UNIVERSITY of L'AQUILA

Department of Engineering

Master Thesis
in
Mathematical Modelling in Engineering

Analysis and Simulation of Hydrodynamic Model for Charge transport in Ionic Pumps

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Contents

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To Mary.

Chapter 1

Biological Introduction

This thesis is meant to analyze, research, develope hypotetical results in order to reach a deeper understanding of the labor in a cell membrane. Basically the transport process. This process details how the cell interact within its enviroment. The absorption and emission of ions (transport components) encompass very important work done by the different kinds of cells, going from the communication between neurons, muscle contractions, to the modus operandi of snake's venom.

Therefore the interest for this subject is quite wide. Having a deep knowledge about the transport process in the cells, developing an accurate model of it, can help to the growth and rise in many fields, going from medicine applications, to biology, chemistry, biocomputing, among others.

1.1 Cells

The cell is the functional basic unit of life. It was discovered by Robert Hooke and is the functional unit of all known living organisms. It is the smallest unit of life that is classified as a living thing, and is often called the building block of life[6]. Some organisms, such as most bacteria, are unicellular (consist of a single cell). Other organisms, such as humans, are multicellular. Humans have about 100 trillion or 1014 cells; a typical cell size is 10 µm and a typical cell mass is 1 nanogram.

The word cell comes from the Latin cellula, meaning, a small room. The descriptive term for the smallest living biological structure was coined by Robert Hooke in a book he published in 1665 when he compared the cork cells he saw through his microscope to the small rooms monks lived in. There are two types of cells: eukaryotic and prokaryotic. Prokaryotic cells are usually independent, while eukaryotic cells are often found in multicellular organisms. The major difference between prokaryotes and eukaryotes is that eukaryotic cells contain membrane-bound compartments in which specific metabolic activities take place.

All cells, whether prokaryotic or eukaryotic, have a membrane that envelops the cell, separates its interior from its environment, regulates what moves in and out (selectively permeable), and maintains the electric potential of the cell. Inside the membrane, a salty cytoplasm takes up most of the cell volume. All cells possess DNA, the hereditary material of genes, and RNA, containing the information necessary to build various proteins such as enzymes, the cell's primary machinery. There are also other kinds of biomolecules in cells.

1.2 Cell membrane

Every living cell has to obtain nutrients and raw materials from its surroundings, usually ions [1], this for the biosynthesis and production of energy; on the other hand, it should take out all the excess, or not needed material. In some few cases, this process is carried out only by pure diffusion (from the region of high concentration to the region of low concentration) through the lipid double layer (about 7.5nm thick), which constitutes the *cell membrane*, but in general it needs the action of some special proteins attached to it. This proteins are known as Ion channels and Ion pumps, different and important types of transport is due to them.

The process of transporting chemicals can occur against gradient of concentration and/or electrical gradient, in these cases it requires the use of energy. When the work requires the use of energy, the transport is known as *active transport* as in the ion pumps, where this energy is given by the ATP molecules. If the transport is due to the generated field; hence, the use of an extra energy is not needed, this is called *passive transport*, as the one in the Ion channels. Exist some different processes to move ions, for example the use of proteins called ionophores, that mask the charge of the ion, and allows diffusing into the cell.

[1] in some cases the cell need to obtain complex molecules

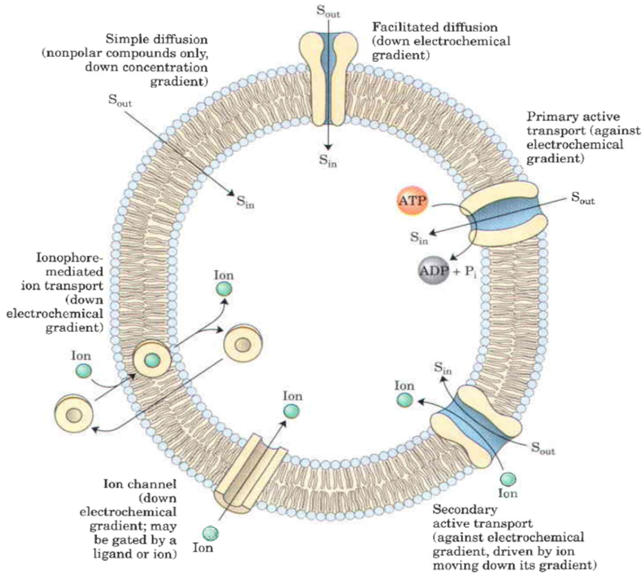


Figure 1.1: Summary of the basic processes of ion movement [2]

As mention above the **ion pumps** is an active process where proteins in the cell membrane choose specific ions and transports them from one specific side to the other. Another important process in the transport of ions in the cell membrane are the so called **ion channel**, proteins that act as small tubes of passive transport, permeable only to some specific ions,. Figure 1.1 shows the summary of the processes.

