} = w_2 \\ \epsilon & \textit{with probability } \frac{1}{2} \quad \textit{if } \widetilde{w} = w_1 \\ -\epsilon & \textit{with probability } \frac{1}{2} \quad \textit{if } \widetilde{w} = w_1 \end{cases}$$

Hence,

$$\widetilde{w} + \widetilde{\epsilon} = \begin{array}{lll} & & \overset{\frac{1}{2}}{\nearrow} \; w_1 - \epsilon \\ & \overset{p}{\nearrow} & \overset{\frac{1}{2}}{\rightarrow} \; w_1 + \epsilon \\ & \overset{1-p}{\searrow} & w_2 \end{array} \tag{3.12}$$

Now, for small ϵ *we can take a Taylor approximation to derive* $E\{U(\widetilde{w}+\widetilde{\epsilon})\} = p\{\frac{1}{2}U(w_1-\epsilon)+\frac{1}{2}U(w_1+\epsilon)\} + (1-p)U(w_2) \approx pU(w_1) + (1-p)U(w_2) + \frac{1}{2}pU''(w_1)\epsilon^2$, *and* $E\{U(\widetilde{w} - \pi_U)\} = pU(w_1 - \pi_U) + (1-p)U(w_2 - \pi_U) \approx$ $\approx pU(w_1) + (1-p)U(w_2) - [pU'(w_1) + (1-p)U'(w_2)]\pi_U$. *Combining these relations, we have*

$$\pi_U \approx -\frac{\frac{1}{2}pU''(w_1)\epsilon^2}{pU'(w_1) + (1-p)U'(w_2)}. \tag{3.13}$$

Now, let A *and* B *be two utility functions with* $A \supseteq_{AP} B$, *but*

$$-\frac{A''(w_1)}{A'(w_2)} < -\frac{B''(w_1)}{B'(w_2)}.$$

For example, for sufficiently large $w_1 - w_2$, *for the following functions* $A = -e^{-aw}, B = -e^{-bw}; a > b$, *we have the above properties. It follows from (3.13), that if* p *is small enough, we will have* $\pi_A < \pi_B$. *Even though* A *is uniformly more risk averse than* B, *the lottery places low likelihood on the event,* $\widetilde{w} = w_1$, *in which the insurance is relevant. The premium is determined by a tradeoff between the benefits of insurance at one wealth level,* w_1, *and the costs at another,* w_2.

Definition 3.24 *A is strongly more risk averse than B, written* $A \supseteq B$, *if and only if*

$$\inf_w \frac{A''(w)}{B''(w)} \geq \sup_w \frac{A'(w)}{B'(w)}.$$

Letting λ be a constant that separates $\inf A''/B''$ from $\sup A'/B'$ we can equivalently define $A \supseteq B$ as follows.

Definition 3.25 *We have $A \supseteq B$ if and only if $\exists\lambda, (\forall w_1, w_2)$*

$$\frac{A''(w_1)}{B''(w_1)} \geq \lambda \geq \frac{A'(w_2)}{B'(w_2)}$$

The ordering $A \supseteq B$ is independent of any arbitrary scaling of A and B. In the above equivalent definition, though, the choice of separating constant λ is dependent on the scaling of A and B.

Theorem 3.26 *If $A \supseteq B$ then $A \supseteq_{AP} B$, but the converse is not true.*

Proof. The implication follows immediately from rearranging the definition. The counter example can be verified for $A(w) = -e^{-aw}$ and $B(w) = -e^{-bw}$, with $a > b$ and large enough $w_1 - w_2$.

Theorem 3.27 *The following three conditions are equivalent*

(i) $\exists\lambda > 0, \forall x, y$
$\frac{A''(x)}{B''(x)} \geq \lambda \geq \frac{A'(y)}{B'(y)}$.

(ii) $(\exists G, \lambda > 0)$, $G' \leq 0, G'' \leq 0, A = \lambda B + G$.

(iii) $(\forall \widetilde{w}, \widetilde{\epsilon})$, $E\{\widetilde{\epsilon}|\widetilde{w}\} = 0$,
$E\{A(\widetilde{w} + \widetilde{\epsilon})\} = E\{A(\widetilde{w} - \pi_A)\}$ *and* $E\{B(\widetilde{w} + \widetilde{\epsilon})\} = E\{B(\widetilde{w} - \pi_B)\}$ *imply that* $\pi_A \geq \pi_B$.

Proof.
(i) $\Rightarrow$ *(ii)*. Define G by $A = \lambda B + G$, where A and B are scaled to satisfy (i). Differentiating, we obtain $G' = A' - \lambda B' \leq 0$ and $\frac{A''}{B''} = \lambda + \frac{G''}{B''} \geq \lambda$ implies that $G'' \leq 0$.
(ii) $\Rightarrow$ *(iii)* The following chain uses the nonincreasing property of G:
$E\{A(\widetilde{w} - \pi_A)\} = E\{A(\widetilde{w} + \widetilde{\epsilon})\} = E\{\lambda B(\widetilde{w} + \widetilde{\epsilon}) + G(\widetilde{w} + \widetilde{\epsilon})\}$
$\leq E\{\lambda B(\widetilde{w} + \widetilde{\epsilon})\} + E\{G(\widetilde{w})\} = E\{\lambda B(\widetilde{w} - \pi_B)\} + E\{G(\widetilde{w})\} \leq E\{\lambda B(\widetilde{w} - \pi_B)\} + E\{G(\widetilde{w} - \pi_B)\} =$
$= E\{A(\widetilde{w} - \pi_B)\}$. Since A is monotone, $\pi_A \geq \pi_B$.
(iii) $\Rightarrow$ *(i)*. Let's choose the lottery (3.12). Using (3.13), we can see that, for small enough ϵ, $\pi_A \geq \pi_B$ for all lotteries only if $(\forall x, y, p)$

$$-\frac{pA''(x)}{pA'(x) + (1-p)A'(y)} \geq -\frac{pB''(x)}{pB'(x) + (1-p)B'(y)}.$$

If $\frac{A''(x)}{B''(x)} \leq \frac{A'(y)}{B'(y)}$ for some x and y, then for p sufficiently small we have a contradiction. Q.E.D.

Theorem 3.27 provides a constructive technique for such pair of utility functions.
Let B be a utility function and choose G to be a decreasing concave function. Now, A as defined by (ii) will be a utility function strongly more risk averse than B, as long as it is monotone. For example, we can take $A(x) = x - e^{-x}$ and $B(x) = x - be^{-1}, 0 < b < 1$.

Notice that if $B'(x) \to 0$ as x approaches the upper limit of the domain of B, then there is no utility function more risk averse than B, since $G' < 0$ implies that $B' + \lambda G'$ is negative for all λ and x sufficiently large. Of course, more risk averse functions than B can be found on restricted domains.

3.3.1 Application 1

A number of problems of uncertain choice involve tradeoffs between "return"and "risk". One way of formalizing this is to consider a choice between two lotteries, $\widetilde{x}$ and $\widetilde{y}$, where $\widetilde{y}$ is distributed as $\widetilde{x}$ plus a "return"$\widetilde{v} \geq 0$ and an additional risk $\widetilde{\epsilon}$, where $E\{\widetilde{\epsilon}|\widetilde{x}+\widetilde{v}\} = 0$. If an individual chooses $\widetilde{x}$ over $\widetilde{y}$ then implicitly he is judging that the return $\widetilde{v}$ does not justify bearing the additional risk $\widetilde{\epsilon}$. In effect, the return-risk tradeoff is not sufficiently favorable. Similarly, intuition would suggest that if the agent B finds such a tradeoff unacceptable then so must do any agent A, who is more risk averse than B. While this is not true for Arrow-Pratt ordering, the strong measure of risk aversion justifies the above case. Let $A \supseteq B$. Now, if $E\{B(\widetilde{x})\} > E\{B(\widetilde{y})\}$ then
$E\{A(\widetilde{x})\} = \lambda E\{B(\widetilde{x})\} + E\{G(\widetilde{x})\} > E\{B(\widetilde{y})\} + E\{G(\widetilde{x}+\widetilde{v})\} = E\{B(\widetilde{y})\} + E\{G(\widetilde{y})\} = E\{A(\widetilde{y})\}$.

3.3.2 Application 2

In simple portfolio problems it seems natural to hope that more risk averse individuals will take less risky positions. This intuition is not supported by the Arrow-Pratt risk aversion measurement but by strong risk aversion measurement.
Consider the two assets portfolio problem where $\widetilde{x}$ and $\widetilde{y}$ denote the returns on the two risky assets. If α denotes the proportion of total wealth, w, invested in $\widetilde{x}$ then the total return is given by

$$\widetilde{w} = w[(1-\alpha)\widetilde{x} + \alpha\widetilde{y}] = w[\widetilde{x} + \alpha\widetilde{z}],$$

where $\widetilde{z} = \widetilde{y} - \widetilde{x}$.
The first order condition for the utility function B, is given by $E\{B'(\widetilde{w})(\widetilde{z})\} = 0$. Below we normalize $w = 1$ and we will assume that $\widetilde{y}$ is an asset that offers a higher return, but a greater risk than $\widetilde{x}$, i.e., $E\{\widetilde{z}|x\} \geq 0, \forall x$.
In this situation, we expect that if A is more risk averse than B, then A will hold less of $\widetilde{y}$ than B. If α is A's optimal portfolio and β denotes B's optimum, then we would expect $\alpha < \beta$.
To see that, this result does not follow from Arrow-Pratt measure, let $A = G(B)$, where G is monotone and concave. Differentiating, we have
$\frac{\partial}{\partial\alpha}E\{A(\widetilde{w}_\alpha)\}|_{\alpha=\beta} = E\{A'(\widetilde{w}_\beta)\widetilde{z}\} = E\{G'(\widetilde{w}_\beta)B'(\widetilde{w}_\beta)\widetilde{z}\}$ and $E\{B'(\widetilde{w}_\beta)\widetilde{z}\} = 0$
by the optimality of β for B.
For $\alpha < \beta$, we must have the slope of $E\{A(\widetilde{w}_\beta)\}$ negative at $\alpha = \beta$, and for this result to hold in general it must hold for all monotone, concave G functions.
For simplicity, let $\widetilde{x}$ and $\widetilde{z}$ be independent with
$\widetilde{z} = \begin{cases} 2 & \text{with probability } 0.5 \\ -1 & \text{with probability } 0.5 \end{cases}$; and $\widetilde{x} = \begin{cases} 1 & \text{with probability } 0.5 \\ 0 & \text{with probability } 0.5 \end{cases}$.

The first order condition for B now takes the form $E\{B'(\widetilde{w}_\beta)\widetilde{z}\} = B_{02} + B_{12} - \frac{1}{2}(B_{01} + B_{11}) = 0$ where $B_{ij} = B'(i + \beta j) \geq 0$.
With similar notation for G' we must have
$E\{G'(\widetilde{w}_\beta)B'(\widetilde{w}_\beta)\widetilde{z}\} = G_{02}B_{02} + G_{12}B_{12} - \frac{1}{2}(G_{01}B_{01} + G_{11}B_{11}) < 0$,
where $G_{ij} \geq 0$, and setting $\beta = \frac{1}{4}$, we must have
$B_{01} \geq B_{02} \geq B_{11} \geq B_{12} \geq 0$ and $G_{01} \geq G_{02} \geq G_{11} \geq G_{12} \geq 0$.
Finally, since the positively weighted B values are not uniformly dominant, a counterexample exists. One such example is $B_{01} = 4, B_{02} = 3, B_{11} = 2, B_{12} = 0$, and $G_{01} = G_{02} = 10, G_{11} = G_{12} = 0$, for which $E\{G'(\widetilde{w}_\beta)B'(\widetilde{w}_\beta)\widetilde{z}\} = 10 > 0$, which implies $\alpha > \beta$.

Now, suppose that A is more risk averse than B in the strong sense. From theorem 3.27, there exist $\lambda > 0$ and a decreasing concave function G, such that $A = \lambda B + G$.
Examining marginal effect, we have
$\frac{\partial}{\partial\alpha}E\{A(\widetilde{w}_\alpha)\}|_{\alpha=\beta} = E\{A'(\widetilde{w}_\beta)\widetilde{z}\} = E\{[\lambda B'(\widetilde{w}_\beta) + G'(\widetilde{w}_\beta)]\widetilde{z}\} = E\{G'(\widetilde{w}_\beta)\widetilde{z}\} \leq 0$,
where the last inequality is a consequence of the fact that G' is negative and decreasing while $E\{\widetilde{z}|x\} > 0$. It follows that $\alpha \leq \beta$. In other words, if A is more risk averse than B in the strong sense then A will choose a less risky portfolio with a lower expected return.

3.3.3 Decreasing/Increasing absolute risk aversion

Note that $\supseteq$ denotes the "more risk averse "relation.

Definition 3.28 *The utility function U, displays decreasing absolute risk aversion,*

$$DARA \text{ iff } (\forall x, y > 0)\ U(x) \supseteq U(x+y)$$

and increasing absolute risk aversion

$$IARA \text{ iff } (\forall x, y > 0)\ U(x+y) \supseteq U(x).$$

Definition 3.29 *The utility function U, displays decreasing relative risk aversion,*

$$DRRA \text{ iff } (\forall x, y > 0)\ U(x) \supseteq U([1+y]x)$$

and increasing relative risk aversion

$$IRRA \text{ iff } (\forall x, y > 0)\ U([1+y]x) \supseteq U(x).$$

Notice that these definitions are strictly stronger than the corresponding definitions for the Arrow-Pratt increasing or decreasing absolute/relative risk aversion.

Theorem 3.30 *The following three conditions are equivalent:*

1. *U exhibits DARA (IARA)*
2. $\exists a, \forall x,\ \frac{U'''(x)}{U''(x)} \leq a \leq \frac{U''(x)}{U'(x)}\ \left(\frac{U'''(x)}{U''(x)} \geq a \geq \frac{U''(x)}{U'(x)}\right)$.
3. $\exists a, \forall x, y > 0,\ \frac{U''(x+y)}{U''(x)} \leq e^{ay} \leq \frac{U'(x+y)}{U'(x)}\ \left(\frac{U''(x+y)}{U''(x)} \geq e^{ay} \geq \frac{U''(x+y)}{U'(x)}\right)$.

Proof. Since arguments are all similar, we pick the DARA case.
(1) $\Rightarrow$ (2) From the definition, $\forall x, y > 0$, $\exists\lambda(y)$, $\frac{U''(x+y)}{U''(x)} \leq \lambda(y) \leq \frac{U'(x+y)}{U'(x)}$. For small y, we have

$$1 + \frac{U'''(x)}{U''(x)}y \leq \lambda(0) + \lambda'(0)y \leq 1 + \frac{U''(x)}{U'(x)}y.$$

Letting $\lambda(0) = 1$ and $a = \lambda'(0)$, it yields the desired result.
(2) $\Rightarrow$ (3) From (2) we have that

$$\frac{\partial}{\partial x}\ln U'(x) \geq a \geq \frac{\partial}{\partial}\ln(-U''(x)).$$

It implies

$$\frac{U'(x+y)}{U'(x)} \geq e^{ay} \geq \frac{U''(x+y)}{U''(x)}.$$

(3) ⇒ (1) This is immediate, since (3) is a restatement of the definition. Q.E.D.

For example, $U(x) = x - e^{-ax}, a > 0$ displays DARA.
Notice that the constant absolute and relative risk aversion utility functions, $-e^{ax}$, $(1/(1-\beta))x^{1-\beta}; \beta \geq 0, \beta \neq 1$, and $\log x$ are also constant ARA and RRA in the strong sense as well. In an analogous way, we can also study relative risk aversion's equivalent conditions in the strong sense.

3.4 Risk aversion with random initial wealth [19]

Suppose that individuals invest their wealth in a safe and a risky asset. Denote the random rate of return of the risky asset by $\widetilde{x}$. Individual i with initial wealth y invests B_i in the risky asset and $y - B_i$ in the safe asset. His optimal choice $\widehat{B}(\widetilde{x}, y)$ is the value of B_i in $[0, y]$ which maximizes

$$EU_i(y + B_i\widetilde{x}).$$

For two individuals, if $R^{U_1} = -\frac{U_1''}{U_1'} > R^{U_2}$ holds on the relevant domain of U_i's, then for any y and non-degenerate random variable $\widetilde{x}$,

$$\widehat{B}_1(\widetilde{x}, y) < \widehat{B}_2(\widetilde{x}, y) \text{ and } \pi_1(\widetilde{x}, y) > \pi_2(\widetilde{x}, y).$$

Suppose that each individual receives a nonnegative random income $\widetilde{y}$ and that he also possesses some initial wealth δ which is non-stochastic and positive. Before knowing $\widetilde{y}$ he invests the non-random wealth δ in a safe and a risky asset.

Definition 3.31 *We can define $\widehat{B}_i(\widetilde{x}, \widetilde{y})$ analogously to $\widehat{B}_i(\widetilde{x}, y)$ as the value of B_i in $[0, \delta]$ which maximizes*

$$EU_i(\widetilde{y} + \delta + B_i\widetilde{x}).$$

Also the risk premium can be defined as

$$EU_i(\widetilde{y} + \widetilde{x}) = EU_i(\widetilde{y} + E\widetilde{x} - \pi_i(\widetilde{x}, \widetilde{y})).$$

If $R^{U_1} = -\frac{U_1''}{U_1'} > R^{U_2}$ holds on the relevant domain, can we say the similar results with the case of non-random initial wealth?
To extend this results, we assume the following :

- $\widetilde{x}, \widetilde{y}$ are independent.
- The utility functions must be taken from a restricted class, specifically it can be shown that it is sufficient that either utility function be non-increasingly risk averse.

In the analysis for this section, the utility functions $U_i, i = 1, 2$ will have as their domain $(\underline{z}, \overline{z})$. Each U_i is assumed to be concave and twice differentiable. The random variable $\widetilde{y}$, with probability measure μ, takes values from $(\underline{y}, \overline{y})$, where $\overline{y} - \underline{y} < \overline{z} - \underline{z}$.
Let $\underline{x} = \underline{z} - \underline{y}$ and $\overline{x} = \overline{z} - \overline{y}$. For each $x \in (\underline{x}, \overline{x})$, $V_i(x)$ is defined by

$$V_i(x) = EU(\widetilde{y} + x),$$

and we assume that the expectation exists and V_i is twice differentiable on $(\underline{x}, \overline{x})$. The Arrow-Pratt risk aversion measure of V_i is denoted by R^{V_i}.

Proposition 3.32 *Let $\widetilde{y}$ be a fixed random variable. The inequalities $\widehat{B}_1(\widetilde{x}, \widetilde{y}) \leq \widehat{B}_2(\widetilde{x}, \widetilde{y})$ and $\pi_1(\widetilde{x}, \widetilde{y}) \geq \pi_2(\widetilde{x}, \widetilde{y})$ hold for all $\widetilde{x}$ independent of $\widetilde{y}$ if and only if $R^{V_1}(x) \geq R^{V_2}(x)$ for all $x \in (\underline{x}, \overline{x})$. If $R^{V_1}(x) > R^{V_2}(x)$ for all $x \in (\underline{x}, \overline{x})$, then*

$$\widehat{B}_1(\widetilde{x}, \widetilde{y}) < \widehat{B}_2(\widetilde{x}, \widetilde{y}) \text{ and } \pi_1(\widetilde{x}, \widetilde{y}) > \pi_2(\widetilde{x}, \widetilde{y}) \tag{3.14}$$

will hold for all $\widetilde{x}$ independent of $\widetilde{y}$.

The proof is an immediate corollary of Theorem 3.14.

Example 3.33 *Restrict $\widetilde{y} + \widetilde{x}$ to the interval $(0, 1)$ and let $U_1(y) = y - \frac{1}{2}y^2$ and $U_2(y) = y - \frac{1}{22}y^{11}$. It can be shown that $R^{U_1}(y) > R^{U_2}(y)$ holds on the relevant interval. Now let $\widetilde{y}$ be a random variable which takes on the values 0.01 and 0.99 each with probability $\frac{1}{2}$, and let $x = 0$. By calculating, we can see that $R^{V_1}(x) = \frac{-EU_1''(\widetilde{y}+x)}{EU_1'(\widetilde{y}+x)} < R^{V_2}(x)$.*
Since the utility functions are polynomial, there exists a neighborhood of x for which $R^{V_2}(x) > R^{V_1}(x)$. This example gives a counterexample where conditions (3.14) do not hold and the agent is risk averse in Arrow-Pratt's sense in the domain.

For the following theorem, we work with the weak form of these inequalities. However, with some modification, we can obtain the results with strong inequalities.

Theorem 3.34 *$R^{U_1}(z) > R^{U_2}(z)$ for all $z \in (\underline{z}, \overline{z})$ and either R^{U_1} or R^{U_2} is a non-increasing function of z on $(\underline{z}, \overline{z})$, then*

$$R^{V_1}(x) > R^{V_2}(x) \tag{3.15}$$

holds on $(\underline{x}, \overline{x})$.

Lemma 3.35 *For any z_a, z_b in $(\underline{z}, \overline{z})$, we define r by $r = [\frac{U_1'(z_a)}{U_1'(z_b)}]/[\frac{U_2'(z_a)}{U_2'(z_b)}]$. If, for all $z_a \geq z_b$ in $(\underline{z}, \overline{z})$,*

$$\{[R^{U_1}(z_a) - R^{U_2}(z_a)] + [R^{U_1}(z_b) - R^{U_2}(z_b)]\}r + (1-r)[R^{U_1}(z_b) - R^{U_2}(z_a)] \geq 0 \tag{3.16}$$

then (3.15) holds on $(\underline{x}, \overline{x})$

Proof of the lemma. We can check that the (3.16) is equivalent to
$-[U_1''(z_a)U_2'(z_b) + U_1''(z_b)U_2'(z_a)] \geq -[U_2''(z_a)U_1'(z_b) + U_2''(z_b)U_1'(z_a)]$.
This is symmetric in z_a and z_b. Therefore, we can take $z_\alpha \geq z_\beta$, and we let $z_\alpha = z_a$ and $z_\beta = z_b$.
$-[U_1''(z_\alpha)U_2'(z_\beta) + U_1''(z_\beta)U_2'(z_\alpha)] \geq -[U_2''(z_\alpha)U_1'(z_\beta) + U_2''(z_\beta)U_1'(z_\alpha)]$. Thus the above inequality holds for all $z_\alpha, z_\beta \in (\underline{z}, \overline{z})$ if and only if (3.16) holds for all $z_a \geq z_b \in (\underline{z}, \overline{z})$.
Let $x \in (\underline{x}, \overline{x})$ and $\widetilde{y_\alpha}$ and $\widetilde{y_\beta}$ be two independent random variables, each of which has the same distribution as $\widetilde{y}$. We take the random variable as $z_\alpha = x + \widetilde{y_\alpha}$ and $z_\beta = x + \widetilde{y_\beta}$.
Substituting z_α, z_β, taking expectations in both sides of the inequality, we obtain
$-[EU_1''(x + \widetilde{y}_\alpha)EU_2'(x + \widetilde{y}_\beta) + EU_1''(x + \widetilde{y}_\beta)EU_2'(x + \widetilde{y}_\alpha)] \geq$
$-[EU_2''(x + \widetilde{y}_\alpha)EU_1'(x + \widetilde{y}_\beta) + EU_2''(x + \widetilde{y}_\beta)EU_1'(x + \widetilde{y}_\alpha)]$. Since $\widetilde{y_\alpha}$ and $\widetilde{y_\beta}$ have the same distribution,
$-2EU_1''(x + \widetilde{y})EU_2'(x + \widetilde{y}) \geq -2EU_2''(x + \widetilde{y})EU_1'(x + \widetilde{y})$,

which is equivalent to (3.15). Q.E.D.