To understand the basic process of transport let's consider now media separated by a permeable interface. If one of the media has bigger chemical concentration than the other, the solute will tend to move from high concentration to low concentration, this is called simple diffusion. Now let's imagine that we have to different media with ions, positive on one side and negative on the other side, and naturally they will try to recombine. This difference of charge in the interface (permeable membrane) will create an electric potential, called membrane potential. This potential creates a force exerted on the ions going through the membrane, making some ions to move one direction and stopping the others moving in the other direction. The combination of these two factors is called electrochemical gradient. This transport process is mediated and controlled by the membrane proteins called Transporters.

Transporters fall within two categories, **carriers** and **channels**. Carriers (ion pumps) are proteins that are highly selective, and the transport velocities are well below compared to the one of free diffusion. An important property of carriers is that if the substrate concentration is above some critical value further increase of concentration will not produce greater transport rate. Channels (ion channels) are proteins that allow movement of ions at greater rates, even thousands of times compared with the one done by carriers, and in contrast with the previous, they don't saturate at any value, so an increase in the concentration of the substrate will increase the rate of ion transport. Around 25% of energy consumption of a human at rest is due to transport processes.

1.2.1 Membrane potential

The differences in concentration of various ions between extracellular and intracellular environment that are caused by ion transport create a potential difference between the membrane.

For understanding how, imagine that you have two media, separated by a membrane. This two media are electrically neutral. Now set a transport process, that is permeable to just one kind ion, let's say positive, then after a while, some ions pass the membrane and the neutrality condition is not longer satisfied, *i.e.* in one side of the membrane you will have accumulation of positive charges, and in the other side accumulation of negative charges. This will create a capacitor-like piece of membrane, and of course an electric potential through the membrane. Now, this electric field will generate a force that opposes further diffusion of more positive charges, reaching then a balance condition. The potential at which the balance condition is achieved is called the **Nernst potential**.

Let's derive an expression for this potential in terms of the concentrations. Consider the Drift Diffusion equation that gives the current J of ions of charge q due to a gradient of concentration n and an electric potential Φ :

$$J = -D\nabla n - \frac{\mu_0 z n}{|z|}\nabla\Phi$$

Where μ_0 is the mobility of the ion, D the diffusion constant and z is the valence of the ion. Using the Einstein relation $D = \mu_0 \frac{K_B T}{q}$, with T the absolute temperature and considering a one-dimensional case, we get:

$$J = -D\left(\frac{dn}{dx} + \frac{qn}{K_B T}\frac{d\Phi}{dx}\right)$$

Assuming that the interior of the membrane is at $x = 0$ and the exterior at $x = L$, and using the boundary conditions $n(0) = n_{int}$ and $n(L) = n_{ext}$ we can solve the previous equation:

$$\ln(n_{ext}) - \ln(n_{int}) + \frac{q}{K_B T}(\Phi(L) - \Phi(0))$$

i.e.

$$\Phi(0) - \Phi(L) = \frac{K_B T}{q} \ln\left(\frac{n_{ext}}{n_{int}}\right)$$

Defining $\Phi(0) - \Phi(L) = V_s$ we can write the Nernst Potential as:

$$V_s = \frac{K_B T}{q} \ln\left(\frac{n_{ext}}{n_{int}}\right)$$

The variety of potentials we can encounter across the membrane produce different behaviors in the cells. Thus we classify them as:

1.2.2 Resting Potential

Conventionally, resting membrane potential can be defined as a relatively stable, ground value of transmembrane voltage in animal and plant cells. In principle, there is no difference between resting membrane potential and dynamic voltage changes like action potential from biophysical point of view: all these phenomena are caused by specific changes in membrane permeabilities for potassium, sodium, calcium, and chloride, which in turn result from concerted changes in functional activity of various ion channels, ion transporters, and exchangers.