Proof of the theorem. Pratt [28] has shown that $r \le 1$ for $z_a \ge z_b$ if $R^{U_1}(z) \ge R^{U_2}(z)$ on $(\underline{z}, \overline{z})$. Therefore, the first term in (3.16) is nonnegative. If R^{U_1} is a non-increasing function of z, then we write
$R^{U_1}(z_b) - R^{U_2}(z_a) = [R^{U_1}(z_b) - R^{U_1}(z_a)] + [R^{U_1}(z_a) - R^{U_2}(z_a)]$. Therefore, (3.16) is nonnegative. If R^{U_2} is a non-increasing function of z, in the same way, we prove that (3.16) is nonnegative and we apply the lemma. Q.E.D.

Corollary 3.36 *If $R^{U_1}(z) > R^{U_2}(z)$ for all $z \in (\underline{z}, \overline{z})$ and either R^{U_1} or R^{U_2} is a non-increasing function of z on $(\underline{z}, \overline{z})$, then (3.14) holds when $\widetilde{x}, \widetilde{y}$ are independent and range over the values such that wealth always lies in $(\underline{z}, \overline{z})$.*

The above corollary follows immediately from the preceding theorem and proposition.

Corollary 3.37 *If $R^U(z)$ is a non-increasing [decreasing] function of z, then $B(\widetilde{x}, \widetilde{y} + z)$ is a non-decreasing (increasing) function of z and $\pi(\widetilde{x}, \widetilde{y} + z)$ is a non-increasing (decreasing) function of z.*

Proof. If $z_1 \le z_2$, let $U_i(z) = U_i(z + z_i)$. Apply corollary 3.36.

In Section 3.3, Ross' main theorem gives a condition under which U_1 is more risk averse than U_2 implies $\pi_1(\widetilde{x}, \widetilde{y}) > \pi_2(\widetilde{x}, \widetilde{y})$ when $E[\widetilde{x}|y] = 0$ for all y. He assumes that $\widetilde{x}$ and $\widetilde{y}$ are uncorrelated in the strong sense that $E[\widetilde{x}|y] = 0$ for all y. If in this section we had restricted to cases in which $E\widetilde{x} = 0$, the hypothesis would have satisfied those of Ross in section 3.3. However, even if $\widetilde{x}$ and $\widetilde{y}$ are uncorrelated in the strong sense, if $\widetilde{x}$ and $\widetilde{y}$ are not independent, Proposition 3.32 does not hold. As a result, the function V is irrelevant if $\widetilde{x}$ and $\widetilde{y}$ are not dependent, even if they are uncorrelated in the strong sense of Ross.

3.5 Proper risk aversion [30]

Denote a decision maker's initial wealth by w if it is certain, by $\widetilde{w}$ if it is uncertain. Let $\widetilde{x}$ and $\widetilde{y}$ be possible additional risks. Assume that the decision maker has a probability distribution under which $\widetilde{w}, \widetilde{x}, \widetilde{y}$ are independent. Assume that he has a von Neumann-Morgenstern utility function U. Recall that "risk averse"is defined by the condition $\widetilde{w} \preceq E\widetilde{w}$ for all $\widetilde{w}$ and is equivalent to concavity of U. Also an intuitive definition of "decreasingly risk averse"is that a certain decrease in wealth never makes an undesirable gamble desirable, i.e.,

$$w + \widetilde{x} + y \preceq w + y \text{ whenever } w + \widetilde{x} \preceq w \text{ and } y < 0. \tag{3.17}$$

Definition 3.38 *U is fixed-wealth proper if $w + \widetilde{x} + \widetilde{y} \preceq w + \widetilde{y}$ whenever $w + \widetilde{x} \preceq w$ and $w + \widetilde{y} \preceq w$.*

In other words, if lotteries $\widetilde{x}, \widetilde{y}$ are individually unattractive, the compound lottery offering both together is less attractive than either alone.

Definition 3.39 *We call U proper if the same condition holds for uncertain $\widetilde{w}$ also, that is, if*

$$\widetilde{w} + \widetilde{x} + \widetilde{y} \preceq \widetilde{w} + \widetilde{y} \text{ whenever } \widetilde{w} + \widetilde{x} \preceq \widetilde{w} \text{ and } \widetilde{w} + \widetilde{y} \preceq \widetilde{w}. \tag{3.18}$$

Obviously, proper implies fixed wealth proper, and fixed wealth proper implies decreasing risk aversion.

Definition 3.40 *The certainty equivalent $C(\widetilde{x},\widetilde{z})$ of a gamble $\widetilde{x}$ in the presence of another gamble $\widetilde{z}$ (including initial wealth) is defined as the sure amount to which $\widetilde{x}$ is indifferent: $\widetilde{z}+\widetilde{x}\sim\widetilde{z}+C(\widetilde{x},\widetilde{z})$.*

The idea of the definition is the same as cash equivalent (Definition 3.8), but the above one is more general, allowing x in Definition 3.8 to be a random variable. Setting $C(\widetilde{x})=C(\widetilde{x},\widetilde{w})$ for convenience, we can write the properness condition as

$$C(\widetilde{x}+\widetilde{y})\leq C(\widetilde{y}) \text{ whenever } C(\widetilde{x})\leq 0 \text{ and } C(\widetilde{y})\leq 0.$$

Dependence between $\widetilde{w}$ and $(\widetilde{x},\widetilde{y})$ seems difficult to allow, because even when the preference is proper, risk aversion can easily be decreased by either a stochastic decrease or added noise in $\widetilde{w}$. To exemplify, suppose $\widetilde{w}$ has two possible values w_1 and w_2, with $w_1<w_2$, and suppose $\widetilde{x}=0$ when $\widetilde{w}=w_1$, while respectively either $\widetilde{x}=-\delta<0$ or $\widetilde{x}=\delta$ with equal probability when $\widetilde{w}=w_2$. Then for certain values of w_1,w_2 and δ, the "derived"utility function $\widehat{U}(y)=EU(\widetilde{w}+y)$, which applies to risk independent of $\widetilde{w}$, may easily be more risk averse than $V(y)=EU(\widetilde{w}+\widetilde{x}+y)$, because changes around w_2 have less relative importance to the former than changes around $w_2-\delta$ to the latter.

Theorem 3.41 *For preferences in accord with expected utility, if there is decreasing risk aversion and the condition*

$$\widetilde{w}+\widetilde{x}+\widetilde{y}\preceq\widetilde{w} \text{ whenever } \widetilde{w}+\widetilde{x}\sim\widetilde{w}\sim\widetilde{w}+\widetilde{y} \tag{3.19}$$

is satisfied, then

$$C(\widetilde{x}+\widetilde{y})\leq C(\widetilde{x})+C(\widetilde{y}) \text{ whenever } C(\widetilde{x})\leq 0 \text{ and } C(\widetilde{y})\leq 0 \tag{3.20}$$

and

$$C(\widetilde{x},\widetilde{y}+\widetilde{w})\leq C(\widetilde{x},\widetilde{w}) \text{ whenever } C(\widetilde{x},\widetilde{w})\leq 0 \text{ and } C(\widetilde{y},\widetilde{w})\leq 0 \tag{3.21}$$

are each equivalent to proper risk aversion, if they hold for all independent $\widetilde{w},\widetilde{x}$ and $\widetilde{y}$, (to proper risk aversion on an interval if $\widetilde{w},\widetilde{w}+\widetilde{x},\widetilde{w}+\widetilde{y},\widetilde{w}+\widetilde{x}+\widetilde{y}$ are restricted to this interval). The same equivalences hold for fixed-wealth properness if $\widetilde{w}$ is restricted to certainties w.

A significant fact used in the proof is that decreasing risk aversion, property (3.17), implies the same property for uncertain $\widetilde{w}$,

$$\widetilde{w}+\widetilde{x}+y\preceq\widetilde{w}+y \text{ whenever } \widetilde{w}+\widetilde{x}\preceq\widetilde{w} \text{ and } y<0 \quad (\star).$$

Remarks First, the foregoing result implies that the above generalization of (3.17) to uncertain $\widetilde{w}$ is equivalent to (3.17), unlike proper or fixed-wealth proper conditions. Second, this equivalence says that U is decreasingly risk averse iff every utility function derived from it is. Third, U is proper iff every utility function derived from U is fixed-wealth proper.

Proof of the theorem. It is immediate that (3.20) or (3.21) $\Rightarrow$ (3.18) $\Rightarrow$ (3.19) plus decreasing risk aversion.
To show that (3.19) plus decreasing risk aversion implies (3.20), let $x=C(\widetilde{x})$ and $y=C(\widetilde{y})$, and suppose $x\leq 0, y\leq 0$. Since $\widetilde{w}+\widetilde{x}\sim\widetilde{w}+x$, replacing $\widetilde{w}$ by $\widetilde{w}+x$ and $\widetilde{x}$ by $\widetilde{x}-x$ in $(\star)$ and then

exchanging x and y gives $\widetilde{w}+x+y+\widetilde{x}-x \preceq \widetilde{w}+x+y$, $\widetilde{w}+y+x+\widetilde{y}-y \preceq \widetilde{w}+y+x$. Applying (3.19) at $\widetilde{w}+x+y$ to these relations gives $\widetilde{w}+\widetilde{x}+\widetilde{y} \preceq \widetilde{w}+x+y$, which is equivalent to (3.20). To complete the proof, we show that properness (3.18) implies (3.21). Since $\widetilde{w}+x+\widetilde{y} \preceq \widetilde{w}+x$ by ($\star$), applying (3.18) to $\widetilde{w}+x, \widetilde{x}-x$ and $\widetilde{y}$ gives $\widetilde{w}+\widetilde{x}+\widetilde{y} \preceq \widetilde{w}+x+\widetilde{y}$, which is equivalent to (3.21). Q.E.D.

We have also proved that proper preference implies that

$$\widetilde{w}+\widetilde{x}+\widetilde{y} \preceq \widetilde{w}+C(\widetilde{x})+\widetilde{y} \preceq \widetilde{w}+C(\widetilde{x})+C(\widetilde{y}),$$

whenever $C(\widetilde{x}) \leq 0$ and $C(\widetilde{y}) \leq 0$.

3.5.1 An analytical sufficient condition

Recall that, for exponential utility, preferences among risks are unaffected by wealth (since $-e^{-c(w+a)}$ is a positive multiple of $-e^{-cw}$). It is known that mixtures of concave exponential utilities have decreasing risk aversion [29]. We show here that they are also proper. The most general mixture of exponential utilities, which is called completely proper, is

$$U(w) = \int_0^\infty [g(s) - e^{sw}] dF(s), \tag{3.22}$$

where g is an arbitrary function, F is non-decreasing, e^{-sw} is to be replaced by w when $s = 0$. Taking the difference eliminates most of the arbitrariness in g and gives the equivalent form

$$U(w) = U(w_1) + \int_0^\infty [g(e^{-sw_1} - e^{sw}] dF(s), \tag{3.23}$$

for any w_1 where $U(w_1)$ is finite. Assuming convergence at more than one point, one can show by monotonicity and concavity in w (e.g., [8, p.409]), that (3.23), and hence (3.22) gives a finite U with positive odd derivatives and negative even derivatives on some interval (w_0, ∞), possibly $(-\infty, \infty)$, and $U(w) = -\infty$ for $w < w_0$. A positive function with positive even derivatives and negative odd derivatives is called "completely monotone".

Theorem 3.42 *A completely proper utility function is proper everywhere that it is finite.*

One point distribution F gives exponential functions. The risk-averse power functions $U(w) \sim d(w-a)^d, d \leq 1$, and $\log(w-a)$ are completely proper on (a, ∞); they satisfy (3.23) with $dF(s) \sim s^{-d-1}e^{as}ds$.
The two point distribution F gives $U(w) = a - be^{-sw} - ce^{-tw}$, where b, c, s, and t are nonnegative constants and w replaces $-e^{-sw}$ if $s = 0$.

Example 3.43 *For $s > t > 0$ and $b, c > 0$, the utility $a - be^{-sw} + ce^{tw}$ is proper risk averse on the interval $w_0 = \frac{1}{s+t}\log\frac{b}{c} < w < \frac{1}{s+t}\log(\frac{b}{c}\frac{s^2}{t^2}) = w_1$, but not of the form (3.22), and it is improper below w_0 although its derivatives alternate in sign for $w < w_1$. It is increasing and its risk aversion is decreasing everywhere. It is risk seeking above w_1.*

Proof of the example. The signs of the derivatives and the formula $r(w) = -t + bs(s + t)/(bs + cte^{sw+tw})$ for the local risk aversion are easily obtained. To show properness on (w_0, w_1), let $k = (c/b)Ee^{s\widetilde{w}+t\widetilde{w}}$ and $b = Ee^{s\widetilde{w}}$ without loss of generality, and observe that
$EU(\widetilde{w}+\widetilde{x}) - EU(\widetilde{w}) = -Es^{-s\widetilde{x}} + kEe^{t\widetilde{x}} + 1 - k.$
If $\widetilde{w}+\widetilde{x} \sim \widetilde{w} \sim \widetilde{w}+\widetilde{y}$, then $Ee^{-s\widetilde{x}} = kEe^{t\widetilde{x}} + 1 - k$ and similarly for $\widetilde{y}$, and substitution gives
$EU(\widetilde{w}+\widetilde{x}+\widetilde{y}) - EU(\widetilde{w}) = k(1-k)(Ee^{t\widetilde{x}} - 1)(Ee^{t\widetilde{y}} - 1).$
For $\widetilde{w}+\widetilde{x} \preceq w_1$, risk aversion implies $E\widetilde{x} \geq 0$ and hence $Ee^{t\widetilde{x}} \geq e^{tE\widetilde{x}} \geq 1$. Similarly $Ee^{t\widetilde{y}} \geq 1$ for $\widetilde{w}+\widetilde{y} \preceq w_1$. Since $k > 1$ for $\widetilde{w} > w_0$, $EU(\widetilde{w}+\widetilde{x}+\widetilde{y}) - EU(\widetilde{w})$ is not positive and the condition (3.19) for properness is satisfied on the interval (w_0, w_1). Q.E.D.

Proof of the theorem. If U has the form (3.22), then so does every derived utility $\widehat{U}(x) = EU(\widetilde{w}+x)$. Hence, it is sufficient to show that every U of the form (3.22) is proper at 0. Suppose $\widetilde{x} \preceq 0$ and $\widetilde{y} \preceq 0$. Let $x(s)$ and $y(s)$ be the certainty equivalents of $\widetilde{x}$ and $\widetilde{y}$ for constant risk aversion, that is $e^{-sx(s)} = Ee^{-s\widetilde{x}}$ and $e^{-sy(s)} = Ee^{-s\widetilde{y}}$; and $x(s)$ and $y(s)$ are decreasing in s (Theorem 3.14). Except in trivial cases, there exist constants c and d such that $x(c) = y(d) = 0$. Since $\widetilde{x}, \widetilde{y}$ are symmetric we can take $c \geq d$. Then one can show that, because $x(s)$ and $y(s)$ are decreasing, $e^{sx(s)} \underset{>}{\overset{<}{=}} 1$ and $e^{sy(s)} \underset{>}{\overset{<}{=}} e^{-cy(c)}$ for $s \underset{>}{\overset{<}{=}} c$.
By (3.23) with $w_1 = 0$ and (3.21),
$EU(\widetilde{x}) - U(0) = \int_0^\infty E[1 - e^{-s\widetilde{x}}]dF(s) = \int_0^\infty [1 - e^{-sx(s)}]dF(s) \leq 0$
and similarly for $\widetilde{y}$.
We then have $EU(\widetilde{x}+\widetilde{y}) - U(0) = \int_0^\infty E[1 - e^{-s\widetilde{x}-s\widetilde{y}}]dF(s) = \int_0^\infty [1 - e^{-sx(s)-sy(s)}]dF(s) \leq$
by independence $\leq \int_0^\infty [e^{-sy(s)} - e^{-sx(s)-sy(s)}]dF(s) \leq e^{-cy(c)} \int_0^\infty [1 - e^{-sx(s)}]dF(s) \leq 0$. Q.E.D.

3.6 Standard risk aversion [20]

Definition 3.44 *Under the von Neumann-Morgenstern utility function U, two risks $\widetilde{x}$ and $\widetilde{y}$ (which may have non-zero means) aggravate each other, starting from initial wealth w, if and only if*

$$\frac{1}{2}EU(w+\widetilde{x}) + \frac{1}{2}EU(w+\widetilde{y}) \geq \frac{1}{2}U(w) + \frac{1}{2}EU(w+\widetilde{x}+\widetilde{y}). \tag{3.24}$$

A risk $\widetilde{x}$ aggravates a reduction in wealth of size ϵ if and only if

$$\frac{1}{2}EU(w+\epsilon) + \frac{1}{2}EU(w+\widetilde{x}) \geq \frac{1}{2}U(w) + \frac{1}{2}EU(w-\epsilon+\widetilde{x}). \tag{3.25}$$

For infinitesimal reduction in wealth (ϵ small), (3.25) becomes

$$EU'(w+\widetilde{x}) \geq U'(w). \tag{3.26}$$

Thus, a risk aggravates an infinitesimal reduction in wealth if and only if it raises expected marginal utility.

Definition 3.45 *A risk $\widetilde{x}$ is loss-aggravating, starting from initial wealth w, if and only if it satisfies (3.26).*

When absolute risk aversion is decreasing, every undesirable risk is loss aggravating, but not every loss-aggravating risk is undesirable. Also, note that mutual aggravation guarantees that if one risk is a bad thing in the absence of the other risk, it will remain a bad thing in the presence of the other risk. For example, if $EU(w+\widetilde{y}) \leq U(w)$, then (3.24), mutual aggravation between $\widetilde{x}$ and $\widetilde{y}$ implies that $EU(w+\widetilde{x}+\widetilde{y}) \leq EU(w+\widetilde{x})$.

Definition 3.46 *The von Neumann-Morgenstern utility function has standard risk aversion iff for any pair of independent risks $\widetilde{x}$ and $\widetilde{y}$, and any initial wealth w, the combination of (3.26) and $EU(w+\widetilde{y}) \leq U(w)$ implies (3.24).*

Other than the allowance for random initial wealth, the only difference between the definition of proper risk aversion and the standard risk aversion is that for the proper risk aversion, both risks must be undesirable to guarantee the mutual aggravation, while for the standard risk aversion, it is enough for one risk to be undesirable, with the other risk loss-aggravating.

Proposition 3.47 *The von Neumann-Morgenstern utility function has a proper risk aversion iff for any triple of mutually independent variables $\widetilde{w}, \widetilde{x}$ and $\widetilde{y}$, the pair of inequalities*

$$EU(\widetilde{w}+\widetilde{x}) \leq EU(\widetilde{w}) \text{ and } EU(\widetilde{w}+\widetilde{y}) \leq EU(\widetilde{w}) \quad (\star)$$

implies $E[U(\widetilde{w}+\widetilde{x}+\widetilde{y}) - U(\widetilde{w}+\widetilde{x}) - U(\widetilde{w}+\widetilde{y}) + U(\widetilde{w})] \leq 0$.

Proof. $(\Leftarrow)(\star)$ imply $E[U(\widetilde{w}+\widetilde{x}+\widetilde{y}) - U(\widetilde{w}+\widetilde{y})] = E[U(\widetilde{w}+\widetilde{x}+\widetilde{y}) - U(\widetilde{w}+\widetilde{x}) - U(\widetilde{w}+\widetilde{y}) + U(\widetilde{w})] + E[U(\widetilde{w}+\widetilde{x}) - U(\widetilde{w})] \leq 0$.
$(\Rightarrow)$ If v has decreasing absolute risk aversion and $Ev(\pi^*+\widetilde{x}) \leq v(\pi^*)$, with $\pi^* \geq 0$, then
$E[v(\pi^*+\widetilde{x}) - v(\widetilde{x})] = \int_0^{\pi^*} Ev'(\zeta+\widetilde{x})d\zeta \geq \int_0^{\pi^*} Ev'(\zeta)d\zeta = v(\pi^*) - v(0)$. $\widetilde{x}$ is undesirable at π^*. By the decreasing absolute risk aversion which guarantees that every undesirable risk is loss-aggravating, the above inequality holds for any $\zeta \leq \pi^*$.
Using this result, $(\star)$ and the definition of proper risk aversion, we have
$$E[U(\widetilde{w}+\widetilde{x}+\widetilde{y}) - U(\widetilde{w}+\widetilde{x}) - U(\widetilde{w}+\widetilde{y}) + U(\widetilde{w})] =$$
$$= E[U(\widetilde{w}+\widetilde{x}+\widetilde{y}) - U(\widetilde{w}+\widetilde{y}) - U(\widetilde{w}+\widetilde{x}+\widetilde{y}+\pi_y^*) + U(\widetilde{w}+\widetilde{y}+\pi_y^*)] +$$
$$+ E[U(\widetilde{w}+\widetilde{x}+\widetilde{y}+\pi_y^*) - U(\widetilde{w}+\widetilde{x}) - U(\widetilde{w}+\widetilde{x}+\pi_x^*+\widetilde{y}+\pi_y^*) + U(\widetilde{w}+\widetilde{x}+\pi_x^*)] +$$
$$+ E[U(\widetilde{w}+\widetilde{x}+\pi_x^*+\widetilde{y}+\pi_y^*) - U(\widetilde{w}+\widetilde{y}+\pi_y^*)] + E[U(\widetilde{w}) - U(\widetilde{w}+\widetilde{x}+\pi_x^*)] \leq 0,$$
where π_x^* and π_y^* are the risk premia for $\widetilde{x}$ and $\widetilde{y}$ at $\widetilde{w}$. The first bracketed term on the right hand of the equation is negative using the above result with $v(\zeta) = EU(\zeta+\widetilde{w}+\widetilde{y})$ and $\pi^* = \pi_y^*$. The second bracketed term is negative using the above result again with $v(w) = EU(\zeta+\widetilde{w}+\widetilde{x})$ and $\pi^* = \pi_x^*$. In both cases, v inherits decreasing absolute risk aversion from U (which has decreasing absolute risk aversion as a consequence of the definition of proper risk aversion, with $\widetilde{y} = -\epsilon, \epsilon > 0$), while the definition together with the undesirability of $\widetilde{x}$ and $\widetilde{y}$ guarantees that $Ev(\pi^*+\widetilde{x}) \leq v(\pi^*)$. Note that $\widetilde{x}$ and $\widetilde{y}$ can be interchanged and can be replaced by $\widetilde{x}+\pi_x^*$ and $\widetilde{y}+\pi_y^*$ in the definition of proper risk aversion. The third is negative using the definition of proper risk aversion. Using the definition of risk premium, the fourth bracket is zero. Q.E.D.