As we expect the concentrations of ions and the membrane transport proteins influence the value of the resting potential. The resting potential of a cell can be most thoroughly understood by thinking of it in terms of equilibrium potentials. To understand better, consider a cell with only two permeant ions, potassium and sodium. Consider a case where these two ions have equal concentration gradients directed in opposite directions, and that the membrane permeabilities to both ions are equal. K+ leaving the cell will tend to drag the membrane potential toward EK. Na+ entering the cell will tend to drag the membrane potential toward the reversal potential for sodium ENa. Since the permeabilities to both ions were set to be equal, the membrane potential will, at the end of the Na+/K+ tug-of-war, end up halfway between ENa and EK. As ENa and EK were equal but of opposite signs, halfway in between is zero, meaning that the membrane will rest at 0 mV. Note that even though the membrane potential at 0 mV is stable, it is not an equilibrium condition because neither of the contributing ions are in equilibrium. Ions diffuse down their electrochemical gradients through ion channels, but the membrane potential is upheld by continual K+ influx and Na+ efflux via ion transporters.

1.3 Action Potential

Is a short-lasting event in which the electrical membrane potential of a cell rapidly rises and falls, following a consistent trajectory. Action potentials occur in several types of animal cells, called excitable cells, which include neurons, muscle cells, and endocrine cells, as well as in some plant cells. In neurons, they play a central role in cell-to-cell communication. In other types of cells, their main function is to activate intracellular processes. In muscle cells, for example, an action potential is the first step in the chain of events leading to contraction.

Action potentials are generated by special types of voltage-gated ion channels embedded in a cell's plasma membrane.[7] These channels are shut when the membrane potential is near the resting potential of the cell, but they rapidly begin to open if the membrane potential increases to a precisely defined threshold value.

As example in the neurons when the channels open, they allow an inward flow of sodium ions, which changes the electrochemical gradient, which in turn produces a further rise in the membrane potential. This then causes more channels to open, producing a greater electric current, and so on. The process proceeds explosively until all of the available ion channels are open, resulting in a large upswing in the membrane potential. The rapid influx of sodium ions causes the polarity of the plasma membrane to reverse, and the ion channels then rapidly inactivate. As the sodium channels close, sodium ions can no longer enter the neuron, and they are actively transported out of the plasma membrane. Potassium channels are then activated, and there is an outward current of potassium ions, returning the electrochemical gradient to the resting state. After an action potential has occurred, there is a transient negative shift, called the after hyperpolarization or refractory period, due to additional potassium currents. This is the mechanism which prevents an action potential traveling back the way it just came.

1.4 Ion selective channels

Ion channels have many differences with other transport proteins, essentially in three ways. The first difference is that the rate of flux of a typical channel is about 10^7 or 10^8 $\frac{ions}{s}$, that is almost the value of unrestricted diffusion. By contrast, for example the ATPase (a carrier protein) has a rate of about $100\frac{ions}{s}$. The second main difference is that they don't saturate, as mentioned before. And the third, they can be *"gated"* (open or closed) in response of a cellular event [2].

Ligand-gated channels: A binding of the channel protein with a small molecule, inside or outside the cell, changes the properties of the protein and closes/opens the channel.

Volt-gated ion channel: The process of gating is different, the movement of ions through the channel modifies the membrane potential (created by the difference of charge in the two extremes of the channel), and when a critical value is reached, the channel closes/opens.

This two are the most common gating processes in channels, but they are more, like temperature gating, in which a hot or cold temperature triggers the mechanism, light gating where the light triggers the opening or closing. Mechanical gating, where a deformation of the protein makes it to open or close.

The ability of channels to open and remain opened milliseconds, makes this molecular devices effective for fast signal transmissions *e.g.* in neurons.

1.4.1 Function measurement

Because the rapid opening and closing of the ion channel it's almost impossible to measure its properties using standard chemical processes. Now, the basic function of the channel, ion movement, gives a very good alternative to make measurements by using voltage or current changes using the appropriate apparatus. This practice was first used by Erwin Nernst and Bert Sakmann in 1976. The method consists of applying a micropipette to the cell membrane containing few channels, then patch of the membrane of the cell and putting in an aqueous solution, then varying in this the voltage and current and measure what happens. See Figure 1.2 below