Proposition 3.48 *If $U'(w) > 0$ and $U''(w) < 0$ over the entire domain of U, then U is standard if and only if both the absolute risk aversion $-U''(w)/U'(w)$ and absolute prudence $-U'''(w)/U''(w)$ are monotonically decreasing over the entire domain of U.*

Proof.
Necessity of Decreasing Absolute Risk Aversion. Specialize the risk $\widetilde{x}$ in (3.24) and (3.26) to a non-random negative quantity $-\epsilon$. Concavity of U ensures that $\widetilde{x} = -\epsilon$ satisfies (3.26). Trivially, $-\epsilon$ and $\widetilde{y}$ are statistically independent. When $\widetilde{x} = -\epsilon$, $EU(w+\widetilde{y}) \leq U(w)$ implies (3.24). In words, if the agent would reject a risk $\widetilde{y}$ at the initial wealth w, then the agent would reject the risk at any lower level of initial wealth. Pratt [28] has shown that this implies decreasing absolute risk aversion. Consider a small risk $\widetilde{y}$ for which $EU(w+\widetilde{y}) \leq U(w)$ holds with equality. Then $\mu = \frac{-U''(w)}{U'(w)}\frac{\sigma^2}{2} + o(\sigma^2)$, where μ is the mean of $\widetilde{y}$ and σ^2 is the variance of $\widetilde{y}$.
Moreover, $EU(w-\epsilon+\widetilde{y}) - U(w-\epsilon) \leq EU(w+\widetilde{y}) - U(w) \leq 0$ implies that $\mu \leq \frac{-U''(w-\epsilon)}{U'(w-\epsilon)}\frac{\sigma^2}{2} + o(\sigma^2)$. Combining the above two results and dividing by $\sigma^2/2$, we have $\frac{-U''(w-\epsilon)}{U'(w-\epsilon)} \geq \frac{-U''(w)}{U'(w)} + \frac{o(\sigma^2)}{\sigma^2}$. If one chooses a small enough risk $\widetilde{y}$ so that $\sigma^2 \to 0$, then $\frac{o(\sigma^2)}{\sigma^2} \to 0$. Therefore $\frac{-U''(w-\epsilon)}{U'(w-\epsilon)} \geq \frac{-U''(w)}{U'(w)}$ for any w and any $\epsilon > 0$.

Necessity of Decreasing Absolute Prudence. Specialize the second risk $\widetilde{y}$ to a non-random negative quantity, $\epsilon < 0$. Monotonicity ensures that $\widetilde{y} = -\epsilon$. Trivially, $-\epsilon$ and $\widetilde{x}$ are statistically independent. When $\widetilde{x} = -\epsilon$, (3.26) implies (3.24), $EU(w-\epsilon+\widetilde{x}) - U(w+\widetilde{x}) - EU(w-\epsilon) + U(w) \leq 0$.
Equivalently, for all $w, \epsilon > 0$, and $\widetilde{x}$ satisfying $EU'(w+\widetilde{x}) \geq U'(w)$, $\int_{w-\epsilon}^{w} [EU'(\zeta+\widetilde{x}) - U'(\zeta)]d\zeta \geq 0$. This means there cannot be any interval $[w-\epsilon, w]$ on which $EU'(\zeta+\widetilde{x}) - U'(\zeta)$ is monotonically increasing from a negative value to zero. Therefore, by continuity, $EU'(w+\widetilde{x}-\epsilon) \geq U'(w-\epsilon)$ for any w and $\epsilon > 0$. In words, if a risk $\widetilde{x}$ is loss-aggravating at the initial wealth w, then it is loss-aggravating at any lower level of initial wealth. By the same argument as above, this, with some modification, implies that $\frac{-U'''(w-\epsilon)}{U''(w-\epsilon)} \geq \frac{-U'''(w)}{U''(w)}$ for any w and any $\epsilon > 0$.

The contribution of Decreasing Absolute Risk Aversion. As shown by [21], decreasing absolute risk aversion implies that $EU'(w+\widetilde{y}) \geq U'(w)$ whenever $EU(w+\widetilde{y}) \leq U(w)$. Using this consequence of decreasing absolute risk aversion, the combination of (3.26), $EU(w+\widetilde{y}) \leq U(w)$ and statistical independence between $\widetilde{x}$ and $\widetilde{y}$ implies

$$E[U'(w+\widetilde{x}) - U'(w)][U'(w+\widetilde{y}) - U'(w)] \geq 0. \tag{3.27}$$

The contribution of Decreasing Absolute Prudence. Here we will prove that (3.27) implies (3.24). The convexity of $\ln(-U'')$ implied by the decreasing absolute prudence ensures that
$\ln(-U''(w+x+y)) - \ln(-U''(w+y)) \leq \ln(-U''(w+x)) - \ln(-U''(w))$, if $xy \geq 0$. If $xy \leq 0$, then the inequality is reversed. Equivalently, $\frac{U''(w+x+y)}{U''(w)} \geq \frac{U''(w+x)}{U''(x)}\frac{U''(w+y)}{U''(w)}$, if $xy \geq 0$.
We take integral to yield $\int_0^x \int_0^y \frac{U''(w+\chi+\xi)}{U''(w)} d\xi\chi \geq (\int_0^x \frac{U''(w+\chi)}{U''(w)} d\chi)(\int_0^y \frac{U''(w+\xi)}{U''(w)} d\xi)$ for any pair of x and y. Performing the integration yields,

$$\frac{U(w+x+y) - U(w+x) - U(w+y) + U(w)}{U''(w)} \geq \frac{U'(w+x) - U'(w)}{U''(w)} \frac{U'(w+y) - U'(w)}{U''(w)}$$

for any pairs of x and y. Taking expectations of both sides of it and multiplying both sides by $U''(w)$, we obtain, for any pair of random variable $\widetilde{x}$ and $\widetilde{y}$,
$E[U(w+\widetilde{x}+\widetilde{y}) - U(w+\widetilde{x}) - U(w+\widetilde{y}) + U(w)] \leq \frac{E[(U'(w+\widetilde{x})-U'(w))(U'(w+\widetilde{y})-U'(w))]}{U''(w)}$.
This gives the desired result that (3.27) implies (3.24). Q.E.D.

Proposition 3.49 *For any random variable $\widetilde{w}$, if U is standard then the derived utility function $\widehat{U}$ defined by $\widehat{U}(x) = EU(\widetilde{w} + x)$ is also standard.*

Proposition 3.50 *If $U'(w) > 0$ and $U''(w) < 0$ and both the absolute risk aversion $-U''(w)/U'(w)$ and absolute prudence $-U'''(w)/U''(w)$ are monotonically decreasing over the entire domain of U, then U is proper.*

Proposition 3.51 *If $U'(w) > 0$ and $U''(w) < 0$ over the entire domain of U, then any loss-aggravating risk always lowers the absolute value of the optimal level of investment in any other independent risk if and only if U is standard.*

For the proofs of the above propositions, see [20].

4 Multivariate risk aversion

4.1 Risk aversion with many commodities [17]

Theorem 3.14 implies that U_1 is more risk averse than U_2 if and only if U_1 is an increasing concave transformation of U_2.
In this section we would like to extend this result. A difficulty encountered in generalizing the Arrow-Pratt theory of risk aversion is that n dimensional von Neumann-Morgenstern utility functions may represent different preference ordering on the set of commodity bundles. Let's illustrate the above statement with an example. Let $\widehat{x} = (\widehat{x_1}, \widehat{x_2})$ and $\widetilde{x} = (\widetilde{x_1}, \widetilde{x_2})$ be two distinct points in Ω_2, the nonnegative part of the Euclidean two-dimensional space with the property that $U_1(\widetilde{x}) > U_1(\widehat{x})$ and $U_2(\widehat{x}) > U_2(\widetilde{x})$. Consumer 1, with utility function U_1, and consumer 2, with utility function U_2, are both faced with the choice of receiving $\widetilde{x}$ with certainty or a gamble between $\widetilde{x}$ and $\widehat{x}$. Clearly, consumer 1 prefers $\widetilde{x}$ with certainty to any gamble between $\widetilde{x}$ and $\widehat{x}$. However, consumer 2 prefers any gamble between $\widetilde{x}$ and $\widehat{x}$. Consumer 1 acts as if she is more risk averse than consumer 2. However, this behavior occurs because of the differences in the ordinal preferences represented by U_1 and U_2. This difficulty never arises in one dimension since in that case all monotonically increasing utility functions represent the same ordinal preference.

Proposition 4.1 *(Univariate case) Let $r_i(x)$ and $\pi_i(x, z)$ be the absolute risk aversion function and the risk premium corresponding to the twice continuously differentiable and monotonically increasing utility function $U_i(x), i = 1, 2$. The following conditions are equivalent:*

- $r_1(x) \geq (>) r_2(x)$.
- $\pi_1(x, z) \geq (>) \pi_2(x, z)$.
- *There exists an increasing, (strictly) concave, twice continuously differentiable function k such that $U_1(x) = k(U_2(x))$.*

Proof. Because of the assumptions made about U_1 and U_2, there exists a monotonically increasing and continuously differentiable function $k = U_1 \circ U_2^{-1}$ such that $U_1 = k \circ U_2$. Differentiating we get $U_1' = k'U_2'$ and $U_1'' = k''(U_2')^2 + k'U_2''$. Using the above equation, we get $k'' = \frac{U_1'' - k'U_2''}{(U_2')^2} = [\frac{U_1''}{U_1'} - \frac{U_2''}{U_2'}][\frac{U_1'}{(U_2')^2}]$. Therefore, $k'' \leq (<) 0$ if and and only if $r_1(x) \geq (>) r_2(x)$.Q.E.D.

Motivated by the above proposition, the following definition generalizes the concept of risk aversion to utility functions of more than one variable. Throughout this section, it is assumed that U_i is strictly concave and has continuous second derivatives.
We also assume that $\partial U_i/\partial x_j > 0, i = 1, 2, j = 1, .., n$. Finally U_1 and U_2 are assumed to represent the same preferences, there exists a function k such that $U_1 = k \circ U_2$, and $k' > 0$.

Definition 4.2 *U_1 is at least as risk averse as U_2, $[U_2 R U_1]$ if $U_1 = k(U_2)$ where $k' > 0$ and k is concave. U_1 is more risk averse than U_2, $[U_1 P U_2]$, if k is strictly concave.*

Let $x, y \in \Omega_n$ and let z be a random variable which takes values in $[0, \infty)$. Consider gambles in Ω_n, the n dimensional Euclidean space, of the form $x + zy$. These gambles lie on the line, originating at x, through the point $x + y$. For the utility function U_i we will study the risk premium $\pi(x, y, z)$, $E\{U_i(x + zy)\} = U_i(x + y(E(z) - \pi(x, y, z)))$, associated with the random variable z, the point x and the direction y.

Proposition 4.3 *If U_1 is at least as risk averse as (more risk averse than) U_2 then, for every $x, y \in \Omega_n$ and every gamble $z \geq 0$, the risk premium $\pi_1(x, y, z)$ for U_1 is at least as large as (larger than) the corresponding risk premium $\pi_2(x, y, z)$ for U_2.*

Proof. For $z \in [0, \infty)$, let $v^i_{x,y}(z)$ be defined by $v^i_{x,y}(z) = U_i(x + zy)$. When x and y are fixed, $v^i_{x,y}(z)$ is a function of the one dimensional variable z, and $\pi_i(x, y, z)$ is analogous to the one dimensional Arrow-Pratt risk premium. $v^1_{x,y}(z) = k(v^2_{x,y}(z))$ where k is (strictly) concave. By Proposition 4.1, the (strict) concavity of k implies that $\pi_1(x, y, z) \geq (>)\pi_2(x, y, z)$. Q.E.D.

Proposition 4.4 *Suppose that U_1 and U_2 both represent the preference ordering $\succeq$. If there exists y such that, for all z, $\pi_1(0, y, z) \geq (>)\pi_2(0, y, z)$ then U_1 is at least as risk averse as (more risk averse than) U_2.*

Proof. Since U_1 and U_2 represent the same preference, there exists a function such that $U_1 = k \circ U_2$. For this same k, $v^1_{0,y}(z) = k(v^2_{0,y}(z))$. Applying Proposition 4.1, $\pi_1(0, y, z) \geq (>)\pi_2(0, y, z)$, for all z, implies that k is (strictly) concave. Q.E.D.

In spite of the fact that the value of the directional risk premium varies with the direction, risk aversion comparisons of utility functions, obtained by comparing directional risk premia are independent of the direction. This result is obtained because comparisons are restricted to utility functions representing the same ordinal preference.

Let $\triangle_n = \begin{vmatrix} U_{11} & \dots & U_{1n} \\ \vdots & & \vdots \\ U_{n1} & \dots & U_{nn} \end{vmatrix}$ and $\triangle^b_n = \begin{vmatrix} U_{11} & \dots & U_{1n} & U_1 \\ \vdots & & \vdots & \vdots \\ U_{n1} & \dots & U_{nn} & U_n \\ U_1 & \dots & U_n & 0 \end{vmatrix}$.

It has been assumed that all utility functions are strictly concave. This assumption implies that $(-1)^n \triangle^b_n \geq 0$. Let's assume that $(-1)^n \triangle^b_n > 0$, then we define

$$\rho(x_1, .., x_n) = \frac{(-1)^n \triangle_n}{\{(-1)^n \triangle^b_n\}^{n/n+1}}$$

.

Proposition 4.5 *Suppose U_1 and U_2 represent the preference $\succeq$. Then U_1 is at least as risk averse as [more risk averse than] U_2 if and only if $\rho^1(x_1,x_2) \geq [>]\rho^2(x_1,x_2)$ for all x_1, x_2.*

Proof In order to simplify notation, in this part of the proof we will denote U_1 by U^1, and $\partial U/\partial x_1$ by U_1. Since U^1 and U^2 represent the same preference, $U^1 = k \circ U^2$, where $k' > 0$. Therefore $U^1_i = k'U^2_i$ and $U^1_{ij} = k''U^2_i U^2_j + k'U^2_{ij}, i,j = 1,2$.
Now let

$$A_i = \begin{vmatrix} U^i_{11} & U^i_{12} \\ U^i_{21} & U^i_{22} \end{vmatrix}, B_i = \begin{vmatrix} U^i_{11} & U^i_1 U^i_2 \\ U^i_{21} & (U^i_2)^2 \end{vmatrix}, C_i = \begin{vmatrix} (U^i_1)^2 & U^i_{12} \\ U^i_2 U^i_1 & U^i_{22} \end{vmatrix}, D_i = \begin{vmatrix} (U^i_1)^2 & U^i_1 U^i_2 \\ U^i_2 U^i_1 & (U^i_2)^2 \end{vmatrix}.$$

By computing, we can check

$$B_i + C_i = -\begin{vmatrix} U^i_{11} & U^i_{12} & U^i_1 \\ U^i_{21} & U^i_{22} & U^i_2 \\ U^i_1 & U^i_2 & 0 \end{vmatrix} = [U^i_2, -U^i_1] \begin{bmatrix} U^i_{11} & U^i_{12} \\ U^i_{21} & U^i_{22} \end{bmatrix} \begin{bmatrix} U^i_2 \\ -U^i_1 \end{bmatrix},$$

$D_i = 0$ and

$$\rho_i = \frac{A_i}{\{-[B_i + C_i]\}^{\frac{2}{3}}}.$$

Combining the results,

$$[B_2 + C_2] = [\tfrac{1}{k'}][B_1 + C_1] - \tfrac{k''}{k'}[U^2_2, -U^2_1] \begin{bmatrix} [U^2_1]^2 & U^2_1 U^2_2 \\ U^2_2 U^2_1 & [U^2_2]^2 \end{bmatrix} \begin{bmatrix} U^2_2 \\ -U^2_1 \end{bmatrix} = [\tfrac{1}{k'}][B_1 + C_1].$$

Using $U^1_{ij} = k''U^2_i U^2_j + k'U^2_{ij}, i,j = 1,2$ and $D_i = 0$, since the determinant is a linear function of each column, we get $A_1 = [k']^2 A_2 + k'k''[B_2 + C_2]$.
Solving for k'' and substituting the result we have obtained gives

$$k'' = \frac{-k'}{\{-[B_2 + C_2]\}^{\frac{1}{3}}}[\rho_1 - \rho_2].$$

Since, $k'/\{-[B_2 + C_2]\}^{\frac{1}{3}} > 0$, $k'' \leq (<)0$ if and only if $\rho_2 - \rho_1 \leq (<)0$. Q.E.D.

Moreover, ρ is invariant under linear transformations.

Proof Let $w = a + bU$. Then

$$\frac{\begin{vmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{vmatrix}}{\begin{vmatrix} w_{11} & w_{12} & w_1 \\ w_{21} & w_{22} & w_2 \\ w_1 & w_2 & 0 \end{vmatrix}^{\frac{2}{3}}} = \frac{b^2 \begin{vmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{vmatrix}}{[b^3 \begin{vmatrix} U_{11} & U_{12} & U_1 \\ U_{21} & U_{22} & U_2 \\ U_1 & U_2 & 0 \end{vmatrix}]^{\frac{2}{3}}} = \frac{\begin{vmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{vmatrix}}{\begin{vmatrix} U_{11} & U_{12} & U_1 \\ U_{21} & U_{22} & U_2 \\ U_1 & U_2 & 0 \end{vmatrix}^{\frac{2}{3}}}. \text{ Q.E.D.}$$

It is possible to introduce other two dimensional risk aversion measures, and to provide the same justification for these measures as has been given for ρ. Specifically, for any $z \in \Omega_1$ and $y \in \Omega_2$, let the directional risk aversion measure $r_{0,y}(z)$ be defined by

$$r_{0,y}(z) = -\frac{v''_{0,y}(z)}{v'_{0,y}(z)} = \frac{\sum_{i=1}^2 \sum_{j=1}^2 U_{ij}(zy) y_i y_j}{\sum_{i=1} 2U_i(zy) y_i},$$

where $v_{x,y}(z) = U(x + zy)$.
The measure of $r_{0,y}$ is simply the Arrow-Pratt measure of risk aversion of the one dimensional utility

function $v_{0,y}(z) = U(zy)$. Hence, $r_{0,y}$ is invariant under linear transformations. A justification for calling $r_{0,y}$ a two dimensional risk aversion measure is provided by the following corollary to propositions 4.3 and 4.4.

Corollary 4.6 *Suppose U_1 and U_2 both represent the preference ordering $\succeq$. Let y be any vector in Ω_2. Then U_1 is at least as risk averse as (more risk averse than) U_2 if and only if $r^1_{0,y}(z) \geq (>) r^2_{0,y}(z)$, for all z.*

4.1.1 An approach to compare the risk averseness of two utility functions with different ordinal preferences

The comparison can be restricted to a class of gambles, call it Γ, that excludes gambles for which the ordinal preferences disagree about the relative ranking of the prizes. As an example, consider a consumer, and let Γ be the class of gambles for which income is random. Assume that prices are fixed and that for each possible income level the consumer chooses consumption to maximize utility within the budget. For such gambles the risk averseness of all utility functions can be compared regardless of the preference they represent. The reason is that the consumers' actions will be determined by their utility of income function (the indirect utility function with fixed prices) which is a function of one variable.

Proposition 4.7 *U_1 is at least as risk averse as (more risk averse than) U_2 if and only if for every $p \in \Omega_n, \underline{U_1}(p, I)$ is at least as risk averse as (more risk averse than) $\underline{U_2}(p, I)$.*

Proof. The assumption that U_1 and U_2 represent the same preferences implies that $x(p, I)$ is the same for U_1 and U_2. The fact that $U_1 = k \circ U_2$, with $k' > 0$, implies
$\underline{U_1}(p, I) = U_1(x(p, I)) = k(U_2(x(p, I))) = k(\underline{U_2}(p, I))$.
If U_1 is at least as risk averse as U_2 then k is concave. By Proposition 4.1, $\underline{U_1}$ is more risk averse than $\underline{U_2}$. On the other hand, if $\underline{U_1}$ is more risk averse than $\underline{U_2}$ for some p then k is concave, again because of the Proposition 4.1, U_1 is at least as risk averse as U_2. Q.E.D.

4.1.2 An approach to the comparison of risk aversion by Yaari

Consider two mutually exclusive events, E and $\sim E$, such that the probability of E is $q \in (0, 1)$. A gamble is then a pair $(z_1, z_2) \in \Omega_n \times \Omega_n$ such that the player is awarded z_1 if E occurs and z_2 if $\sim E$ occurs.

Definition 4.8 *For any gamble (z_1, z_2), the acceptance set $A(z_1, z_2)$, associated with the utility function U, is the set of gambles which yield at least as high an expected value of U as (z_1, z_2). Formally,*

$$A(z_1, z_2) = \{(y_1, y_2) : qU(y_1) + (1 - q)U(y_2) \geq qU(z_1) + (1 - q)U(z_2)\}.$$

We can prove that the definition of risk aversion using the acceptance set $A^i(z, z)$ corresponding to U^i is equivalent to the Arrow-Pratt definition of risk aversion in the one-dimensional utility case.

Definition 4.9 *U_1 is [Yaari] at least as risk averse as U_2 $[U_1 Y U_2]$ if for all $z \in \Omega_n, A^2(z, z) \supseteq A^1(z, z)$. U_1 is [Yaari] more risk averse than U_2 $[U_1 V U_2]$ if for all $z \in \Omega_n, A^2(z, z) \supset A^1(z, z)$.*

From a geometrical point of view, if U_1 is more risk averse than U_2 in Yaari's sense, the hyperplane supporting the set $A^2(z, z)$ at (z, z) also supports the set $A^1(z, z)$ at (z, z). Thus if the risk averseness of U_1 and of U_2 are to be compared at (z, z) using Yaari's approach, the supporting hyperplane to the sets $A^1(z, z)$ and $A^2(z, z)$ must be the same.

Proposition 4.10 *If U_1 is at least as risk averse as U_2 in Yaari's sense, then U_1 and U_2 must represent the same ordinal preferences.*

Proof. Suppose U_1 and U_2 represent difference preference orderings. Then there exist x and y such that $U_1(x) > U_1(y)$ but $U_2(x) < U_2(y)$. Then $(x, y) \in A^1(y, y)$ but $(x, y) \notin A^2(y, y)$. Q.E.D.

The following proposition shows the equivalence of the Yaari definition and the definition 4.2 (which was introduced by Kihlstrom and Mirman [17]). Recall that the certainty equivalent of a gamble (z_1, z_2) is the certain payoff z which yields the same expected utility as the gamble. Note that in the Yaari framework the certainty equivalent for a particular gamble (z_1, z_2) is the point (z, z) on the frontier of the acceptance set $A(z_1, z_2)$.

Proposition 4.11 *Suppose U_1 and U_2 represent the preference $\succeq$. U_1 is at least as risk averse as U_2 (in Kihlstrom and Mirman's sense) if and only if $U_1 Y U_2$ (risk aversion in Yaari's sense).*

Proof. First suppose U_1 is at least as risk averse as U_2, and suppose $(y_1, y_2) \in A^1(z, z)$.
Then $k(qU_2(y_1) + (1-q)U_2(y_2)) \geq qk(U_2(y_1)) + (1-q)k(U_2(y_2)) \geq k(U_2(z))$.
Since k is monotonically increasing this inequality implies $(y_1, y_2) \in A^2(z, z)$.
Now suppose $U_1 Y U_2$. Then, by Proposition 4.10, U_1 and U_2 represent the same preferences, hence $U_1 = k \circ U_2, k' > 0$. Suppose k is not concave over the range of values taken by U_2. Then there exist y_1 and y_2 such that $k(qU_2(y_1) + (1-q)U_2(y_2)) < qk(U_2(y_1)) + (1-q)k(U_2(y_2))$. Let z be the certainty equivalent to (y_1, y_2) for U_1. Then $(y_1, y_2) \in A^1(z, z)$. However, $k(U_2(z)) = qk(U_2(y_1)) + (1-q)k(U_2(y_2)) > k(qU_2(y_1) + (1-q)U_2(y_2))$, which implies that $(y_1, y_2) \notin A^2(z, z)$, a contradiction. Q.E.D.

4.2 Risk independence and multi-attribute utility functions [16]

For multi-attribute utility functions, we can define conditional risk aversion functions as the same likes of the Arrow-Pratt risk aversion $r(x)$. More specifically, consider the utility function $U(x_1, ..., x_n)$ for attributes $X = X_1 \times X_2 \times ... \times X_n$. Let's denote $X_1 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_n$ as $X_{\bar{i}}$.

Definition 4.12 *The conditional risk aversion for X_i, which is denoted by $r_i(x)$, is defined by*

$$r_i(x) = -\frac{U_i''(x)}{U_i'(x)}, \tag{4.1}$$

where $U_i'(x)$ and $U_i''(x)$ are the first and second partial derivatives of U with respect to x_i. We have assumed that U is increasing in each X_i and the first and second partial derivatives exist and are continuous.

Definition 4.13 *X_i is risk independent of $X_{\bar{i}}$ if $r_i(x)$ does not depend on $x_{\bar{i}}$.*

In other words, X_i is risk independent of other attributes if the riskiness (as measured by r_i) of lotteries involving only uncertain amounts of X_i does not depend on the fixed amounts of the other attributes. That is, if all of the risk is associated with only one attribute and the other attributes are all fixed, the decision-maker's attitudes toward the risk will depend only on that attribute involving the risk.

Definition 4.14 *Let $x_{\bar{i}}$ represent any amount of $X_{\bar{i}}$ and let $x_{\bar{i}}^0$ be a specific amount. Then we can define the conditional utility function for X_i given $x_{\bar{i}} = x_{\bar{i}}^0$ to mean any positive linear transformation of $U(x_i, x_{\bar{i}}^0)$.*

Lemma 4.15 *If X_i is risk independent of $X_{\bar{i}}$, then*

$$U(x_i, x_{\bar{i}}) = f(x_{\bar{i}})U(x_i, x_{\bar{i}}^0) + g(x_{\bar{i}}),$$

where $f(x_{\bar{i}}) > 0$.

Proof. Given $r_i(x_i, x_{\bar{i}}) = r_i(x_i, x_{\bar{i}}^0)$, it follows from equation (4.1) that
$\frac{\partial}{\partial x_i}[\log U_i'(x_i, x_{\bar{i}})] = \frac{\partial}{\partial x_i}[\log U_i'(x_i, x_{\bar{i}}^0)]$, so by partial integration and exponentiation,
$U_i'(x_i, x_{\bar{i}})e^{a(x_{\bar{i}})} = U_i'(x_i, x_{\bar{i}}^0)e^b$, where $a(x_{\bar{i}})$ and b are integration constants.
Integrating again, $U(x_i, x_{\bar{i}})e^{a(x_{\bar{i}})} + c(x_{\bar{i}}) = U(x_i, x_{\bar{i}}^0)e^b + d$.
After arranging, and putting $f(x_{\bar{i}}) = e^{b-a(x_{\bar{i}})}$ and $g(x_{\bar{i}}) = [d - c(x_{\bar{i}})] \times e^{-a(x_{\bar{i}})}$, we obtain the desired result. Q.E.D.

The lemma becomes almost obvious when we consider that $r_i(x)$ specifies the conditional utility function for X_i uniquely up to positive transformations and that $r_i(x)$ does not depend on $x_{\bar{i}}$.

4.2.1 Utility functions and risk independence

In this section we derive the functional form of a utility function with two attributes given that each attribute is risk independent of the other.

Theorem 4.16 *Assume that X is risk independent of Y and Y is risk independent of X. Let $U(x, y)$ be the utility function for attributes X and Y, and U is increasing, twice continuously differentiable in each attribute then $U(x, y)$ can be expressed by*

$$U(x, y) = U(x, y_0) + U(x_0, y) + kU(x, y_0)U(x_0, y),$$

where k is an empirically evaluated constant and $U(x, y_0)$ and $U(x_0, y)$ are consistently scaled conditional utility functions.

Proof. For reference, let us define the origin of $U(x, y)$ by $U(x_0, y_0) = 0$. Since X is risk independent of Y, from Lemma 4.15, we know $U(x, y) = f_1(y)U(x, y_0) + g_1(y) \quad (\star)$.
Similarly, Y is risk independent of X, so $U(x, y) = f_2(x)U(x_0, y) + g_2(x) \quad (\star\star)$.
Evaluating $(\star)$ at $x = x_0$ gives $U(x_0, y) = g_1(y)$ and $(\star\star)$ at $y = y_0$ gives $U(x, y_0) = g_2(x)$.
Substituting the above equations back into $(\star)$ and $(\star\star)$ correspondingly, we obtain:
$f_1(y)U(x, y_0) + U(x_0, y) = f_2(x)U(x_0, y) + U(x, y_0)$,
which, after rearranging, is

$$\frac{f_1(y) - 1}{U(x_0, y)} = \frac{f_2(x) - 1}{U(x, y_0)}, x \neq x_0, y \neq y_0.$$

In the above equation, a function of x is equal to a function of y; therefore, they both must equal a constant. We have

$$f_2(x) = kU(x, y_0) + 1.$$

Substituting $f_2(x) = kU(x, y_0) + 1$, $U(x, y_0) = g_2(x)$ into $(\star\star)$, gives
$U(x, y) = U(x, y_0) + U(x_0, y) + kU(x, y_0)U(x_0, y)$. Q.E.D.

The above theorem simplifies the assessment of $U(x, y)$ provided the requisite of risk independent assumptions hold. The assessment of two attribute utility function is reduced to assessing two one-attribute conditional utility functions.

4.2.2 Conditional risk premium

Consider the lottery represented by $(\widetilde{x}_i, x_{\bar{i}})$, where $\sim$ represents a random outcome, and let $p_i(x_i)$ represent the probability density function describing this outcome.

Definition 4.17 *The conditional certainty equivalent for $\widetilde{x}_i$ given $x_{\bar{i}}$ is defined as the amount of X_i, call it $\widehat{x}_i$, such that the decision maker is indifferent between $(\widehat{x}_i, x_{\bar{i}})$ and $(\widetilde{x}_i, x_{\bar{i}})$. The conditional risk premium π_i for this lottery is defined as the amount such that the decision maker is indifferent between $(\overline{x_i} - \pi_i, x_{\bar{i}})$ and $(\widetilde{x}_i, x_{\bar{i}})$, where $\overline{x_i}$ is the expected value of $\widetilde{x}_i$.*

It should be clear that $\pi_i = \overline{x_i} - \widehat{x}_i$. In general, there is no reason why the conditional certainty equivalent and conditional risk premium for $\widetilde{x}_i$ would not depend on $x_{\bar{i}}$. However, from Lemma 4.15, it follows that when X_i is risk independent of $X_{\bar{i}}$, the conditional risk premium and conditional certainty equivalent for $\widetilde{x}_i$ will not depend on $x_{\bar{i}}$.
To be specific, consider the lottery $(\widetilde{x}, \widetilde{y})$, where we have verified that X and Y are risk independent of each other. We can now calculate the expected utility of $(\widetilde{x}, \widetilde{y})$ using Theorem 4.16, to find
$E[U(\widetilde{x}, \widetilde{y})] = \int\limits_x \int\limits_y [U(x, y_0) + U(x_0, y) + kU(x, y_0)U(x_0, y)]p(x, y)dxdy =$
$= E[U(\widetilde{x}, y_0)] + E[U(x_0, \widetilde{y})] + kE[U(\widetilde{x}, y_0)U(x_0, \widetilde{y})]$,
where $p(x, y)$ is the joint probability density function for $(\widetilde{x}, \widetilde{y})$.
When X and Y are probabilistically independent, the above equation becomes
$E[U(\widetilde{x}, \widetilde{y})] = E[U(\widetilde{x}, y_0)] + E[U(x_0, \widetilde{y})] + kE[U(\widetilde{x}, y_0)]E[U(x_0, \widetilde{y})]$.
But since X and Y are risk independent, we can reduce it to
$E[U(\widetilde{x}, \widetilde{y})] = U(\widehat{x}, y_0) + U(x_0, \widehat{y}) + kU(\widehat{x}, y_0)U(x_0, \widehat{y})$,
where $\widehat{x}$ and $\widehat{y}$ are respectively the conditional certainty equivalents for $\widetilde{x}$ and $\widetilde{y}$.
It follows from here that

$$E[U(\widetilde{x}, \widetilde{y})] = U(\widehat{x}, \widehat{y}) = U(\overline{x} - \pi_x, \overline{y} - \pi_y),$$

where π_x and π_y are conditional risk premia.

4.3 A matrix measure (extension of Pratt measure of local risk aversion) [7]

4.3.1 Multivariate risk premia

Let the risk faced by an individual be denoted by Z, a random vector taking values in E^n, the n-dimensional Euclidean space. An element $x = (x_1, ..., x_n)$ is a commodity vector, while U is the von Neumann-Morgenstern utility function consistent with the individual's preferences.
We assume that U is strictly increasing in each commodity and that $EU(x + Z)$ is finite.

Definition 4.18 *A family of risk premium functions* $\pi(x;Z)$*: For a given risk* Z *a vector* $\pi = (\pi_1, ..., \pi_n)' \in E^n$ *is assigned to each commodity vector* x*. If* $E(Z) = 0$*, the vector* π *must satisfy*

$$U(x - \pi) = EU(x + Z). \tag{4.2}$$

Pratt [28] noted that π is unique in the univariate case; in the multivariate case a vector π exists which satisfies equation (4.2) but uniqueness does not hold. A simple example of this is $U(x_1, x_2) = x_1 x_2$, where equation (4.2) is satisfied if $\pi_1\pi_2 - \pi_1 x_2 - \pi_2 x_1 = \sigma_{12}$. Clearly, additivity of u and the covariance structure of Z are of critical concern when examining risk premia in the multivariate setting.

4.3.2 Multivariate absolute risk aversion locally

Definition 4.19 *As we have seen the univariate absolute risk aversion, in an analogous way we can extend the result. The absolute risk aversion matrix can be defined as* $\mathbf{R} = [-U_{ij}/U_i]$*.*

Consider a Taylor series expansion of the function on each side of equation (4.2). First, for $U_{ij}(x) = \partial^2 U(x)/\partial x_i \partial x_j$ continuous we obtain

$$U(x - \pi) = U(x) - \sum_{i=1}^{n} \pi_i U_i(x) + \frac{1}{2} \sum_{i,j=1}^{n} \pi_i \pi_j U_{ij}(x - \theta\pi) \tag{4.3}$$

for some $0 \leq \theta \leq 1$. Second, if $\mathbf{A} = varZ = [\sigma_{ij}]$ exists
$EU(x + Z) = E[U(x) + \sum\limits_{i=1}^{n} Z_i U_i(x) + \frac{1}{2} \sum\limits_{i,j=1}^{n} Z_i Z_j U_{ij}(x) + \sum\limits_{i,j,k=1}^{n} O(Z_i Z_j Z_k)]$

$$EU(x + Z) = U(x) + \frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij} U_{ij}(x) + O(tr\mathbf{A}), \tag{4.4}$$

where $tr\mathbf{A} = \sum_{i=1}^{n} \sigma_{ii}$. By substituting (4.3) and (4.4) into equation (4.2), we have the approximate solution

$$u'\pi = \sum_{i=1}^{n} \pi_i U_i(x) = \frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij} U_{ij}(x) = -\frac{1}{2} tr\mathbf{UA}, \tag{4.5}$$

where $\mathbf{U} = [U_{ij}(x)]$ is the $n \times n$ Hessian matrix and $u = (U_i(x))$ is an n-vector. This equation restricts π to lie in an n-dimensional hyperplane.
Let's denote $\widehat{\pi}^0$ as a particular approximate risk premium vector. It has the form $\widehat{\pi} = -\frac{1}{2} u'^{-} tr\mathbf{UA}$, where for any $m \times n$ matrix $\mathbf{A}$, the $n \times m$ matrix $\mathbf{A}^{-}$ is its generalized inverse.
Therefore, $\widehat{\pi}_i^0 = -\frac{1}{2} \sum\limits_j U_i^{-1} U_{ij} \sigma_{ij}$, that is $\widehat{\pi}^0 = \frac{1}{2} dg\mathbf{RA}$, where

$$\mathbf{R} = [r^{ij}] = [-\frac{U_{ij}}{U_i}] = [\text{diag}u]^{-1}\mathbf{U}.$$

(For any $n \times n$ matrix $\mathbf{B}$, dg$\mathbf{B}$ is the n-vector of the main diagonal elements (b_{ii}), while for any n-vector c, diag$c$ is the $n \times n$ diagonal matrix with (i, i)th element c_i.)
It follows from Theorem 2.4.1 of Rao and Mitra [32] for the general representation of a generalized inverse that all approximate risk premium vectors have the form
$\widehat{\pi} = \widehat{\pi}^0 - \frac{1}{2} a\text{tr}\mathbf{UA} + \frac{1}{2} a' u (U_i^{-1} \mathbf{U_i'} \mathbf{A_i})$, where a is an arbitrary n-vector and $\mathbf{U_i}$, $\mathbf{A_i}$ are the i-th columns

of $\mathbf{U}$ and $\mathbf{A}$.
Consider risk Z with positive variance on only one variable, say Z_i. If the marginal risk premia π_j are to be zero for all marginal variables $Z_j, j \neq i$ and equal to the Arrow-Pratt quantity $-\frac{1}{2}(U_{ii}/U_i)\sigma_i^2$ for $i = j$, then $\widehat{\pi} = \widehat{\pi}^0$. Therefore, it appears appropriate to call $\mathbf{R} = [-U_{ij}/U_i]$ the absolute risk aversion matrix.

Example 4.20 *Consider the following family of utility functions with constants θ_1 and θ_2: $U(x_1, x_2) = -\theta_1[e^{-x_1} + e^{-x^2}] - \theta_2 e^{-x_1 - x_2}$. One can compute $r^{ii}(x) = 1$ and $r^{ij}(x) = \theta_2/(\theta_1 e^{x_j} + \theta_2)$ for $i \neq j$. Suppose the current fortune is $(\mu_1, \mu_2)'$ and a gamble X is available which will leave the fortune at a normally distributed random bivariate level with mean vector $(\mu_1, \mu_2)'$ and covariance matrix $\mathbf{A}$. Then in the special case $\theta_1 = \theta_2 = 1$,*

$$\widehat{\pi}^0 = \frac{1}{2}(\sigma_{11} + \sigma_{12}/(e^{\mu_2} + 1), \sigma_{22} + \sigma_{12}/(e^{\mu_1} + 1))'.$$

The approximate risk premium depends on the present fortune, just as the actual premium does.

Lemma 4.21 *The system of partial differential equations, $f_j(x) = a^{(j)}(x_j), j = 1, .., n$, has the general solution $f(x) = \sum_{j=1}^n \int a^{(j)}(x_j)dx_j + c$, where c is an arbitrary constant.*

Theorem 4.22 *The absolute risk aversion matrix $\mathbf{R}$ is diagonal if and only if the utility function is additive, $U(x) = \sum_{j=1}^n h^{(j)}(x_j)$. If $\mathbf{R}$ is diagonal, the commodities are mutually risk independent.*

Proof. Sufficiency is immediate. To show necessity, suppose $i \neq j$. Then $r^{ij}(x) = 0$ implies $\partial \log U_i(x)/\partial x_j = 0$ and, hence, $U_i(x)$, and, consequently, $U_{ii}(x)$ is not a function of x_j for $i \neq j$. Therefore, the commodities are mutually risk independent and applying Lemma 4.21 we obtain that U is additive. Q.E.D.

4.3.3 Positive risk premium

In this section we examine the relationship between concavity of the multivariate utility function and positivity of a risk premium vector, especially in its approximate form.

Theorem 4.23 *Let Z be an n-dimensional random vector with expectation $EZ = 0$. If there exists a nonnegative risk premium vector π for all two point gambles Z, then U is concave. Let U be a concave utility function on E^n. Then there exists a non-negative n-vector π such that equation (4.2) is satisfied.*

Proof. Let π be a nonnegative risk premium vector with respect to a two point gamble Z. Since U is a utility function, $U(x - \pi) \leq U(x)$ whenever π is nonnegative. But by definition of π, $U(x - \pi) = EU(x+Z)$. Therefore, $EU(x+Z) \leq U(x)$, which, since Z is an arbitrary two point gamble, guarantees that U is concave. The proof of the second statement follows easily by using the continuity assured by concavity on considering $\pi = t\pi^0$ for varying $t > 0$ and any fixed nonnegative $\pi^0 \neq 0$. Q.E.D.

Theorem 4.24 *If there is any approximate risk premium vector which is nonnegative regardless of the risk, then U is concave. If U is concave, there is a positive approximate risk premium vector.*

Proof. If there is any risk premium vector π which is nonnegative regardless if the risk given by $\mathbf{A}$ then for all $\mathbf{A}$, by equation (4.5), $0 < u'\pi = \frac{1}{2}tr\mathbf{UA}$. In particular it holds for all $\mathbf{A}$ of the rank one form tt'. Therefore, $0 < -\frac{1}{2}tr\mathbf{U}tt' = \frac{1}{2}t'\mathbf{U}t$ for all $t \neq 0$. Then the Hessian matrix $\mathbf{U}$ is negative definite, hence U is concave.
Conversely, if U is concave, then $\mathbf{U}$ is negative definite. Any $\mathbf{A}$ may be written in its spectral decomposition form as $\sum_{i=1}^{r} \lambda_i p_i p_i'$ where the rank of $\mathbf{A}$ is $r \leq n$ and $\lambda_i > 0, i = 1, .., r$. Then $tr\mathbf{UA} = \sum_{i=1}^{r} \lambda_i tr\mathbf{U}p_i p_i' = \sum_{i=1}^{r} \lambda_i p_i \mathbf{U} p_i < 0$. Hence, $u'\pi > 0$ and so π may be chosen to be positive. Q.E.D.

4.3.4 Constant and proportional multivariate risk aversion

Theorem 4.25 *The absolute risk aversion matrix* $\mathbf{R}$ *does not depend on the fortune* x *if and only if the utility function* U *is equivalent to*

$$U(x) = \sum_{i=1}^{n} [\alpha_i x_i \beta_i exp(-\mu_i x_i)] + exp(-\sum_{i=1}^{n} \gamma_i x_i), \tag{4.6}$$

where only one of the real coefficient α_i, β_i*, and* γ_i *can be nonzero for each* $i, i = 1, .., n$.

For the proof, see [7, page 900]. Moreover, we can verify that a utility function of the form (4.6) provides risk premia which are constant over x not only locally, but also globally. Also the approximate risk premia will equal the actual risk premia if and only if the risk Z has a multivariate normal distribution.

Definition 4.26 *We will say that an individual has proportional multivariate local risk aversion if* $x_j r^{ij}(x)$ *does not depend on* x *for* $i, j = 1, ..., n$ *and will denote by* $\mathbf{P}$ *the proportional risk aversion matrix,* $[x_j r^{ij}(x)]$.

Theorem 4.27 *Let* $x > 0$. *The proportional risk aversion matrix* $\mathbf{P}$ *is constant if and only if the utility function* U *is equivalent to*

$$U(x) = \sum_{i \notin I} a_i x_i^{1-r^{ii}} + \sum_{i \in I} b_i \log x_i,$$

where $I = \{i : r^{ii} = 1\}$ *and* a_i, b_i *are positive real numbers.*

For the proof, see [7, page 902].