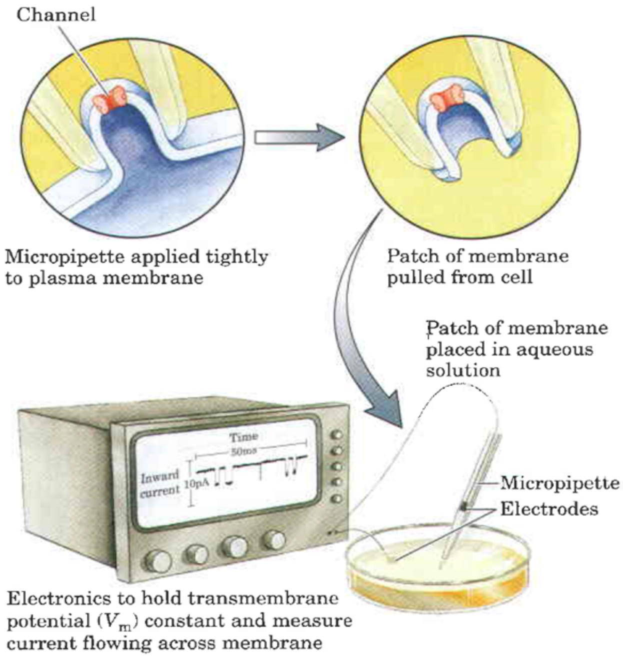


Figure 1.2: Patch-clamping method [2]

1.4.2 Basic ion channel structure

Let's take as an example the structure of potassium K^+ ion channel of the bacteria *Streptomyces lividans*, first determined by Roderick MacKinnon in 1998. This protein serves as prototype for studying every other channel, including voltage gated neuron ion channels, discovered later. It was measured that the rate at which this channel works is about $10^7\ \frac{ions}{s}$, approaching the upper limit of free diffusion. In the figure below we can see the structure of the hole protein, including the bottom part, the channel, first in (a) it can be seen eight protein helices (red and blue helices) that form a cone facing the wider end toward outer of the cell. The ending segments, in gray, converge to a smaller aperture to make the selectivity filter, the channel. In the upper view (b) we can see that the helices make a sort of tunnel, just big enough for letting a potassium ion go through[2]. See Figure 1.3.

At the entrance of the channel there are negatively charged amino acids which increase the local concentration of cations (In this case K^+ ions), after approaching the channel, the narrow gray part in previous picture, the potas-

[2]This images have been done in VMD, Visual Molecular Dynamics program, with data from www.pdb.org -Protein Data Base -)

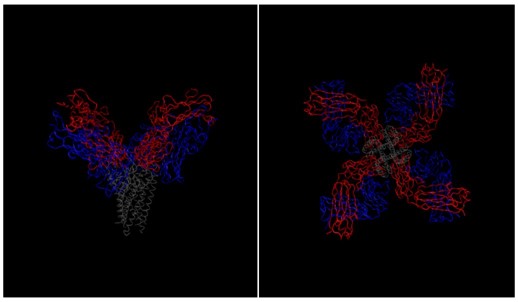


Figure 1.3: Potassium channel protein structure

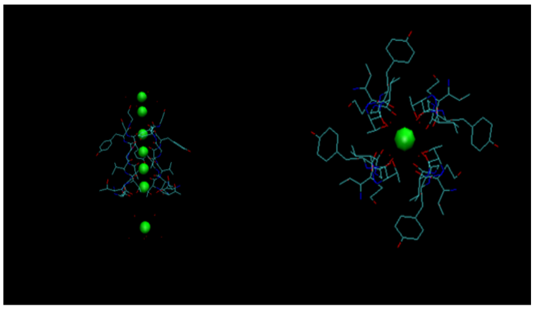


Figure 1.4: Carbonyl oxygen (red) cage surrounding the ions (green). Side and top view.

sium ions still have their hydration sphere (water molecules that surround the positive charged potassium). At this point, about two thirds of the size of the protein, the narrowed part takes out the hydration sphere and replace this by carbonyl oxygen atoms in the structure of the tunnel, forming a sort of cage surrounding the K^+ ion (picture below). The electrical repulsion between the potassium and the oxygen is the key of the selectivity of ions; moreover, this repulsion is the responsible for moving the ion inside the cell. See Figure 1.4.

The Figure 1.5 shows a visualization of Ion passing through the channel. Side and top view.