4.4 Extension of the Arrow measure of risk aversion [23]

In this part, we present the generalization of Arrow's univariate gamble to the multivariate case. Even for the same preference ordering, the multivariate risk premium is not unique, which complicates the comparison of the risk premia of different individuals. In an analogous way to Arrow's, the probability premium is introduced. The advantage of the probability premium approach is that it provides a unique solution and, therefore, an unambiguous criterion for the comparison of the risk aversion.

Definition 4.28 *A single vector risk z is defined as*

$$z = \begin{cases} h \text{ with probability } p \\ -h \text{ with probability } 1-p \end{cases}$$

where $h' = (h_1, .., h_n) \in R^n$ is some vector.

We assume that an individual with initial consumption x and with twice continuously differentiable utility function $U(x)$ (assumed to be increasing in each argument) is confronted with a single vector z. The vector h can be chosen as long as the following inequality holds:

$$U(x+h) > U(x) > U(x-h). \tag{4.7}$$

Condition (4.7) ensures that receiving vector h is preferred to remaining with the initial x, which is preferred to losing the vector h. Due to Jensen's inequality, any risk averter will reject the gamble z if $p = 1/2$, i.e., if z is a fair gamble. As in Arrow's analysis in the univariate case, we can find probability p which the individual is indifferent between receiving the bet z and remaining with the initial wealth x.

$$U(x) = pU(x+h) + (1-p)U(x-h). \tag{4.8}$$

If the individual is risk averse, then $p > \frac{1}{2}$.
The following theorem is a multivariate generalization of the univariate result of Arrow. The theorem proves the relationship between the probability "premium"(given by Arrow, Proposition 3.3) and the concept of being more risk averse.

Theorem 4.29 *Let U_1 and U_2 be two utility functions, where U_1 is a monotonic transformation of U_2. Let h be some single vector risk and p_1, p_2 be probabilities satisfying equation (4.8) associated with U_1 and U_2. The following two conditions are equivalent:*

- *$p_1 \geq p_2$ for every x and every h, satisfying the inequalities in (4.7).*
- *U_1 is more risk averse than U_2 according to Kihlstrom and Mirman's definition.*

Therefore, the larger p the more risk averse is the individual.

For the proof, see [23, page 894].

Relation to the work of Duncan

The gamble defined in this section is a special case of the general risk introduced by Duncan, just as in the univariate case Arrow's gamble is a special case of the general risk introduced by Pratt. However, while in the univariate case both the Arrow's premium defined as probability (in equation 4.8) and Pratt's premia (defined in equation 3.10 or Definition 3.12) depend solely on the absolute risk aversion, the relationship between Duncan's result and those of this section is not so obvious.

Theorem 4.30 *Let U_1 and U_2 be two utility functions representing the same preferences. Let z be some risk and π_1, π_2 be the risk premia, such that $U(x + Ez - \pi) = EU(x + z)$, associated with the risk, corresponding to U_1 and U_2, respectively. The following two conditions are equivalent:*

- *U_1 is more risk averse than U_2 according to Kihlstrom and Mirman's definition.*
- *$U_1(x - \pi_2) \geq U_1(x - \pi_1)$, for all x and all risks z.*

For the proof, see [23, page 896].

4.5 Notes and comments about risk premium [26]

Utility is assumed to be twice continuously differentiable and monotonically increasing all over. $\pi_U(x, \widetilde{z})$ is not uniquely determined from equation $U(x + E\widetilde{z} - \pi) = E\{U(x + \widetilde{z})\}$. Using the fact that the solution to (3.10) has $n - 1$ degrees of freedom, we can introduce another way of comparing risk premia.

Definition 4.31 *π^1 is higher than π^2, $\pi^1 \succ \pi^2$, if $\pi_i^1 > \pi_i^2$, whenever $\pi_j^1 = \pi_j^2$, for the $n - 1$ values $i \neq j$. π^1 is equivalent to π^2, $\pi^1 \sim \pi^2$, if $\pi_i^1 = \pi_i^2$ whenever $\pi_j^1 = \pi_j^2$ for all $i \neq j$.*

Proposition 4.32 *$\pi^1 \succ \pi^2$ if and only if $a_2 > a_1$ and $\pi^1 \sim \pi^2$ if and only if $a_1 = a_2$, where $\pi^i = \pi_U(x, \widetilde{z}_i)$ and $a_i = E\{U(x + \widetilde{z}_i)\}$*

The proof is a direct outcome of the definition of the relations "$\sim$"and "$\succ$"and of the monotonicity of U.

Proposition 4.33 *For small risks, $\pi^1 \succ \pi^2$ if and only if*

$$\sum U_k(\pi_k^1 - \pi_k^2) > 0,$$

and $\pi^1 \sim \pi^2$ if and only if $\sum U_k(\pi_k^1 - \pi_k^2) = 0$, where π_k^i is the k-th component of π^i, U_k is the partial derivative of U with respect to x_k.

Proof. By equation (3.10), π^i is a solution of $a_i = U(x - \pi^i)$. For small risks $U(x - \pi^i) \approx U(x) - \sum U_k \pi_k^i$ and therefore $a_2 - a_1 = \sum U_k(\pi_k^1 - \pi_k^2)$. By applying Proposition 4.32, the proof is complete. Q.E.D.

Let's restrict to the same preference ordering.

Proposition 4.34 *Suppose U and V are two utility functions defined on the same preference field, then $\pi_V \succ \pi_U$ if and only if V is more risk averse (in KM's sense) than U. $\pi_V \sim \pi_U$ if and only if V is a linear transformation of U.*

Proof. Since U and V are defined on the same preference field then some h exists such that $V = h \circ U, h' > 0$. According to KM's definition, V is more risk averse than U if h is strictly concave, i.e., $h'' < 0$. By definition $U(x - \pi_U) = E\{U(x + \widetilde{z})\}$ and $V(x - \pi_V) = E\{V(x + \widetilde{z})\}$ but since $V = h \circ U$, $U(x - \pi_V) = h^{-1}(E\{V(x + \widetilde{z})\})$. Due to Jensen's inequality, $E\widetilde{y} < h^{-1}(Eh(\widetilde{y}))$ if and only if h is strictly concave. Using Proposition 4.32, we can see that $\pi_V \succ \pi_U$ if and only if V is more risk averse (in KM's sense) than U. Again using Proposition 4.32 and the fact that $E\widetilde{y} = h^{-1}(Eh(\widetilde{y}))$ if and only if h is linear, we can prove the other part. Q.E.D.

Proposition 4.35 *Suppose U and V are two utility functions defined on the same preference field, i.e., $V = h \circ U, \quad h' > 0$, then, for small risks, $\pi_V \succ \pi_U$ if and only if $\sum U_k(\pi_{Vk} - \pi_{Uk}) > 0$ and $\pi_V \sim \pi_U$ if and only if $\sum U_K(\pi_{Vk} - \pi_{Uk}) = 0$.*

Proof. For small risks, $U(x - \pi_U) \approx U(x) - \sum U_k \pi_{Uk}$ and also $EU(x + \widetilde{z}) \approx U(x) + \frac{1}{2}E(\widetilde{z}' H_U \widetilde{z})$, where H_U is the hessian matrix of U. From these equations , we can derive $\sum U_k \pi_{Uk} = -\frac{1}{2}E(\widetilde{z}' H_U \widetilde{z})$, and similarly $\sum V_k \pi_{Vk} = -\frac{1}{2}E(\widetilde{z}' H_V \widetilde{z})$.

Since $V_k = h'U_k$ and $V_{rs} = h'U_{rs} + h''U_rU_s$, we have
$\sum V_k\pi_{Vk} = h'\sum U_k\pi_{Vk} = -\frac{1}{2}h'E(\widetilde{z}'H_U\widetilde{z}) - \frac{1}{2}h''E(\widetilde{z}'G_U\widetilde{z})$, where $G_U = \alpha\alpha'$, and α' is the vector of the n partial derivatives of U.
Dividing the equation by h' and simplifying we get:

$$\sum U_k(\pi_{Vk} - \pi_{Uk}) = -\frac{1}{2}\frac{h''}{h'}E(\widetilde{z}'G_U\widetilde{z}).$$

Since G_U is a positive semi-definite matrix, then $E(\widetilde{z}'G_U\widetilde{z})$ is always positive. Note also that $h'' = 0$ if and only if h is linear. Q.E.D.

The last two propositions suggest $-h''/h'$ as a relative index of risk aversion with many commodities.

Proposition 4.36 *Suppose V_1 and V_2 are utility functions defined on the same preference field as U, i.e., $V_1 = h_1 \circ U$, $h_1' > 0$ and $V_2 = h_2 \circ U$, $h_2' > 0$. V_1 is more risk averse than V_2 (in KM's sense) if and only of $r_1 > r_2$ where $r_i = -h_i''/h_i'$.*

Proof. Since V_1 and V_2 are defined on the same preference field, then some T exists such that $V_1 = T \circ V_2, T' > 0$. Obviously $T = h_1 \circ h_2^{-1}$. But $T' = h'_1/h'_2$, and $T'' = (h''_1h'_2 - h''_2h'_1)/(h'_2)^2 = (r_2 - r_1)(h'_1/h'_2)$, or $r_1 - r_2 = -(T''/T')$. So $r_1 > r_2$ if and only if T is strictly concave. Q.E.D.

4.6 Risk aversion using indirect utility [15]

Let $\psi(y, p_1, .., p_{n-1})$ be the indirect utility function, with $y = Y/p_n, p_i = P_i/p_n, i = 1, .., n-1$, where Y denotes money income and P_i is the money price of the i-th commodity. Let z be an n-dimensional random vector, where we will denote z_1 is the deviation of y from an arbitrary value y^0, z_{i+1} is the random deviation of p_i from an arbitrary value p_i^0 and in general z_i and $z_j, i \neq j, i, j = 1, .., n$ need not be independent. Let $\widetilde{z}$ denote the joint probability distribution of z.

Definition 4.37 *(Risk premium by Karni) Let the risk premium function, $\pi(y^0, p_1^0, ..., p_{n-1}^0, \widetilde{z})$ be defined as the income compensation such that the decision maker is indifferent between receiving a combined income and relative risk $\widetilde{z}$, and a non random income and relative price vector:*

$$\psi(y^0 + E\{z_1\} - \pi, p_1^0 + E\{z_2\}, .., p_{n-1}^0 + E\{z_n\}) = E\{\psi(y^0 + z_1, p_1^0 + z_2, .., p_{n-1}^0 + z_n)\}. \quad (4.9)$$

We will confine our discussion to the case where the expectation on the right-hand side of (4.9) exists and is finite. The existence and uniqueness of π follow from the fact that ψ is a continuous and decreasing function, which for any given income y and any positive relative price vector $(p_1, ..., p_{n-1})$ ranges over all possible values of ψ.
From (4.9), it follows that for any n-dimensional real vector $(\mu_1, .., \mu_n) = \mu$,

$$\pi(y^0, p_1^0, ..., p_{n-1}^0, \widetilde{z}) = \pi(y^0 + \mu_1, p_1^0 + \mu_2, ..., p_{n-1}^0 + \mu_n, \widetilde{z}'),$$

where $\widetilde{z}'$ is the joint probability distribution of $z - \mu$. Therefore, without loss of generality, we can consider neutral risk, $E\{z\} = 0$.
From now on, in this section, we will denote z_1 the random deviation of y from its mean value $\overline{y}$, and z_{i+1} the random variation of p_i from its mean value $\overline{p_i}$, $i = 1, ..., n-1$. Also, p will denote the $n-1$ dimensional vector $(p_1, ..., p_{n-1})$, $\overline{p} = (\overline{p_1}, ..., \overline{p_{n-1}})$ and $\widetilde{z}$ denote the joint probability distribution of $z = (z_1, .., z_n)$.

4.6.1 Local risk aversion

We would like to obtain measures of decision maker's aversion to small risks, such that $Pr\{(z_1, ..., z_n) \in B\} = 1$, where B is an n-dimensional ball of center $(\overline{y}, \overline{p_1}, ..., \overline{p_{n-1}})$ with radius ϵ. Let ψ_1 and ψ_{ij} denote the first and second partial derivatives of ψ with respect to its i and j arguments respectively. Consider the Taylor expansions of the functions on both sides of (4.9):
$\psi(\overline{y} - \pi, \overline{p}) = \psi(\overline{y}, \overline{p}) - \pi\psi_1(\overline{y}, \overline{p}) + O(\pi^2)$ and, if $var[z] = V = [\sigma_{ij}]$ exists,
$E\{\psi[(\overline{y}, \overline{p}) + z]\} = \psi(\overline{y}, \overline{p}) + \frac{1}{2}\sum\sum_{i,j} \sigma_{ij}\psi_{ij}(\overline{y}, \overline{p}) + o(trV)$,
where $trV = \sum_{i=1}^{n} \sigma_{ii}$. Setting these expressions equal to one another, solving for π, we obtain an approximate solution:

$$\pi[(\overline{y}, \overline{p}), \widetilde{z}] \approx -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}\frac{\psi_{ij}}{\psi_1}(\overline{y}, \overline{p}). \tag{4.10}$$

Let

$$\mathbf{R} = [r_{ij}] = \left[\frac{-\psi_{ij}}{\psi_1}\right]. \tag{4.11}$$

$\mathbf{R}$ is the matrix measure of local risk aversion. The diagonal elements of $\mathbf{R}$, $-\frac{\psi_{jj}}{\psi_1}$, are proportional to the risk premium per unit of variance σ_{jj} when $z_i = 0$ with probability 1 for all $i \neq j$. The off-diagonal elements, $-\frac{\psi_{ij}}{\psi_1}$ with $i \neq j$, have the interpretation of the excess of risk premium, when z_i, z_j are random and $z_k = 0$ $(k \neq i, k \neq j)$ are non random, per unit of covariance (σ_{ij}). The following theorem points out a property of $\mathbf{R}$ in relation to π.

Theorem 4.38 *Let $z = (z_1, .., z_n)$ be a random vector in $z_1 \geq -y, z_i > -p_{i-1}$ $(i = 2, .., n)$, with expectations $E\{z\} = 0$. Let $\psi(y, p)$, $y \geq 0$, $p > 0$, be an indirect utility function, and let $\mathbf{R}$ be given by (4.11).Then the following conditions are equivalent:*

a) $\mathbf{R}$ is positive semi-definite [positive definite] for all $y \geq 0, p > 0$.

b) $\pi[(y, p), \widetilde{z}] \geq [>]0$ for all $y \geq 0, p > 0$ and all $\widetilde{z}$ with mean (y, p).

c) $\psi(y, p)$ is [strictly] concave.

Proof. To show that $a \Leftrightarrow c$ we note that ψ is [strictly] concave if and only if its Hessian H is negative [definite] semi definite for all $y \geq 0, p > 0$. But $\mathbf{R} = -H/\psi_1$ and $\psi_1 > 0$ by the properties of the indirect utility function. Hence, H is negative [definite] semi-definite, if and only if $\mathbf{R}$ is positive [definite] semi-definite.
To show that $b \Rightarrow c$, by the properties of the indirect utility function $\psi(y - \pi, p) \leq [<]\psi(y, \pi)$ for all $\pi \geq [>]0$. But, by definition, $\psi(y - \pi, p) = E\{\psi[(y, p) + z]\}$. Therefore, $E\{\psi[(y, p) + z]\} \leq [<]\psi(y, p)$ for any $\widetilde{z}$, with mean (y, p). Hence, by Jensen's inequality ψ is [strictly] concave.
Finally, $c \Rightarrow b$ follows directly from [strict] concavity of ψ and Jensen's inequality. Q.E.D.

4.6.2 Comparative risk aversion

Let χ and ψ be indirect utility functions with the same domain. Comparison of global attitudes toward multivariate risks, using the comparison of risk premia, is directional dependent in general. In the following definition, we are using the risk premium introduced by Karni (equation 4.9).

Definition 4.39 *An indirect utility function* χ *is globally [strictly] more risk averse than an indirect utility function* ψ *if and only if,* $\pi_\chi[(y,p),\widetilde{z}] \geq [>]\pi_\psi[(y,p),\widetilde{z}]$ *for all* $y \geq 0, p \in R^{n-1}_{++}$ *and every joint probability distribution* $\widetilde{z}$ *with mean* (y,p) *of n-dimensional vectors z such that* $E\{z\} = 0$*, and* $z_1 \geq -y, z_i > -p_{i-1}(i = 2,..,n)$*.*

The local interpretation is obtained if $\widetilde{z}$ represents small risks.

In the local case, the relationship between $\mathbf{R}_\chi, \mathbf{R}_\psi$, the respective measures of local risk aversion, and corresponding risk premia is given in the following theorem.

Theorem 4.40 *Let* $\mathbf{R}_\chi(y,p), \mathbf{R}_\psi(y,p), \pi_\chi[(y,p),\widetilde{z}]$ *and* $\pi_\psi[(y,p),\widetilde{z}]$ *be respectively, the matrix measures of local risk aversion and the risk premium (by Karni) functions corresponding to the indirect utility functions* χ *and* ψ*. Let z be an n-dimensional small random vector with joint probability distribution* $\widetilde{z}$*, with* $z_1 \geq -y^0, z_i > -p^0_{i-1}, i = 2,...,n$ *and expectation* $E\{z\} = 0$*. Then* $\pi_\chi[(y^0,p^0),\widetilde{z}] \geq [>]\pi_\psi[(y^0,p^0),\widetilde{z}]$*, for* $y^0 \geq 0, p^0 > 0$ *and* $\widetilde{z}$ *with mean* (y^0,p^0) *if and only if* $\mathbf{R}_\chi(y^0,p^0) - \mathbf{R}_\psi(y^0,p^0)$ *is positive [definite] semi definite.*

Outline of the proof. From (4.10) and (4.11),
$\pi_\chi[(y^0,p^0),\widetilde{z}] - \pi_\psi[(y^0,p^0),\widetilde{z}] = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}[r^\chi_{ij}(y^0,p^0) - r^\psi_{ij}(y^0,p^0)] =$
$= \frac{1}{2}tr\{V[\mathbf{R}_\chi(y^0,p^0) - \mathbf{R}_\psi(y^0,p^0)]\} \geq [>]0$, where the last inequality follows from the fact that the covariance matrix V is a real symmetric positive [definite] semi-definite matrix.
To prove necessity, choose V such that the correlation coefficients of (z_i, z_j) are all equal to one, which leads to contradiction. Q.E.D.

The following theorem states that if the local property pointed out in the above theorem holds everywhere, then χ is globally more risk averse than ψ using the risk premium definition by Karni. Since both utility functions are continuous and monotonic increasing in y, we define the p-inverse as follows: $t = \psi(y,p), \quad y = \psi^{-1}(t,p)$.

Theorem 4.41 *Let* $\mathbf{R}_\chi(y,p), \mathbf{R}_\psi(y,p), \pi_\chi[(y,p),\widetilde{z}]$ *and* $\pi_\psi[(y,p),\widetilde{z}]$ *be, respectively, the matrix measures of local risk aversion and the risk premium (by Karni) functions corresponding to the indirect utility functions* χ *and* ψ*. Let z be an n-dimensional random vector with a joint probability distribution* $\widetilde{z}$*, of mean* $(y^0, p^0 = (p^0_1,..,p^0_{n-1}))$ *on* $z_1 \geq -y^0, z_i > -p^0_{i-1}, i = 2,...,n$*, with expectation* $E\{z\} = 0$*. Then the following conditions are equivalent :*

- $\chi(\psi^{-1}(t,p),p) = \phi(t,p)$ *is [strictly] concave.*
- $\mathbf{R}_\chi(y,p) - \mathbf{R}_\psi(y,p)$ *is positive [definite] semi-definite for all* $y \geq 0, p > 0$*.*
- $\pi_\chi[(y,p),\widetilde{z}] \geq [>]\pi_\psi[(y,p),\widetilde{z}]$ *for all* $y \geq 0, p > 0$ *and all n-dimensional joint probability distributions* $\widetilde{z}$ *with mean (y,p).*

For the proof, see [15, page 1396].

4.7 Alternative representations and interpretations of the relative risk aversion [11]

U is assumed cardinal, twice continuously differentiable, strongly increasing and strongly quasi-concave on a convex open set $X \subset R_+^n$. By the strong quasi-concavity of U, we mean: $(-1)^i |U_{xx}^B|_i > 0, i = 1, .., n$, where $|U_{xx}^B|_i$ is the principal minor of order $(i+1)$ of the bordered Hessian matrix of U. Also, strongly increasing means $U_x = \partial U/\partial x >> 0$. The following Lagrangian function corresponds to utility maximization under given positive prices p and income y, using normalized prices $q = (1/y)p$:

$$L = U(x) - \mu(q^t x - 1).$$

The conditions for an optimum are given by

$$U_x = \mu q = \frac{\mu}{y} p; \quad q^t x = 1, \tag{4.12}$$

where $\mu \in R_{++}$, $U_x = (U_1, ..., U_n)^t = \partial U/\partial x$. Under certainty, the consumer is assumed to maximize the expected utility.
Using results on duality, preferences under certainty may be represented by the following five alternatives:

(i) $u = U(x)$, direct utility function;

(ii) $u = U[x(p, y)] = V(p, y)$ indirect homogeneous utility function;

(iii) $u = V(\frac{1}{y}p, 1) = W(\frac{1}{y}p) = W(q)$, indirect utility function;

(iv) $y = E(p, u)$, expenditure function;

(v) $1 = E(\frac{1}{y}p, u) = F(q, u)$, unit cost price frontier.

The assumptions on U imply the following restrictions on the other functions:

- (ii) V is derived from U by substitution of demand function $x(p, y)$ which is the solution to (4.12). V is strongly increasing in y, strongly decreasing and strongly quasi convex in p, and is zero-homogeneous in $(p, y) \in C \subset R_+^{n+1}$, where C is an open convex cone.

- (iii) W is strongly decreasing and strongly quasi-convex on an open convex set $Q(X) \subset R_+^n$. W is derived by using the zero-homogeneity of V.

- (iv) E is derived by using the implicit function theorem on V. E is positively homogeneous and concave in p, and strongly increasing and twice differentiable in $(p, u) \in C_n \times I$, where $C_n(C \cap R^n)$ is an open convex cone, and $I \subset R$ an open interval.