The study and understanding of ion channel proteins is very important, since they are present in all cells, especially in neurons, and they are key component of neuron electrical transmissions. Most of the neurotoxins in nature (and some neurological diseases) act upon ion channels, so it's really promising that a good understanding of this proteins help to develop medicines. Moreover nowadays studies have shown that it's possible to make

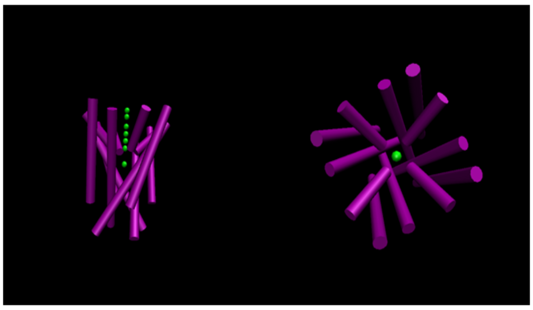


Figure 1.5: Ion channel and ions.

connections between neurons and silicon, widening the possibilities of the achieving the dream of connecting brains to computers, see for example [2].

As said before, the volt-gated channels are one of the most common ion channels in cells, especially in neurons, so it is worth it to study how the membrane potential behaves.

1.5 Ion Pump or Ion transporter

Is a transmembrane protein that moves ions across a plasma membrane against their concentration gradient, in contrast to ion channels, where ions go through passive transport. These primary transporters are enzymes that convert energy from various sources, including ATP (called ATPases), sunlight, and other redox reactions, to potential energy stored in an electrochemical gradient. This energy is then used by secondary transporters, including ion carriers and ion channels, to drive vital cellular processes, such as ATP synthesis.

The family of active transporters called P-type ATPases are cation transporters that are reversibly phosphorylated by ATP. ATPases are a class of enzymes that catalyze the decomposition of adenosine triphosphate (ATP) into adenosine diphosphate (ADP) and a free phosphate ion. This dephosphorylation reaction releases energy, which the enzyme (in most cases) harnesses to drive other chemical reactions that would not otherwise occur. This process is widely used in all known forms of life. Some such enzymes are integral membrane proteins (anchored within biological membranes), and move solutes across the membrane, typically against their concentration gradient. These are called transmembrane ATPases.[6]

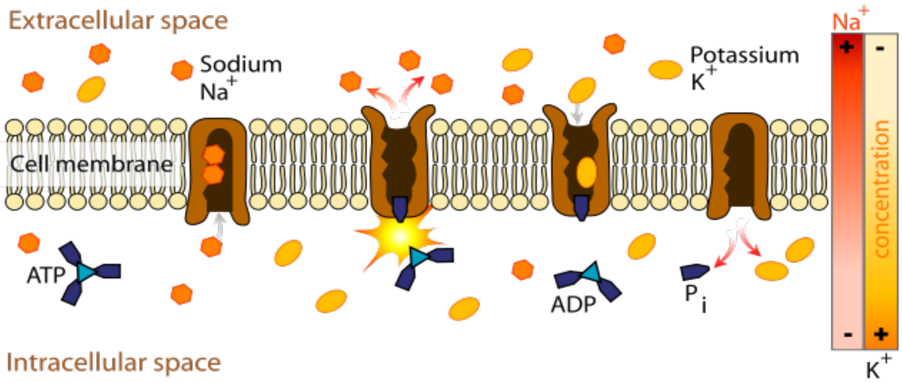


Figure 1.6: Na+/K+ATPase.

Transmembrane ATPases harness the chemical potential energy of ATP, because they perform mechanical work: they transport solutes in a direction opposite to their thermodynamically preferred direction of movement—that is, from the side of the membrane where they are in low concentration to the side where they are in high concentration. That is why this process is considered active transport.

Transmembrane ATPases import many of the metabolites necessary for cell metabolism and export toxins, wastes, and solutes that can hinder cellular processes. An important example is the sodium-potassium exchanger or Na+/K+ATPase (see Figure 1.6), which establishes the ionic concentration balance that maintains the cell potential. Another example is the hydrogen potassium ATPase (H+/K+ATPase or gastric proton pump) that acidifies the contents of the stomach.

1.5.1 ATP Molecule

Is a multifunctional nucleotide used in cells as a coenzyme. It is often called the "molecular unit of currency" of intracellular energy transfer. [13] ATP transports chemical energy within cells for metabolism.

One molecule of ATP contains three phosphate groups, and it is produced by ATP synthase from inorganic phosphate and adenosine diphosphate (ADP) or adenosine monophosphate (AMP).

Metabolic processes that use ATP as an energy source convert it back into its precursors. ATP is therefore continuously recycled in organisms: the human body, which on average contains only 250 grams (8.8 oz) of ATP, [14] turns over its own body weight in ATP each day.

ATP is an unstable molecule in unbuffered water, in which it hydrolyses to ADP and phosphate. This is because the strength of the bonds between the phosphate groups in ATP are less than the strength of the hydrogen bonds (hydration bonds), between its products (ADP + phosphate), and water. Thus, if ATP and ADP are in chemical equilibrium in water, almost all of the ATP will eventually be converted to ADP. A system that is far from equilibrium contains Gibbs free energy, and is capable of doing work [3].

The *pump uses the energy* stored in the ATP molecule which is released when it is dephosphorylated into ADP, through the overall reaction scheme.

$$ATP + 3Na_i^+ 2K_e^+ \rightarrow ADP + P_i + 3Na_e^+ 2K_i^+$$

where the subindexes e and i stands for respectively extracellular and intracellular ions.

Chapter 2

Applications: Neuroelectronic Interfacing

In this section we will talk about the technological applications that arise from the study of ion pumps and ion channels in the cell membrane. In recent years studies have shown that it is possible to connect and transmit electrical impulses between neurons and semiconductors. Its well known that the electrical communication between neurons, i.e. the synapsis, is possible because of the action of ions moving from and into the cell membrane embedded with the ion channels and ion pumps, so taking advantage of this, a coupling with semiconductors is not difficult to build and use. We will describe the basics of this interfacing and some simple but promising, applications.

2.1 Contact cell-chip

A simple union with a cell (nerve) and a sensor transistor in silicon is shown in Figure (2.1). The cell is surrounded by a membrane made from lipids, making it an insulating covering. That lipid bilayer separates the bath with a concentration of 150 mM sodium ions from the cytoplasm with about 150 mM potassium ions. The electrical interaction of cell and chip is the structure of the contact between the lipid bilayer and the oxide. As seen on [8] the cell and the semiconductor are separated by an electrolitic bath, and the average distance between them is about $50nm$. Now, lets remember that the electrical impulses in the nerves are because of the transport of ions, like potasium or sodium, and in the semiconductor are because of the moving electrons, and

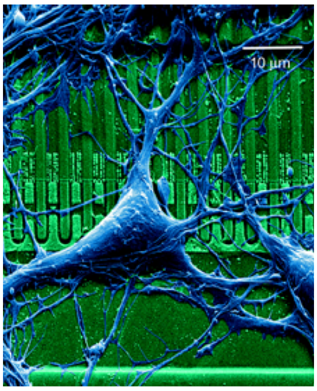


Figure 2.1: Cell and chip. [9]

the interaction depends on the resistance of the distance between oxide[1] and the membrane. The resistance is in average $10M\Omega^2$ as seen in [8][2].

2.2 Ion and electron coupling

In a junction between a chip and a cell we can have two ways of communication, one from cell to chip and the other in the opposite direction. This signals are measure in different ways, one is using a capacitor, which is used to induce responses on the cell, changing the electric potential around the cell, triggering an *action potential*, and in the other a source and drain, used to measure the current in the ion channel, since the gate is controlled by the external cell potential. Lets describe a little bit more the first process. A falling voltage ramp is applied to the capacitor. A displacement current flows across the oxide and gives rise to an ohmic current along the sheet resistance of the cell–chip contact. The resulting negative extracellular voltage $V_j = RI$, with $I = C\frac{\Delta V}{\Delta t}$, which opens the channels in the attached membrane [8]. The process and results are shown in Figure (2.2) and (2.3). The experiment shows that the ionic current in the cell is controlled by an electronic signal in the semiconductor.

Now lets describe the process that is used to test signaling from cells to chips. In this case its used a source and drain in the semiconductor, and the electrolyte between the cell and oxide plays the role of a gate. Using

[1]Its usually used an oxide-semiconductor chip, almost like a MOS, but instead of metal its used an electrolyte.

[2]This values, as seen in the reference, come form measurements of a chip and a single rat nerve cell

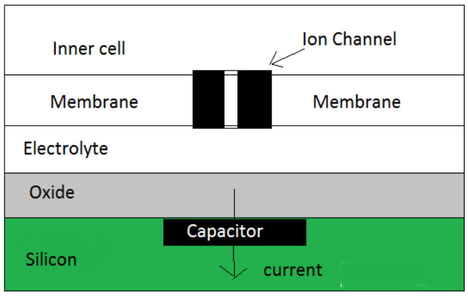


Figure 2.2: Using a capacitor to control the opening and closing of the ion channel