The Lagrangian μ is positive and interpreted as the utility valuation of income, since $\mu = yV_y$, where V_y is the marginal utility of income. The relative risk aversion $r(p, y) = \frac{-yV_{yy}(y)}{V_y(y)}$ is zero homogeneous in (p, y), so that it is a unit-free measure of local aversion to risk.
The following theorem represents alternative derivations of r, corresponding to alternative functional representations of preferences.

Theorem 4.42 *If x, p, y and u satisfy the optimality condition (4.12), then the following alternative representations of relative risk aversion r are equivalent:*

(1) $r(x) = \frac{U_x^t x |U_{xx}|}{\begin{vmatrix} 0 & U_x^t \\ U_x & U_{xx} \end{vmatrix}}$; *if* $|U_{xx}| \neq 0$ *then* $r(x) = -U_x^t x / U_x^t U_{xx}^{-1} U_x$,

where $U_{xx} = \partial U_x / \partial x = \{U_{ij}(x)\}$ *is the Hessian matrix of* U.

(2) $r(p, y) = -\frac{\partial \log V_y(y)}{\partial \log y} = 1 - \frac{\partial \log \mu(p,y)}{\partial \log y} = 2 + \frac{p^t V_{pp}(y) p}{V_p^t p} = 2 + \frac{\lambda V_{\lambda\lambda}(\lambda p, y)}{V_\lambda(\lambda p, y)}|_{\lambda=1}$;

(3) $r(q) = 2 + \frac{\lambda W_{\lambda\lambda}(\lambda q)}{W_\lambda(\lambda q)}|_{\lambda=1}$;

(4) $r(p, u) = 1 - \frac{\partial \mu(p,u)}{\partial u}$;

(5) $r(q, u) = -\frac{\partial \mu(q,u)}{\partial u}$.

For the proof, see [11, p.417]

These alternative expressions for r from equation (1) to (5) have some alternative economic interpretations.

1. From (2) in Theorem 4.42, $-r = \partial \log V_y(y) / \partial \log y = \widehat{\omega} = 1/\phi = y/k$, where $\widehat{\omega}$ is the elasticity of marginal utility of income, called by Frisch the "money flexibility", ϕ is Thiel's and k is Houthakker's "income flexibility". These seem to play an important role in demand analysis.

2. From (2) and (3),
$$2 - r = \frac{-\lambda V_{\lambda\lambda}(\lambda)}{V_\lambda(\lambda)}|_{\lambda=1} = \frac{-\lambda W_{\lambda\lambda}(\lambda)}{W_\lambda(\lambda)}|_{\lambda=1}$$
is seen to be a measure of "price risk aversion"with respect to the proportional variation in all prices (i.e., movements along the income-consumption path in the X-space) around the point $q^0 = (1/y^0)p^0$ (which corresponds to $\lambda = 1$), in analogy to the "income risk aversion"$r = -yV_{yy}(y)/V_y(y)$.

3. From $r(p, y) = 1 - \frac{\partial \log \mu(p,y)}{\partial \log y}$ and the interpretation given to μ, it follows that $1 - r = \partial \log \mu / \partial y$ is the elasticity of utility valuation of income with respect to money income y.

4. By (4), $1 - r = \partial \mu / \partial u$ is also the change in utility-valued income with respect to utility, as nominal prices are fixed and utility varies together with money income y.

5. Finally, by (5), $-r = \partial \mu(q, u) / \partial u$ is the change in utility-valued income with respect to utility, as the unit-cost frontier $F(q, u) = 1$ is shifting, with fixed "real"prices q.

Assuming maximization of expected utility Eu, the consumer is averse to all small risks in x, if and only if the direct utility function is concave at x; by Jensen's inequality, $EU(x + \delta x) < U(x)$ holds for all δx such that $E\delta x = 0$ and $x + \delta x$ in an open neighborhood of x, if and only if U is concave at x.

Theorem 4.43 *U is concave at x, if and only if $r(x) \geq 0$; if $r(x) > 0$, then U is strictly concave. (U is defined at the beginning of the section 4.7.)*

Proof. The assumption of strong quasi-concavity of U implies $(-1)^n|U^B_{xx}| > 0$, where U^B_{xx} is the bordered Hessian. Since $U^x_t x > 0$, r is nonnegative iff $(-1)^n|U_{xx}| \geq 0$. But if U is concave, then $(-1)^n|U_{xx}| \geq 0$, so $r(x) \geq 0$. If U is concave but not strongly concave at x, then $|U_{xx}| = 0$ and $r = 0$. If U is strongly concave, then $r(x) > 0$.
When $r > 0$, U is strongly concave and, therefore, strictly concave, iff U_{xx} is negative definite, or equivalently, iff the characteristic roots of U_{xx} are all negative: $0 > \lambda_1 > ... > \lambda_n$. By a known theorem [25], strong quasi-concavity implies that $(\lambda_2, ..., \lambda_n) < 0$. Thus $\lambda_1 < 0$ is sufficient for strong concavity. On the other hand, $|U_{xx}| = \lambda_1 ... \lambda_n$, and $(-1)^n|U_{xx}| > 0$ if $r > 0$. Therefore, $\lambda_1 < 0$. Q.E.D.

Theorem 4.44 *(i) If W is convex, or $V(p, y)$ is convex in p, then $r(q) = r(p, y) \leq 2$*

(ii) In general, however, $r \leq 2$ is not sufficient for convexity in prices unless $n = 1$, or preferences homothetic (monotone transformation of a homogeneous function).

(iii) W and V cannot be concave in prices, unless $n = 1$ and $r \geq 2$.

An economic interpretation of these propositions is:
(i) A necessary condition for risk loving with respect to price fluctuations is that income risk aversion is not too large.
(ii) This is also sufficient for a single commodity and for homothetic preferences.
(iii) The consumer is not risk averse with respect to all price fluctuations.

Proof of (i)**.** If W is convex, W_{qq} (or V_{pp}) is positive semi-definite, and $q^t W_{qq} q \geq 0$. Since $W^t_q q < 0$, the second term in equality $r(q) = 2 + \frac{q^t W_{qq} q}{W^t_q q}$ is non-positive, and $r(q) \leq 2$. (If $r < 0$, there is no risk loving with respect to income, as well.) Q.E.D.
For the proof (ii) and (iii), see [11, p.420]

4.8 Constant, increasing and decreasing risk aversion with many commodities [18]

In this section, we extend the theory of increasing, decreasing and constant absolute and relative risk aversion to multidimensional utility functions. There are several problems regarding the extension. Firstly, it is only possible to compare the risk aversion of those utility functions which represent the same ordinal preferences. In the theory of increasing Arrow-Pratt risk aversion, the risk aversion of one utility function at a specific wealth level is compared to the risk aversion of the same utility function at a different wealth level. Generalizing this to multi-dimension, it is necessary to compare the risk aversion of one utility function at a specific point to the risk aversion of the same utility function at a different point. This requires ordinal preferences represented by the utility function are the same at each of the two points. This is done by restricting to homothetic preferences, i.e. those preferences for which marginal rates of substitution remain constant along the rays through the origin. The second problem is the ambiguity in term "increasing".
For each homothetic utility function of many variables to which we would like to apply the theory, there must be some "base" utility function which can play a role analogous to the linear homogenous utility functions in the one-dimensional Arrow-Pratt theory. This problem is solved by using the least concave representation of the ordinal preferences.

Given $p \in R_n$ and $x \in R_n^+$, we define the hyperplane $H(p,x) = \{\overline{x} : \overline{x} \in R_n, p \cdot \overline{x} = p \cdot x\}$.
The hyperplane $H(p,x)$ supports $V(x)$ at x if $\overline{x} \in V(x)$ implies $p \cdot \overline{x} \geq p \cdot x$.
We now let $\sum(x) = \{p : H(p,x)$ supports $V(x)$ at $x\}$. We consider preference orderings $\succeq$ defined on R_n^+ with the following properties:

- Continuity. $\succeq$ is continuous if there exists a continuous real valued function u with domain R_n^+ which represents $\succeq$ in the sense that $u(x) \geq u(x')$ if and only if $x \succeq x'$.
- Convexity. $\succeq$ is convex if, for all $x \in R_n^+$, $V(x) = \{\overline{x} : \overline{x} \in R_n^+$ and $\overline{x} \succeq x\}$ is convex.
- Monotonicity. $\succeq$ is monotonic if, whenever $\overline{x_i} \geq x_i$ for all i and $\overline{x_i} > x_i$ for some i, then $\overline{x} = (\overline{x_1}, ..., \overline{x_n}) \succeq (x_1, ..., x_n) = x$.
- Homotheticity. $\succeq$ is homothetic if $\sum(x) = \sum(\lambda x)$ for all $x \in X$ and $\lambda > 0$.

Lemma 4.45 *If $\succeq$ is continuous, convex, monotonic and homothetic, $x \sim x'$ if and only if $\lambda x \sim \lambda x'$ for all $x, x' \in X$ and all $\lambda > 0$.*

Outline of the Proof. If $\lambda x \succ \lambda x'$ when $x \sim x'$, it would be possible to show that there is a vector $p \in \sum(\lambda x')$ which is not in $\sum(x')$. Q.E.D.

Let's call u^* a least concave representation of $\succeq$ if, whenever u is a concave representation of $\succeq$, $u = f \circ u^*$ where f is concave and increasing.

Proposition 4.46 *If $\succeq$ is continuous, convex, monotonic and homothetic, then there exists a homogeneous of degree one utility function u^* which is a least concave representation of $\succeq$.*

Proof. The continuity and monotonicity assumptions imply that for each $x \in R_n^+$ there exists $t \geq 0$ such that $x \sim (t, ..., t)$. We let $u^*(x) = t$. Monotonicity guarantees that $x \succeq x'$ if and only if $u^*(x) \geq u^*(x')$. Furthermore, Lemma 4.45 implies that u^* is homogeneous of degree one.
To prove that u^* is a least concave representation of $\succeq$, let v be a concave representation of $\succeq$. Then v can be written as $v = f \circ u^*$ where f is a monotonically increasing function. If $\overline{u}, \widetilde{u} \in u^*(R_n^+)$, there exists $x \in R_n^+$ and λ such that $u^*(x) = \overline{u}$ and $u^*(\lambda x) = \widetilde{u}$. For $\mu \in (0,1)$,
$f(\mu\overline{u} + (1-\mu)\widetilde{u}) = f(\mu u^*(x) + (1-\mu)u^*(\lambda x)) = f([\mu + (1-\mu)\lambda]u^*(x)) =$
$= f(u^*([\mu + (1-\mu)\lambda]x)) = f(u^*(\mu x + (1-\mu)\lambda x)) = v(\mu x + (1-\mu)\lambda x)$.
Since v is concave,
$v(\mu x + (1-\mu)\lambda x) \geq \mu v(x) + (1-\mu)v(\lambda x) = \mu f(u^*(x)) + (1-\mu)f(u^*(\lambda x)) = \mu f(\overline{u}) + (1-\mu)f(\widetilde{u})$.
Combining the above two, we have that f is concave. Q.E.D.

Notice that when $n = 1$, u^* is just a linear function of x and $u^*(0) = 0$.
If $a > 0$, the transformation au^* has the same properties of u^* mentioned in the above proposition. However, $au^* + b$, where $a, b > 0$, is a least concave representation of $\succeq$, but is not homogeneous of degree one.

Definition 4.47 *u is an increasing (decreasing, constant) absolute risk averse representation of $\succeq$ if $u(x) = h(u^*(x))$ and h is an increasing (decreasing, constant) absolute risk aversion function of a single real variable.*

Definition 4.48 *u is an increasing (decreasing, constant) relative risk averse representation of $\succeq$ if $u(x) = h(u^*(x))$ and h is an increasing (decreasing, constant) relative risk aversion function of a single real variable.*

Notice that when $n = 1$, so that u^* is linear with $u^*(0) = 0$, these definitions coincide with the Arrow-Pratt definitions. In [35], it was shown that if the derived indirect utility function exhibits constant relative or absolute risk aversion as a function of income, then the underlying preferences must be homothetic.

Definition 4.49 *Let $(p, I) \in R^+_{n+1}$. The demand correspondence $\phi(p, I)$ associated with $\succeq$ is defined on R^+_{n+1} by*

$$\phi(p, I) = \{x : x \in R^+_n, p \cdot x \leq I \text{ and } x \succeq \overline{x} \text{ if } p \cdot \overline{x} \leq I\}.$$

One well-known property of the demand correspondence of homothetic preferences is that $Ix \in \phi(p, I)$ if $x \in \phi(p, 1)$.
Let $U(p, I) = u(\phi(p, I))$ be the indirect utility function associated with u.

Theorem 4.50 *If $\succeq$ is continuous, monotonic, convex and homothetic, and if u is an increasing (decreasing, constant) relative risk averse representation of $\succeq$, then $U(p, I)$ is an increasing (decreasing, constant) relative risk averse function of income. Similarly, if u is an increasing (decreasing, constant) absolute risk averse representation of $\succeq$, then $U(p, I)$ inherits this property when considered as a function of I.*

Proof. Suppose that $u = h \circ u^*$ where h is a constant relative risk averse function. Also let $\overline{x} \in \phi(p, 1)$. Then $U(p, I) = h(u^*(I\overline{x})) = h(Iu^*(\overline{x}))$. Since h is a constant relative risk averse function, U is a constant relative risk averse function in I. The same argument applies to the other cases. Q.E.D.

Consider a consumer-investor with wealth W that can either be consumed in the present or saved and invested to yield a random return $\widetilde{x}$. If s is the fraction of wealth saved and invested, there will be $sW\widetilde{x}$ available for future consumption while present consumption is $W(1 - s)$. Let's denote the consumer-investor's utility function as $u(c_1, c_2)$. He will choose $s \in [0, 1]$ so as to maximize $Eu((1-s)W, sW\widetilde{x})$. Let's assume that the optimal choice of s, $\widehat{s}(W)$, is unique. We will be concerned with the effect on $\widehat{s}(W)$ of changes in W. In general, the direction of this effect is ambiguous because of income and substitution effects are in opposite directions. For the purpose of discussing these effects and their influence on $\widehat{s}(W)$, we refer to the case in which x, the return savings, is not random. When x is certain, the choice depends only on the ordinal preference $\succeq$, i.e. it is the same for all u representing $\succeq$. Let $\xi(x)$ denote the optimal proportion of wealth saved by an individual whose preferences are $\succeq$ and who faces a certain return x. Kihlstrom-Mirman [17] has shown that the resulting relationship between x and $\xi(x)$ is critical in determining the influence of risk aversion on savings. Theorem 4.50 asserts that when the ordinal preferences are such that $\xi(x)$ is an increasing (decreasing) function of x, then $\widehat{s}^1(W) < \widehat{s}^2(W)$ [>] for all W, if and only if u^1 is more risk averse than u^2 in Kihlstrom-Mirman's sense.
This suggests that $\widehat{s}(W)$ should increase (decrease) as W increases if u is a decreasing relative risk aversion function and if $\succeq$ is such that $\xi(x)$ increases (decreases) as x increases; i.e. if $\succeq$ is such that the substitution effect outweighs (is outweighed by) the income effect.

Definition 4.51 *When u is an homothetic and twice-continuously differentiable representation of $\succeq$, the elasticity of substitution, σ, is defined by*

$$\sigma = \frac{q}{r}\frac{dr}{dq},$$

where r is defined implicitly by $u_1(c_1, rc_1)/u_2(c_1, rc_1) = q$, and where dr/dq is obtained by implicit differentiation of it.

This elasticity determines the direction of the effect which changes in x have on $\xi(x)$ by determining the relative size of the above mentioned income and substitution effects. Note that when $\succeq$ is homothetic, the function defined by $u_1(c_1, rc_1)/u_2(c_1, rc_1) = q$, relating r, the ratio of c_2 to c_1, to q, the marginal rate of substitution, is independent of c_1.
In Kihlstrom-Mirman [17] it has been shown that if σ is always greater (less) than one then the substitution effect of a change in x is always greater (less) than the income effect and $\xi(x)$ is increasing (decreasing) in x. Also, Kihlstrom-Mirman [17] states that when $\succeq$ is such that σ is greater (less) than one, $\widehat{s}^1(W) < \widehat{s}^2(W)$ [>] for all W if and only if u^1 is more risk averse than u^2. We are now led to conjecture that $s(W)$ will increase (decrease) with W if u is a decreasing relative risk function and if $\succeq$ is such that σ is uniformly greater (less) than one. This conjecture is confirmed by the following theorem.

Theorem 4.52 *Suppose that $\succeq$ is continuous, strictly convex, monotonic, and homothetic on R^2_+. Also assume that u is twice continuously differentiable and $\widetilde{x}$ is a nontrivial random variable. If u is an increasing (decreasing) relative risk averse representation of $\succeq$ and if the elasticity of substitution of $\succeq$ is uniformly greater than one, then $\widehat{s}(W)$ decreases (increases) with W. If the elasticity of substitution is less than one, then $\widehat{s}(W)$ increases (decreases) with W.*

Proof. Suppose that $u = h \circ u^*$ where h is an increasing relative risk averse function. Suppose that $W_1 > W_2$, and that $\widehat{s}(W_i)$ maximizes
$Eh(u^*((1-s)W_i, sW_i\widetilde{x})) = Eh(W_i u^*((1-s), s\widetilde{x}))$.
If we let $u^1(c_1, c_2) = h(W_i u^*(c_1, c_2))$ and $k_i(u^*) = h(W_i u^*)$ then $-\frac{u^* k_i''(u^*)}{k_i'(u^*)} = -\frac{u^* W_i h''(W_i u^*)}{h'(W_i u^*)}$.
Since h is an increasing relative risk averse function, $W_1 > W_2$ and the above equality implies that $-\frac{k_1''(u^*)}{k_1'(u^*)} > -\frac{k_2''(u^*)}{k_2'(u^*)}$ for all u^*. Thus $k_1(u^*) = f(k_2(u^*))$, where f is strictly concave. Therefore, $u^1(c_1, c_2) = f(u^2(c_1, c_2))$.
Since $\widehat{s}(W_i)$ maximizes $Eu^i((1-s), s\widetilde{x})$, and u^1 is more risk averse than u^2, the corollary to [17, Theorem 2] implies that $\widehat{s}(W_1) < \widehat{s}(W_2)$ [>] if the elasticity of substitution exceeds (is less than) one. Q.E.D.

5 Some other concepts related to the risk aversion [18]

5.1 First and second order risk aversion [34]

This section defines a new concept of attitude towards risk. For an actuarially fair random variable $\widetilde{\epsilon}$, $\pi(t)$ is the risk premium the decision maker is willing to pay to avoid $t\widetilde{\epsilon}$. Since Pratt [28], it has been known that in expected utility theory, the risk premium, for small t, is proportional to t^2 and to the variance of $\widetilde{\epsilon}$. Therefore, it approaches zero faster than t; in other words, for small risks the

decision maker is almost risk neutral. Let M be a bounded interval in R and let D be the set of random variables (or lotteries) with outcome in M. For a lottery $\widetilde{x} \in D$, let $F_{\widetilde{x}}(x) = Pr(\widetilde{x} \leq x)$ be the cumulative distribution function of $\widetilde{x}$. Lotteries with a finite number of outcomes are sometimes written as vectors of the form $(x_1, p_1; x_2, p_2; ...; x_n, p_n)$ where $\sum p_i = 1$, and $p_i \geq 0, \forall p_i$. Such a lottery yields x_i with probability p_i. δ_x stands for the lottery $(x, 1)$.
On D there exists a complete and a transitive preference relation $\succeq$. We assume that $\succeq$ is continuous with respect to the topology of weak convergence, and monotonic with respect to the first order stochastic dominance (i.e., $\forall x,\ F_{\widetilde{x}}(x) \leq F_{\widetilde{y}}(x) \Rightarrow \widetilde{x} \succeq \widetilde{y}$).
$V : D \to R$ represents the relation $\succeq$ if $V(\widetilde{x}) \geq V(\widetilde{y}) \Leftrightarrow \widetilde{x} \succeq \widetilde{y}$. The certainty equivalent of $\widetilde{x}$, $CE(\widetilde{x})$, is defined implicitly by $\delta_{CE(\widetilde{x})} \sim \widetilde{x}$. Its existence is guaranteed by the continuity and monotonicity assumptions, and it can be used as a representation of $\succeq$.

Definition 5.1 *The risk premium of a lottery $\widetilde{x}$ is given by $\pi(\widetilde{x}) = E[\widetilde{x}] - CE(\widetilde{x})$. The decision maker is risk averse iff the risk premium π is positive.*

When $E[\widetilde{x}] = 0$, the risk premium is the amount the decision maker is willing to receive to accept not participating in the lottery $\widetilde{x}$.
Let ϵ be a random variable such that $E[\widetilde{\epsilon}] = 0$ and consider the lottery $x + t\epsilon$. Its risk premium π is a function of t, and it is defined by $\delta_{x-\pi(t)} \sim x + t\epsilon$. Of course, $\pi(0) = 0$. Throughout this section we assume that π is continuously twice differentiable with respect to t around $t = 0$, except, possibly, at $t = 0$, where it may happen that only right and left derivatives exist. We assume, in addition, that all these derivatives are continuous in x.
If the decision maker is risk averse, then for every non-degenerate $\widetilde{\epsilon}$, and for every $t \neq 0$, $\delta_x \succ x + t\epsilon$. Therefore, it follows that $\partial\pi/\partial t|_{t=0^+} \geq 0$ and $\partial\pi/\partial t|_{t=0^-} \leq 0$.

Definition 5.2 *The decision maker's attitude towards risk at x is of order 1 if for every $\widetilde{\epsilon} \neq \delta_0$ such that $E[\widetilde{\epsilon}] = 0, \partial\pi/\partial t|_{t=0^+} \neq 0$.*
It is of order 2 if for every such $\widetilde{\epsilon}, \partial\pi/\partial t|_{t=0} = 0$, but $\partial^2\pi/\partial t^2|_{t=0^+} \neq 0$.

This definition says that the decision maker's attitude towards risk is of order one if $\lim_{t\to 0^+} \pi(t)/t \neq 0$, that is, if $\pi(t)$ is not $o(t)$. The attitude is of order 2, $\pi(t) = o(t)$ but not $o(t^2)$.