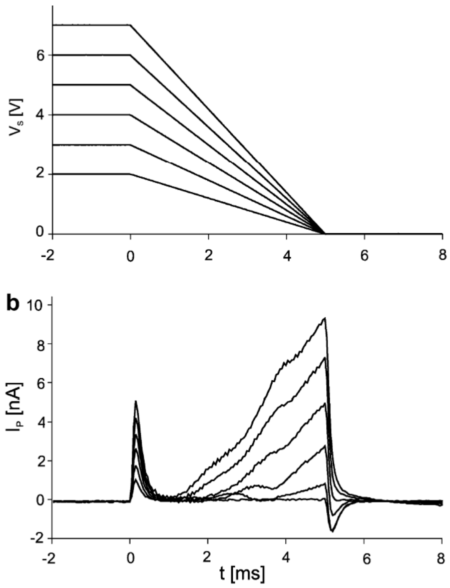


Figure 2.3: Applied voltage to the capacitor and ion channel respond [8]

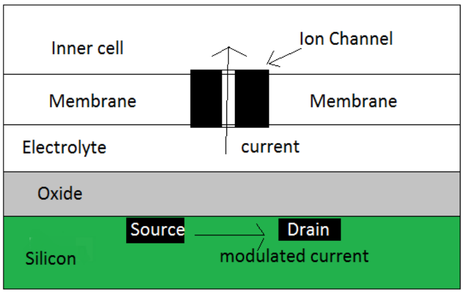


Figure 2.4: Electrolyte Oxide Semiconductor cell-chip junction. Changes in ion current induce changes in source to drain current

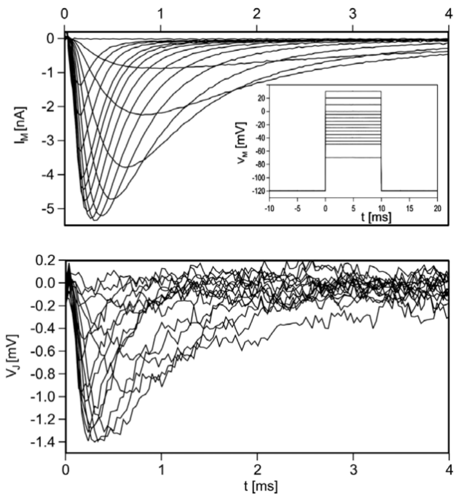


Figure 2.5: Positive voltage pulses are applied to the cell (ion currents induced) and EOS responds to the transients of the extracellular voltage that are proportional to the membrane current [8]

a micropipette, positive voltage pulses are applied to the cell to open the channels and ions move (for example Na^+), current flows into the cell and along the sheet resistance of the cell–chip contact. It gives rise to a negative extracellular voltage that modulates the electron current from source to drain [8]. The opening of the channel causes a voltage $V_j = IR$ in the channel, with the current $I = G\Delta V$ flowing to the inner cell. This is completely analogous to a MOSFET, and the current of electrons in the silicon channel is modulated. The process and results are seen in Figures (2.4) and (2.5).

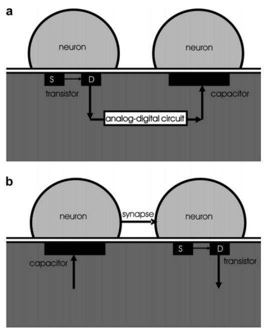


Figure 2.6: (a) chip-prothesis, (b) neuronal memory on chip [8]

2.3 Elementary neuroelectronic devices

With this methods of controlling (EOS capacitor) and recording (EOSFET) the responses of ion channels and chips, two simple devices can be build.

One is what could be thought as a *neuron chip-prothesis*. The chip prothesis is no more that two separated neurons that can communicate via a chip. First we assume that the cells have not synapsis, i.e. no connection between them, then they are putted on a chip, one over an EOS (electrolyte oxide semiconductor) and the other over a capacitor, both of course, on the same chip. This works as follows: one neuron transmits an ion current inside and outside a the cell modulating the gate in the EOSFET, the source and drain records the signal, then it is treated with an analogical.digital circuit which transmits the signal to the capacitor, who changes the voltage in the cell membrane inducing a respond, and subsequently ion current in the other neuron. Making in this way a artificial synapses between cells. This device , as simple as it seems, could have a huge impact in many fields, for example in medicine, where it can be used as a prosthesis for neurons which lost the synapse. This flaw is present in persons who suffer from Alzheimer. As propose in [12] .