Proposition 5.3 *Let $E[\widetilde{\epsilon}] > 0$. If the decision maker's attitude towards risk is of order 2, then for a sufficiently small $t > 0$, $x + t\widetilde{\epsilon} \succ \delta_x$. If his attitude towards risk is of order 1 and is negative (i.e., $\partial\pi/\partial t|_{t=0^+} > 0$), and if $E[\widetilde{\epsilon}]$ is small enough, then for a sufficiently small $t > 0, \delta_x \succ x + t\widetilde{\epsilon}$.*

Proof. Let $\widetilde{\epsilon}' = \widetilde{\epsilon} - E[\widetilde{\epsilon}]$. Define $\pi_t(s)$ implicitly by $\delta_{x+tE[\widetilde{\epsilon}]-\pi_t(s)} \sim x + tE[\widetilde{\epsilon}] + s\widetilde{\epsilon}'$, and define $\pi^*(t)$ by $\delta_{x+\pi^*(t)} \sim x + t\widetilde{\epsilon}$. Obviously, $\partial\pi^*/\partial t = E[\widetilde{\epsilon}] - \partial\pi_t/\partial s|_{s=t}$.
By continuity of $\partial\pi_t/\partial s$ in s and t it follows that $\lim_{t\to 0^+} \partial\pi_t/\partial s|_{s=t} = \partial\pi_0/\partial t|_{t=0^+}$.
Hence, $\partial\pi^*/\partial t|_{t=0^+} = E[\widetilde{\epsilon}] - \partial\pi_0/\partial t|_{t=0^+}$. If the decision maker's attitude towards risk is of order 2, then $\partial\pi_0/\partial t|_{t=0^+} = 0$ and $\partial\pi^*/\partial t|_{t=0^+} = E[\widetilde{\epsilon}] > 0$.
If his attitude is of order 1 and negative then $\partial\pi_0/\partial t|_{t=0^+} > 0$.
For $E[\widetilde{\epsilon}] < \partial\pi_0/\partial t|_{t=0^+}$, $\partial\pi^*/\partial t|_{t=0^+} < 0$. Q.E.D.

Proposition 5.4 *Let the decision maker be an expected utility maximizer. At the points where his utility function is differentiable and $U'' \neq 0$ his attitude towards risk is of order 2, and at the points where the utility function is not differentiable but has (different) side derivatives, his attitude is of order 1.*

For the proof, see [34, page 118].

5.2 The duality theory of choice under risk [37]

In this section a new theory of choice under risk is being proposed. It is a theory which is dual to expected utility theory. The duality theory has the property that utility is linear in wealth, in the sense that applying an affine transformation to the payment levels of two gambles always leaves the direction of preference between them unchanged. (Under expected utility, this is true only when the agent is risk neutral.)
Let V be the set of all random variables defined on some given probability space, with values in the unit interval.
For each $v \in V$, define the decumulative distribution function (DDF) of v, to be denoted by G_v, by $G_v(t) = Pr\{v > t\}, 0 \leq t \leq 1$. G_v is always non-increasing, right-continuous, and satisfies $G_v(1) = 0$. For all $v \in V$, the following relationship holds $\int_0^1 G_v(t)dt = Ev$, where Ev stands for the expected value of v. The values of random variables in V will be interpreted as payments. A preference relation $\succeq$ is assumed to be defined on V.

Axiom 5.5 *Neutrality: Let u and v belong to V, with respective DDF's G_u and G_v. If $G_u = G_v$, then $u \sim v$.*

We may construct a preference relation among DDF's by writing $G(\succeq)H$ if and only if, there exist two elements u and v, of V such that $G_u = G, G_v = H$, and $u \succeq v$. Under Axiom 5.5, the assertions $u \succeq v$ and $G_u(\succeq)G_v$ are equivalent. We assume that the underlying probability space is "rich", in the sense that all distributions with supports contained in the unit interval can be generated from elements of V. Let a family of functions Γ be defined by $\Gamma = \{G : [0,1] \to [0,1] | G \text{ is non-increasing, right continuous, } G(1) = 0\}$. Then the above assumption implies that $G \succeq H$ is meaningful for every pair of functions, G and H, in Γ.

Axiom 5.6 *Complete weak order: $\succeq$ is reflexive, transitive and connected.*

Axiom 5.7 *Continuity (with respect to L_1 - convergence): Let G, G', H, H', belong to Γ; assume that $G \succ G'$. Then, there exists an $\epsilon > 0$ such that $||G - H|| < \epsilon$ and $||G' - H'|| < \epsilon$ imply $H \succ H'$, where $||\ ||$ is the L_1 norm, i.e., $||m|| = \int |m(t)|dt$.*

Axiom 5.8 *Monotonicity (with respect to first-order stochastic dominance): If $G_u(t) \geq G_v(t)$ for all t, $0 \leq t \leq 1$, then $G_u \succeq G_v$.*

With the above axioms 5.5-5.8, we write down an independence axiom and obtain the result that preferences are representable by expected utility comparisons.

Axiom 5.9 *Independence: If G, G' and H belong to Γ and α is a real number satisfying $0 \leq \alpha \leq 1$, then $G \succeq G'$ implies $\alpha G + (1 - \alpha H) \succeq \alpha G' + (1 - \alpha)H$.*

If x and p both lie in the unit interval, then $[x;p]$ will stand for a random variable that takes values x and 0 with probabilities p and $1 - p$, respectively.

Theorem 5.10 *A preference relation $\succeq$ satisfies Axioms 5.5-5.8 and 5.9, if and only if, there exists a continuous and nondecreasing real function ϕ, defined on the unit interval, such that, for all u and v belonging to V, $u \succeq v \Leftrightarrow E\phi(u) \geq E\phi(v)$. Moreover, the function ϕ, which is unique up to a positive affine transformation, can be selected in such a way that, for all t satisfying $0 \leq t \leq 1$, $\phi(t)$ solves the preference equation $[1; \phi(t)] \sim [t, 1]$.*

Proof. Using Fishburn's theorem (Theorem 2.4), it follows from Axioms 5.6-5.8 that the premises of Fishburn's theorem hold, with unit interval acting as the set of consequences and with distributions representing probability measures. The conclusion, therefore, is that a function ϕ satisfying $u \succeq v \Leftrightarrow E\phi(u) \geq E\phi(v)$ exists, uniquely up to a positive affine transformation and, moreover, that equation $[1; \phi(t)] \sim [t, 1]$ provides the construction of ϕ. That ϕ is continuous and nondecreasing follows directly from Axiom 5.7 and Axiom 5.8, in conjunction with $[1; \phi(t)] \sim [t, 1]$. Q.E.D.

The duality theory of choice under the risk is obtained when the independence axiom of expected utility theory (Axiom 5.9) is taken. Instead of independence being postulated for convex combinations which are formed along the probability axis, it will now be postulated for convex combinations which are formed along the payment axis.
Let $G \in \Gamma$, so that G is the DDF of some $v \in V$. Now define a set-valued function, $\widehat{G}$, by writing, for $0 \leq t \leq 1$, $\widehat{G}(t) = \{x | G(t) \leq x \leq G(t-)\}$, where $G(t-) = \lim_{s \to t, s<t} G(s)$ for $t > 0$, and $G(0-) = 1$. $\widehat{G}$ is simply the set-valued function which "fills up"the range of G, to make it coincide with the unit interval. The values of $\widehat{G}$ are closed and for each p, $0 \leq p \leq 1$, there exists some t such that $p \in \widehat{G}$. Using $\widehat{G}$, we define the (generalized) inverse of G, $G^{-1}(p) = \min\{t | p \in \widehat{G(t)}\}$.
Note that G^{-1} belongs to Γ and that, for all $G \in \Gamma$, $(G^{-1})^{-1} = G$. Furthermore, if G and H belong to Γ and $|| \; ||$ stands for the L_1 norm, then $||G - H|| = ||G^{-1} - H^{-1}||$. Apparently, if G is invertible then G^{-1} is just the usual inverse function of G.

Definition 5.11 *If G and H belong to Γ and if $0 \leq \alpha \leq 1$, then $\alpha G \oplus (1-\alpha)H$ is the member of Γ given by*

$$\alpha G \oplus (1-\alpha)H = (\alpha G^{-1} \oplus (1-\alpha)H^{-1})^{-1}.$$

Axiom 5.12 *Dual Independence: If G, G' and H belong to Γ and α is a real number satisfying $0 \leq \alpha \leq 1$, then $G \succeq G'$ implies $\alpha G \oplus (1-\alpha)H \succeq \alpha G' \oplus (1-\alpha)H$.*

Theorem 5.13 *A preference relation $\succeq$ satisfies Axioms 5.5-5.8 and 5.12, if and only if, there exists a continuous and non decreasing real function f, defined on the unit interval, such that, for all u and v belonging to V,*

$$u \succeq v \Leftrightarrow \int_0^1 f(G_u(t))dt \geq \int_0^1 f(G_v(t))dt.$$

Moreover, the function f, which is unique up to a positive affine transformation, can be selected in such a way that, for all p satisfying $0 \leq p \leq 1$, $f(p)$ solves the preference equation $[1; p] \sim [f(p); 1]$.

Proof Define a binary relation $\succeq^*$ on the family Γ of DDF's, as follows: $G \succeq^* H$ if and only if $G^{-1} \succeq H^{-1}$ for all G and H in Γ. Clearly, if u and v are random variables in V, then $u \succeq v \Leftrightarrow G_u^{-1} \succeq^* G_v^{-1}$.
Checking Axioms 5.6-5.8, we find that they hold for $\succeq$ if and only if , they hold for $\succeq^*$.
Furthermore, $\succeq$ satisfies the axiom of dual independence if and only if $\succeq^*$ satisfies the independence axiom. Therefore, from Theorem 5.10, $\succeq$ satisfies axioms 5.5-5.8 and 5.12, if and only if, $\succeq^*$ has the appropriate expected utility representation.
In other words, $\succeq$ satisfies axioms 5.5-5.8 and 5.12, if and only if, there exists a continuous and nondecreasing function f, defined on the unit interval, such that
$u \succeq v \Leftrightarrow -\int_0^1 f(p)dG_u^{-1}(p) \geq -\int_0^1 f(p)dG_v^{-1}(p)$ is true for all u and v in V. Let G be any member of Γ. Then, the equation $-\int_0^1 f(p)dG^{-1}(p) = \int_0^1 f(G(t))dt$ holds, by introducing the change of variable

$p = G(t)$, and this proves the first part of the theorem.
Now applying the second part of Theorem 5.10 to $\succeq^*$, we find that f can be selected so as to satisfy the preference equation $G_{[1;f(p)]} \sim^* G_{[p;1]}$ for $0 \leq p \leq 1$. Note, however, that if G is the DDF of $[x;p]$ then G^{-1} is the DDF of $[p,x]$. Therefore, rewriting $G_{[1;f(p)]} \sim^* G_{[p;1]}$ in terms of the original preference relation $\succeq$, gives $[1;p] \sim [f(p);1]$. Q.E.D

Let v belong to V, with DDF G_v, and let $U(v)$ be defined by

$$U(v) = \int f(G_v(t))dt,$$

with f defined as $[1;p] \sim [f(p);1]$. Theorem 5.13 tells us that the function U is a utility on V when preferences satisfy axioms 5.5-5.8 and 5.12. The hypothesis of the duality theory is that agents will choose among the random variables so as to maximize U.
This is an analogy with the hypothesis of expected utility theory, which is that agent chooses among the random variables so as to maximize the function W, given by $W(v) = E\phi(v) = -\int_0^1 \phi(t)dG_v(t)$, with ϕ defined in $[1;\phi(t)] \sim [t;1]$.

The utility U of the duality theory has two important properties: First, U assigns to each random variable its certainty equivalent. In other words, if v belongs to V, $U(v)$ is equal to that sum of money which, when received with certainty, is considered by the agent equally as good as v. The second property of U is linearity in payments: When the values of a random variable are subjected to some fixed positive affine transformation, the corresponding value of U undergoes the same transformation.

Proposition 5.14 *Under axioms 5.5-5.8 and 5.12, the relationship $v \sim [U(v);1]$ holds for every $v \in V$.*

Proof. It follows from $U(v) = \int f(G_v(t))dt$ that $U([x;1]) = x$ for all x, $0 \leq x \leq 1$. In particular, $U([U(v);1]) = U(v)$ and, by Theorem 1, $[U(v);1] \sim v$. Q.E.D.

Remark In expected utility theory, the following dual to the above proposition exists:
Let $\succeq$ satisfy axioms 5.5-5.9, and let ϕ be defined by $[1;\phi(t)] \sim [t;1]$ and $W(v) = E\phi(v) = -\int_0^1 \phi(t)dG_v(t)$. Then, $v \sim [1;W(v)]$ is true for every $v \in V$.

Proposition 5.15 *Let v belong to V and let a and b be two real numbers, with $a > 0$. Define a function $av + b$ by writing $(av + b)(s) = av(s) + b$ for each state-of-nature s, and assume that $0 \leq av(s) + b \leq 1$ for all s. Then, $U(av + b) = aU(v) + b$.*

Proof. Let G_v and G_{av+b} be the DDF's of v and $av + b$ respectively. Note that, for every $t \in [0,1]$, we have

$$G_{av+b}(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t < av_0 + b, \\ G_v(\frac{t-b}{a}) & \text{for} \quad t \geq av_0 + b, \end{cases}$$

where v_0 is the infimum of the range of v. Hence,
$U(av + b) = av_0 + b + \int_{av_0+b}^1 f(G_{av+b}(t))dt = av_0 + b + \int_{av_0+b}^1 f(G_v(\frac{t-b}{a}))dt.$
Introducing the change of variable $s = (t - b)/a$, we get
$U(av + b) = a[v_0 + \int_{v_0}^1 f(G_v(s))ds] + b = aU(v) + b$. Q.E.D.

Corollary 5.16 *If the preference relation $\succeq$ satisfies axioms 5.5-5.8 and 5.12, then, for all u and v belonging to V, we have $u \succeq v \Leftrightarrow au + b \succeq av + b$, provided $a > 0$ and $au + b, av + b \in V$. In words, under axioms 5.5-5.8 and 5.12, agents always display constant absolute risk aversion as well as constant relative risk aversion.*

Note that under expected utility theory, an agent with constant absolute risk aversion as well as constant relative risk aversion must be risk-neutral, i.e., this agent's preferences always rank random variables by comparing their means. Under duality theory, we have linearity without risk neutrality being implied in any way. Indeed, let us see how risk neutrality is characterized under the duality theory. It follows from $u \succeq v \Leftrightarrow \int_0^1 f(G_u(t))dt \geq \int_0^1 f(G_v(t))dt$ in conjunction with $\int_0^1 G_v(t)dt = Ev$, that under axioms 5.5-5.8 and 5.12, the agent's preference relation $\succeq$ ranks random variables by comparing their means if and only if the function f representing $\succeq$ coincides with the identity, i.e., $f(p) = p$ for $0 \leq p \leq 1$. In other words, risk neutrality is characterized in the duality theory by the function f in $[1;p] \sim [f(p);1]$ being identity. But there is nothing in theorem 5.13 to "force" f to coincide with the identity: Any continuous and nondecreasing function f, satisfying $f(0) = 0$ and $f(1) = 1$ can be obtained in $[1;p] \sim [f(p);1]$, for some preference relation $\succeq$ satisfying axioms 5.5-5.8 and 5.12.

It is interesting to compare the construction of the function f in the duality theory with the construction of the von Neumann-Morgenstern utility ϕ in expected utility theory. Consider the preference equation $[1;p] \sim [t;1]$. We know from $[1;p] \sim [f(p);1]$ and $[1;\phi(t)] \sim [t;1]$ that $f(p)$ is the value of t that solves $[1;p] \sim [t;1]$, while $\phi(t)$ is the value of p that solves $[1;p] \sim [t;1]$. It follows, therefore, that $f = \phi^{-1}$.

5.2.1 Paradoxes

Behavior which is inconsistent with the expected utility theory has been observed systematically, and often such behavior has been branded "paradoxical". As it turns out, behavior which is "paradoxical" under expected utility theory is, in many cases, entirely consistent with duality theory. This does not mean, however, that the duality theory is "paradox free". On the contrary, for each "paradox "of expected utility theory, one can usually construct a "dual paradox" of duality theory, by interchanging the roles of payments and probabilities. Let's illustrate the above with the following example.

Example *A famous "paradox" of expected utility theory is the so-called common ratio effect: Dividing all the probabilities by some common divisor reverses the direction of preferences. Kahnemen and Tversky [14], for example, have found that a great majority of subjects prefer [0.3;1] over [0.4; 0.8] but that an equally large majority prefer [0.4;0.2] over [0.3; 0.25].(The symbol $[x;p]$ stands for a random variable which takes values x and 0 with probabilities p and $1-p$, respectively.) This pattern, which is inconsistent with expected utility theory, is entirely in keeping with the duality theory. Specifically, with the utility function defined as $U(v) = \int f(G_v(t))dt$, we find that U([0.3;1])=0.3, U([0.4;0.8])= (0.4)$f(0.8)$, U([0.3;0.25])= (0.3)$f(0.25)$ and U([0.4;0.2])= (0.4)$f(0.2)$, and these numbers will support the preference pattern $[0.3;1] \succ [0.4;0.8]$ and $[0.4;0.2] \succ [0.3;0.25]$ if $f(0.8) < \frac{3}{4} < \frac{f(0.2)}{f(0.25)}$. This inequality is satisfied, for example, when f is of the form $f(p) = p/(2-p)$, for $0 \leq p \leq 1$. (This f is in fact risk averse, as we will see in the next section.)*

5.2.2 Risk aversion in duality theory

How would the risk aversion be characterized under the duality theory? Under expected utility theory, preferences are represented by a von Neumann-Morgenstern utility ϕ. Under the duality theory, preferences are represented by a function f and $f = \phi^{-1}$. Since the concavity of ϕ is equivalent to the convexity of ϕ^{-1}, and since the concavity of ϕ characterizes risk aversion, we should expect the convexity of f to characterize risk aversion under the duality theory.
Letting $\succeq$ be a preference relation on V, as before, we say that $\succeq$ is risk averse if $v = u + noise$ implies $u \succeq v$: Adding noise can never be $\succeq$ improving.

Definition 5.17 *Let u and v belong to V, with DDF's G_u and G_v, respectively, and consider the inequality $\int_0^T G_u(t)dt \geq \int_0^T G_v(t)dt$. A preference relation $\succeq$ on V is said to be risk averse if $u \succeq v$ whenever the above inequality holds for all T satisfying $0 \leq T \leq 1$, with equality for $T = 1$.*

Theorem 5.18 *Consider the class of preference relations on V satisfying Axioms 5.5-5.8 and 5.12. A preference relation $\succeq$ in this class is risk averse if and only if the function f representing $\succeq$ is convex.*

Proof. Let $\succeq$ satisfy axioms 5.5-5.8 and 5.12, and assume that $\succeq$ is risk averse. Take five real numbers x, y, p, q, r such that $0 \leq y \leq x \leq 1$ and $0 \leq q \leq p \leq r \leq 1$, and construct two random variables, u and v, in the following manner: u takes the values x, y and 0 with probabilities q, $r - q$, and $1 - r$, respectively, and v takes the values x and 0 with probabilities p and $1 - p$, respectively. Assume that $(p - q)x = (r - q)y$. Then, by direct calculation, $\int_0^T G_u(t)dt \geq \int_0^T G_v(t)dt$ holds for all T satisfying $0 \leq T \leq 1$, with equality for $T = 1$. Hence, $u \succeq v$.
By Theorem 5.13, $u \succeq v \Leftrightarrow U(u) \succeq U(v)$. We can see that $U(u) = yf(r) + (x - y)f(q)$ and $U(v) = xf(p)$.
Moreover, $(p - q)x = (r - q)y \Rightarrow yf(r) + (x - y)f(q) \geq xf(p)$ holds for any x, y, p, q, r satisfying the above conditions.
It is trivial when $r = q$. So assume $r > q$. Define $\lambda \in [0, 1]$, by writing $\lambda = (p - q)/(r - q)$ and note that $p = \lambda r + (1 + \lambda)q$. Using this, we obtain that
$y = \lambda x \Rightarrow yf(r) + (x - y)f(q) \geq xf(\lambda r + (1 - \lambda)q)$ must hold for all λ and x in the unit interval. Since $x < 0$, f is convex.
Conversely, let $\succeq$ satisfy axioms 5.5-5.8 and 5.12 and suppose that u and v satisfy $\int_0^T G_u(t)dt \geq \int_0^T G_v(t)dt$ for $0 \leq T \leq 1$, with equality for $T = 1$. Then by Hardy, Littlewood and Polya [12, Theorem 10], the inequality holds for every convex and continuous f. Therefore, using the definition of U, $U(u) \geq U(v)$ (or $u \succeq v$) holds whenever f is continuous and convex. Thus, if the function f of Theorem 5.13 is convex, then $\succeq$ is risk averse. Q.E.D.

The fact that risk aversion is characterized in the duality theory by the convexity of f has a useful interpretation when f happens to be differentiable. Let v belong to V, with DDF G_v, and let $U(v)$ be the utility number assigned to v under the duality theory, i.e., $U(v) = \int f(G_v(t))dt$. If f is differentiable, then the expression for $U(v)$ can be integrated by parts to obtain
$U(v) = \int_0^1 tf'(G_v(t))dF_v(t)$, where F_v is the cumulative distribution of v. Note that
$\int f'(G_v(t))dF_v(t) = 1$ and $\int tdF_v(t)$ is the mean of v.
In $U(v)$, a similar integral is being calculated, each t is given a weight $f'(G_v(t))$. If f is convex, then f' is nondecreasing; i.e., those values of t for which $G_v(t)$ is small receive relatively low weights and those values of t for which $G_v(t)$ is large receive relatively high weights.

5.3 Behavior towards risk with many commodities [35]

The purpose of this part is to show what restrictions on the indifference map (e.g., on the income-consumption curves) are implied by alternative assumptions about consumer behavior under uncertainty and, conversely, what restrictions on consumer behavior under uncertainty are implied by alternative assumptions about the indifference map.

5.3.1 Linearity of income-consumption curves for risk neutrality

It is well known that risk neutrality is equivalent to linearity in income of the Von Neumann-Morgenstern utility function for fixed prices. If the individual is risk neutral at all prices and incomes in some open region, then all income-consumption curves must be straight lines. To see this, observe that since the individual is risk neutral at fixed prices, utility can be written as a linear function of income, $U = a'y + b$. If the individual is risk neutral at another, nearby, set of prices, we can again write utility as a linear function of income, $U = a''y + b''$, where in general $a' \neq a'', b' \neq b''$. If this is to be true for all sets of prices in a given neighborhood, then it must be true that

$$U(y, p^1, ..., p^n) = a(p^1, ..., p^n)y + b(p^1, ..., p^n). \tag{5.1}$$

The linearity of the indirect utility function means that the expenditure function, which gives the minimum level of expenditure E required to obtain a given level of utility $\overline{U}$ at given prices, is linear in $\overline{U}$:

$$E = \frac{\overline{U}}{a} - \frac{b}{a}.$$

The compensated demand curve for the i-th commodity, x^i, is then linear in $\overline{U}$:

$$x^i(p^1, ..., p^n, \overline{U}) = \frac{\partial E}{\partial p^i} = -\frac{\overline{U}}{a}\frac{a_i}{a} - \frac{b_i}{a} + \frac{ba_i}{a^2}, \tag{5.2}$$

where $b_i = \partial b / \partial p^i, a_i = \partial a / \partial p^i$.
The income-consumption curves may be derived by substituting (5.1) into (5.2), or directly from the fact that, if $U(y, p^1, ..., p^n)$ is the indirect utility function,

$$x^i = -\frac{\partial U / \partial p^i}{\partial U / \partial y} = -\frac{a_i}{a}y - \frac{b_i}{b}.$$

These are linear in income, but need not go through the origin, i.e., the indifference map need not be homothetic. If, however, an individual is risk neutral at all prices and incomes, all income-consumption curves must be straight lines through the origin, i.e., the indifference must be homothetic.
The converse is also true: if all income-consumption curves are linear in some open neighborhood of the origin, there exists a cardinal representation of utility which is linear in income. Gorman [10] has shown that if income-consumption curves are straight lines in an open region of the commodity space, then there exist functions $g(p)$ and $h(p)$ such that $x^i = g^i(p) + Uh^i(p)$, where U is the utility index and p is the vector of prices. Multiplying by p^i and summing over all commodities, we obtain the expenditure function, which in turn implies that U is linear in y, for given p.

5.3.2 Extensions to concave and convex utility functions

If an individual has a utility function that is concave (or convex) in income at a given set of prices, p^*, for all values of $y \leq y^*$, then if all income-consumption curves are linear, his utility function will be concave (or convex) in y at any fixed set of prices p for all values of y such that $U(y,p) \leq U(y^*,p^*)$. To see this, assume we performed a betting experiment in which it turned out that $U_{yy}(y,p^*) < 0$ for all values of $y \leq y^*$. One cardinal representation of the utility function is $\widehat{U} = a(p)y + b(p)$. U and $\widehat{U}$ must be related by a monotonic transformation F, such that $U(y,p^*) = F(\widehat{U}(y,p^*))$, so $U_{yy}(y,p^*) = F''(a(p^*))^2$. In order to have $U_{yy}(y,p^*) < 0$ for $y \leq y^*$, we must have $F'' < 0$, for $\widehat{U}(0,p^*) \leq \widehat{U} \leq \widehat{U}(y^*,p^*)$. But this implies that for all p, $U_{yy} < 0$, provided only that $U(y,p) \leq U(y^*,p^*)$.

5.4 Risk aversion over income and over commodities [24]

Let f and g be two real valued and continuous functions defined on some open interval, with g strictly increasing. The standard way of saying that f is more concave than g is to require that there be a concave function φ such that $f = \varphi \circ g$.

Definition 5.19 *We call a function $\overline{g} : I \to R$ a capping function of f at y^* if $\overline{g}(y^*) = f(y^*)$ and $\overline{g}(y) \geq f(y)$ for all $y \in I$. A function $g : I \to R$ is a capping function of f if for each y^* in I, there are scalars r' and r such that the function $\overline{g}$ given by $\overline{g}(y) = r' + rg(y)$ is a capping function of f at y^*.*

Theorem 5.20 *Let f and g be two real-valued and continuous functions defined in an open interval I, with g strictly increasing. Then the following are equivalent:*

a) f is more concave than g.

b) g is a capping function of f.

c) The function f has the representation

$$f(y) = \min_{r \in U}(\phi(r) + rg(y)), \tag{5.3}$$

where U is a set in R and ϕ is a real-valued function defined on U.

Proof. $(a) \Rightarrow (b)$. The function $f \circ g^{-1}$ is concave, so at any point $z^* = g(y^*)$ in $g(I)$, there is a tangent at $(z^*, f \circ g^{-1}(z^*))$ which "caps"the graph of $f \circ g^{-1}$. In other words, for some r' and r, $r' + rz^* = f \circ g^{-1}(z^*)$ and $r' + rz \geq f \circ g^{-1}(z)$ for all z in $g(I)$. Substituting $g^{-1}(z^*)$ for y^*, and $g^{-1}(z)$ for y shows that $r' + rg(y)$ is a capping function for f at y^*.
$(b) \Rightarrow (c)$. If $h(y) = r' + rg(y)$ is a capping function of f at a point y^*, then $\widetilde{h}(y) = r'' + rg(y)$ for $r'' \neq r$ cannot be a capping function at any point in I. Therefore, we can consider $r' = \phi(r)$. Since each point in I admits an affine transformation of g as a capping function, the formula follows immediately.
$(c) \Rightarrow (a)$. The implication follows observing that $f \circ g^{-1}(z) = \min_{r \in U}(\phi(r) + rz)$ is a concave function.
Q.E.D

Note that f is concave if and only if it is more concave than the identity function; so defining g by $g(y) = y$, we obtain the following representation of any concave function f: $f(y) = \min_{r\in U}(\phi(r) + ry)$. In this section, a Bernoulli utility function v is considered and we assume that the function v is nice: the domain of v is R_{++} and v is non-decreasing.

Definition 5.21 *For any positive real number σ, we denote by $L_A(\sigma, y, t, z')$ the lottery*

$$t \bullet \left(y - \frac{1}{\sigma}\ln z'\right) \oplus (1-t) \bullet \left(y - \frac{1}{\sigma}\ln z''\right), \tag{5.4}$$

where $z', z'' \in (0, e^{\sigma y})$ satisfying $tz' + (1-t)z'' = 1$. The nice utility function v is said to be of type A_σ if the agent with this utility function prefers y to any lottery $L_A(\sigma, y, t, z')$; in other words, $v(y) \geq tv(y - (\ln z')/\sigma) + (1-t)v(y - (\ln z'')/\sigma)$.

Lemma 5.22 *Suppose $\sigma > \overline{\sigma}$. Then for every lottery $L_A(\overline{\sigma}, y, t, \overline{z'})$ with $\overline{z'} \neq 1$, there is a lottery $L_A(\sigma, y, t, z')$ such that $y - (\ln z')/\sigma > y - (\ln \overline{z'})/\overline{\sigma}$ and $y - (\ln z'')/\sigma > y - (\ln \overline{z''})/\overline{\sigma}$.*

Proof. Note that $-(\ln \overline{z'})/\overline{\sigma} = -(\ln \overline{z'}^{\sigma/\overline{\sigma}})/\sigma$ and similarly $-(\ln \overline{z''})/\overline{\sigma} = -(\ln \overline{z''}^{\sigma/\overline{\sigma}})/\sigma$. Defining $\widetilde{z'} = \overline{z'}^{\sigma/\overline{\sigma}}$ and $\widetilde{z''} = \overline{z''}^{\sigma/\overline{\sigma}}$, we observe that $M = t\widetilde{z'} + (1-t)\widetilde{z''} = t\overline{z'}^{\sigma/\overline{\sigma}} + (1-t)\overline{z''}^{\sigma/\overline{\sigma}} \geq 1$, since $\sigma > \overline{\sigma}$ and $t\overline{z'} + (1-t)\overline{z''} = 1$. Choosing $z' = \widetilde{z'}/M$ and $z'' = \widetilde{z''}/M$ gives us the desired lottery. Q.E.D.

The above lemma says that any lottery $L_A(\overline{\sigma}, y, t, \overline{z'})$ is dominated by some lottery $L_A(\sigma, y, t, z')$ in the sense that the latter has a higher payoff in every realization. Since v is nondecreasing, it follows that an agent who prefers y to all lotteries of the form $L_A(\sigma, y, t, z')$ must also prefer y to all lotteries of the form $L_A(\overline{\sigma}, y, t, \overline{z'})$. Thus we obtain the next proposition.

Proposition 5.23 *The nice utility function v is of type A_σ if and only if it is of type $A_{\overline{\sigma}}$ for all $\overline{\sigma} \leq \sigma$.*

Proposition 5.24 *Suppose that the nice utility function v is C^2 with $v' > 0$. Then $A_v(y) = -v''(y)/v'(y) \geq \sigma$ for all $y > 0$ if and only if v is of type A_σ.*

Proof. The result follows from two important observations. Firstly, the condition $A_v \geq \sigma$ is equivalent to saying that v is more concave than $g(y) = -e^{-\sigma y}$ since $A_g = \sigma$. Secondly, the concavity of $v \circ g^{-1}$ is equivalent to v being of type A_σ. To see this, note that the map $v \circ g^{-1}$ has as its domain the interval $(-1, 0)$ and $v \circ g^{-1}(w) = v(-\ln(-w)/\sigma)$.
That this map is concave means that for any $t \in [0, 1]$,
$tv(-\ln(-w')/\sigma) + (1-t)v(-\ln(-w'')/\sigma) \leq v(-\ln(-tw' - (1-t)w'')/\sigma)$.
If we define y by $-tw' - (1-t)w'' = e^{-\sigma y}$ and z', z'' by $-w' = e^{-\sigma y}z'$, $-w'' = e^{-\sigma y}z''$ respectively, we have $tz' + (1-t)z'' = 1$. Making this substitution in the above inequality gives $v(y) \geq tv(y - (\ln z')/\sigma) + (1-t)v(y - (\ln z'')/\sigma)$. Q.E.D.

Proposition 5.25 *Suppose that the nice utility function v is C^2 with $v' > 0$. Then $A_v(y^*) = \sigma$ if and only if the following holds:*

(a) For each $\widetilde{\sigma} > \sigma$, there is a neighborhood of 1 such that whenever z' and z'' are in that neighborhood, the agent prefers $L_A(\widetilde{\sigma}, t, y^*, z')$ to the sure income level y^*.

(b) For each $\widetilde{\sigma} < \sigma$, there is a neighborhood of 1 such that whenever z' and z'' are in that neighborhood, the agent prefers the sure income level y^* to $L_A(\widetilde{\sigma}, t, y^*, z')$.

Proof. Define the function $G_{\widetilde{\sigma}}$ by $G_{\widetilde{\sigma}}(z) = v(y^* - (\ln z)/\widetilde{\sigma})$. Then $G''_{\widetilde{\sigma}}(1) = \frac{\widetilde{\sigma}^{-1} v''(y^*) + v'(y^*)}{\widetilde{\sigma}}$. If $A_v(y^*) = \sigma$, this term is negative if $\sigma > \widetilde{\sigma}$ and positive if $\widetilde{\sigma} > \sigma$, as required by (a) and (b). On the other hand, if (a) and (b) hold, it means that $G''_{\widetilde{\sigma}}(1) \leq 0$ whenever $\sigma > \widetilde{\sigma}$ and $G''_{\widetilde{\sigma}}(1) \geq 0$ whenever $\widetilde{\sigma} > \sigma$. This can only happen if $A_v(y^*) = \sigma$. Q.E.D.

Proposition 5.26 *For a nice utility function v, the following are equivalent:*

1. *v is of type A_σ.*
2. *The function $g_\sigma = -e^{-\sigma y}$ is a capping function of v.*
3. *The function v has the representation $v(y) = \min_{r \in U}(\phi(r) - re^{-\sigma y})$, where U is a set in R and ϕ is a real valued function defined on U.*

Proof. That v is of type A_v is equivalent to $v \circ g^{-1}$ being concave, where $g(y) = -e^{-\sigma y}$. The result then follows immediately from theorem 5.20. Q.E.D.

Based on the coefficient of relative risk aversion, we can define an appropriate class of Bernoulli utility functions for which results similar to those holding for functions of type A_σ can be obtained. For the details, see [24].

The following results give the precise way in which an agent's risk aversion over incomes can be related to her risk aversion over consumption bundles.
Let R^l_{++} be the consumption space and u be the Bernoulli utility function. For any price vector p in R^l_{++} and income $y > 0$, the budget set at the price-income situation (p, y) is the set $B(p, y) = \{x \in R^l_{++} : p \cdot x \leq y\}$.
The demand at (p, y) refers to the set $\text{argmax}_{x \in B(p,y)} u(x)$; we denote this set by $\overline{x}(p, y)$.
We say that u is well behaved if the following hold:

(a') $\overline{x}(p, y)$ is nonempty for all (p, y) in $R^l_{++} \times R_{++}$ and obeys the budget identity $p \cdot x' = y$ for $x' \in \overline{x}(p, y)$.

(b') $\forall x \in R^l_{++}$, there is p such that x is in $\overline{x}(p, 1)$.

We say u is very well behaved if, in addition to (a') and (b'), we have:

(c') The demand set $\overline{x}(p, y)$ is a singleton at all (p, y), and the function $\overline{x}$ is continuous.

Assuming that u is well behaved, the function $v : R^l_{++} \times R_{++} \to R$ defined by $v(p, y) = u(\overline{x}(p, y))$ is the indirect utility function generated by u.
Given $w \in R^l_{+} \setminus \{0\}$, we define the normalized price set $Q^w = \{p \in R^l_{++} : p \cdot w = 1\}$.

Definition 5.27 *Let w be an element of $R^l_+ \setminus \{0\}$, and σ be a positive real number. We say that $u : R^l_{++} \to R$ is of type A^w_σ if the agent with this utility function always prefers the (sure) bundle $tx' + (1-t)x''$ to any lottery of the form*

$$t \bullet \left(\frac{1}{\alpha'} x' - \frac{\ln \alpha'}{\sigma} w\right) \oplus (1-t) \bullet \left(\frac{1}{\alpha''} x'' - \frac{\ln \alpha''}{\sigma} w\right),$$

where $\alpha', \alpha'' > 0, t\alpha' + (1-t)\alpha'' = 1$, $x', x'' \in R^l$, and the two possible realizations of the lottery and the bundle $tx' + (1-t)x''$ all belong to the consumption space. In other words,

$$tu\left(\frac{1}{\alpha'}x' - \frac{\ln \alpha'}{\sigma}w\right) + (1-t)u\left(\frac{1}{\alpha''}x'' - \frac{\ln \alpha''}{\sigma}w\right) \le u(tx' + (1-t)x'').$$

Choosing $\alpha' = \alpha'' = 1$, we can see that u is a concave function and the agent is risk averse.

Theorem 5.28 *Suppose $u : R^l_{++} \to R$ is very well behaved and generates the indirect utility function $v : R^l_{++} \times R_{++} \to R$. Then the following are equivalent:*

(a) $v(p, \cdot)$ is of type A_σ for all p in the normalized price set Q^w.

(b) u has the representation

$$u(x) = \min_{(q,r)\in\overline{U}} (\phi(q,r) - re^{-\sigma(q\cdot x)}),$$

where $\overline{U}$ is a subset of $Q^w \times R$ and ϕ is the real valued function defined on $\overline{U}$.

(c) u is of type A^w_σ.

For the proof, see [24].

6 Summary

The risk aversion implied by a von Neumann-Morgenstern utility function u is closely related to the measure of concavity. Assuming maximization of the expected utility EU, the consumer is risk averse to all small risks in x, if and only if the utility function is concave at x. We assume that u is twice continuously differentiable and monotonically increasing. Arrow-Pratt risk aversion $r(x) = -u''(x)/u'(x)$ is invariant under the positive linear transformation which is consistent to von Neumann-Morgenstern utility function property. Also, it has been used in many applications proving its usefulness.
In the univariate case, one of the powerful concepts is the risk premium π, the real number where receiving a risk $\tilde{z}$ or receiving non-random amount $E(\tilde{z}) - \pi$ is indifferent. Let's consider a portfolio selection behavior of an investor when there is one risky and one non-risky asset. Pratt [28] considers two investors with utility functions u_1 and u_2 and shows that investor 1 always invests less in the risky asset than investor 2 if $r_1 > r_2$ or equivalently $\pi_1 > \pi_2$.
Arrow shows that [1] if the agent becomes more risk averse as his wealth rises; i.e., if $r(x)$ is an increasing function, then the amount of money invested in the risky asset decreases as wealth increases.

Suppose that both assets are risky, then there are cases where the measure of Arrow-Pratt risk aversion can not support the intuitions. Therefore, Ross [33] introduced the stronger measure of risk aversion : A is strongly more risk averse than B if and only if

$$\inf_w \frac{A''(w)}{B''(w)} \ge \sup_w \frac{A'(w)}{B'(w)}.$$

For example, consider a choice between two lotteries, $\widetilde{x}$ and $\widetilde{y}$, where $\widetilde{y}$ is distributed as $\widetilde{x}$ plus a "return"$\widetilde{v} \ge 0$ and an additional risk $\widetilde{\epsilon}$, where $E\{\widetilde{\epsilon}|\widetilde{x} + \widetilde{v}\} = 0$. Intuition would suggest that if the

agent B finds such a tradeoff unacceptable then so must do any agent A, who is more risk averse than B. While this is not true for the Arrow-Pratt ordering, the strong measure of risk aversion justifies the above case.
Moreover, from [19], there is an another counter example where $r_1 > r_2$ holds but $\pi_1 > \pi_2$ fails to hold for two independent lotteries $\widetilde{x}, \widetilde{y}$. It can be shown that if u_1 is more risk averse than u_2 ($r_1 > r_2$) and if either u_1 or u_2 is a non-increasing risk aversion, then π_2 is always smaller than π_1 for two independent lotteries.

Pratt and Zeckhauser [30] approached this problem from axiomatic point of view, saying that an undesirable lottery can never be made desirable by the presence of an independent desirable lottery. They proposed the proper utility function u iff the condition

$$\widetilde{w} + \widetilde{x} + \widetilde{y} \preceq \widetilde{w} + \widetilde{y} \text{ whenever } \widetilde{w} + \widetilde{x} \preceq \widetilde{w} \text{ and } \widetilde{w} + \widetilde{y} \preceq \widetilde{w}$$

holds. They gave the analytical necessary and sufficient conditions for utility function u being proper. Kimball has given the concept of the standard risk aversion which implies the proper risk aversion (see Definition 3.46). One of the advantage is an simple characterization of standard risk aversion; Decreasing absolute risk aversion and decreasing absolute prudence is equivalent to u being standard, under assumption $u' > 0$, $u'' < 0$.

In univariate case, the Arrow-Pratt risk premium π is proportional to t^2, so it approaches to zero faster than t. For small risks, the risk averse agent is almost risk neutral. Segant [34] defined first and second order of risk aversion, where the above phenomena can be distinguished, and studied its properties.

Moreover, there are other behaviors which are inconsistent with the expected utility theory, called "Paradoxes". One example of such behavior is the common ratio effect: Dividing all the probabilities by some common divisor reverses the direction of preferences. Yaari [37] proposed the dual theory, where instead of the independence axiom in ordering so-called "Dual independence"is used. This gives a possibility to explain these "Paradoxes".

In the multivariate case, n-dimensional von Neumann-Morgenstern utility functions may represent different preference orderings on the set of commodity bundles. The comparison of risk averseness of utility functions representing different ordinal preferences are confusing because of the differences in these preferences. Therefore, Kihlstrom and Mirman restricted the utility functions to represent the same ordinal preference and defined: U_1 is at least as risk averse as U_2, if $U_1 = k(U_2)$ where $k' > 0$ and k is concave. Also, U_1 is more risk averse than U_2, if k is strictly concave.
Moreover, the risk premium is not unique in the multivariate case. In [17], they have defined the directional risk premium, generalizing the one-dimensional risk premium. To generalize one-dimensional risk aversion function r, they proposed an function $\rho : R^n_+ \to R_+$ and proved that it is an appropriate measure of risk aversion under some assumptions.
Keeney [16] defined the risk independence and the conditional risk premium where all the components except a certain one are taken as constants.
A matrix measure was given by Duncan [7], considering the risk premium π as a vector satisfying

$$U(x - \pi) = EU(x + Z),$$

where $x, Z \in E^n$ and $E(Z) = 0$. Since the risk premium is a vector and the measure of risk aversion is a matrix, it makes impossible to compare the risk averseness in this case.
In an analogous way of how Arrow defined the measure of risk aversion in one-dimension, using

$$U(x) = pU(x+h) + (1-p)U(x-h),$$

H.Levy and A.Levy [23] proved that probability p can be used as the measure of risk aversion in the multivariate case.
Paroush [26] noticed that the risk premium, $U(x + E\widetilde{z} - \pi) = E\{U(x + \widetilde{z})\}$, has $(n-1)$ degrees of freedom. When the $(n-1)$ free variables are forced to be the same for risk premia, he proved that the comparison of the first component of the risk premia is also an alternative way to compare risk averseness.
Karni [15], using the indirect utility function, defined the one-dimensional risk premium for the income y:

$$\psi(y + E\{z_1\} - \pi, p_1 + E\{z_2\}, .., p_{n-1} + E\{z_n\}) = E\{\psi(y + z_1, p_1 + z_2, .., p_{n-1} + z_n)\},$$

where $z = (z_1, .., z_n)$ is a small random variable and $p = (p_1, ..., p_{n-1}) \in R^{n-1}_{++}$ is a price vector. He further studied the relationship between the matrix measure and the above described risk premium.

Let's suppose the risk averseness of two utility functions representing the different ordinal preference. We can restrict the class of gambles, where excludes gambles with different preferences. Using the indirect function with fixed price, it can regarded one-dimensional function of income y. In this case, the Arrow-Pratt results can be applied.

Furthermore, using the least concave representation of utility functions u^*, Kihlstrom and Mirman defined in [18]: u is an increasing absolute risk averse representation of $\succeq$ if $u(x) = h(u^*(x))$ and h is an increasing absolute risk aversion function of a single real variable.

In "Risk aversion over income and over commodities"part, the connection between an agent's risk attitudes over income and his risk attitudes over the goods he consumes with that income is given precisely for the specific class of utility functions.

As we can see, there are many approaches in the univariate cases. Since all of them are the measures of risk aversion, it is difficult to compare these approaches. However, depending on the different conditions or situations, we can proceed with the appropriate one.
In the multivariate case, as Paroush has proved fixing $(n-1)$ components and comparing the only dependent component of the risk premia, we can compare risk averseness of the agents. Therefore, the risk premium of indirect utility function, the directional risk premium and the conditional risk premium are the same in general.
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