Another device is a *neuronal memory on chip*, and works as follows: A neuronal memory element is obtained as a connection between two neurons by synapses, one cell is connected to a capacitor and the other to a transistor to induce and record the signals respectively. The presynaptic nerve cell is stimulated from a capacitor, the signal activates the chemical synapsis and the postsynaptic excitation is recorded with an EOSFET [8]. A scheme of the two devices are shown in Figure (2.6).

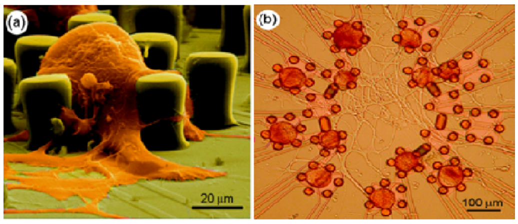


Figure 2.7: (a) polymer fence and neuron, (b) neuron net on chip [9]

The last thing to mention is one of the most important challenges in this arising technology, is the problem of how to arrange the cells within the surface of the chip such that it works as the designer wants. In other words, how to do the cell nets and keep them fix in there places. There are some ways to do it, one is simply by letting the neurons displaced randomly, of course this is easy process, but it has limit applications, since it can be used, for the moment, only to make general measurements and studies. Other way is to fix the neurons physically. Mechanical immobilization of cell bodies is used by picket fences of an organic polymer on capacitor/transistor contacts with neurons grown in the central area as in Figure (2.7).

So, from this promising applications, we see that is important to continue studying how the transport of ions in the cell due to the ion channels works. Owing to the fact that in the cell membrane there are more proteins that can contribute to the transport process, like, for example, ion pumps, its very important and necessary step to describe the action of this proteins. The goal of this document is to introduce a mathematical description of ion pumps, to complement what is known of ion channels and to give a more general picture of the hole process.

Chapter 3

Models

As said in the introduction section, the goal of this work is to introduce and study a very simple mathematical model for the Ion transport in the cell membrane. Before starting this, we will give a briefly mathematical introduction to conservations laws, the Riemann problem, the P-system, the equations that describe the gas dynamics and discuss some existing models for ion channels. In particular we will describe the Hydrodynamic model for ion channels as in [4], this last is the modification of the Nernst-Plank model doing an analogy with the carrier transport in a semiconductor.

The outline of the chapter will be organized in the following way, first we recall the mathematical background needed to study the models. The second part we will introduce and discuss a mathematical model for ion transport in the cells taking into account the ion pumps first using a P-system and later on the Euler equations for gas dynamics.

3.1 Mathematical Introduction

3.1.1 Conservation Laws

In physics, a *conservation law* states that a particular measurable property of an isolated physical system does not change as the system evolves, ie. there are no sink nor source attached to the system which modifies this measurable properties.

A Conservation Law is characterized with the linear form:

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0 \tag{3.1}$$

or the quasilinear

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_U \mathbf{U}_x = 0 \tag{3.2}$$

where $\mathbf{U} \in \mathbb{R}^n$ and $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ for a given set of n equations of conservation laws.

3.1.2 Riemann Problem

A Riemann problem, consists of a conservation law together with piecewise constant data having a single discontinuity, i.e

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})\mathbf{U}_x = 0$$

with the initial condition

$$\mathbf{U}(0,x) = \begin{cases} \mathbf{U}_l & \text{if } x \leq 0, \\ \mathbf{U}_r & \text{if } x > 0. \end{cases}$$

The solution of the problem can compromise the appearance of contact discontinuities, shock and rarefaction waves (no more than two in the result) which will be briefly explain afterward.

Since we can see from the Conservation Law, $\mathbf{U}$ is an eigenvector of the Jacobian of $\mathbf{F}$ in (3.1), ie. our solution is characterized by the eigenvectors and eigenvalues (which for future references we will define as r_k and λ_k respectively) of the Jacobian of $\mathbf{F}$. We should keep in mind that in a Conservation Law the eigenvalues are real and distinct given the hyperbolicity of the equations. Each of the eigenvalues along with its eigenvector, generates a *k-family* of waves.
To describe these *k-family* of waves we should know its basic overall properties.

Definition 1.

K-shock waves: Are convex solutions on which the information travels at a speed s, this speed is given by the 'Rankine-Hugoniot' or 'jump' condition.

K-rarefaction waves: Are smooth solutions on which the discontinuity spreads out in time, allowing a "diffusive" curve in between to fill it up, so the solution in this spreading gap has the form $\mathbf{U}(\frac{x}{t})$.

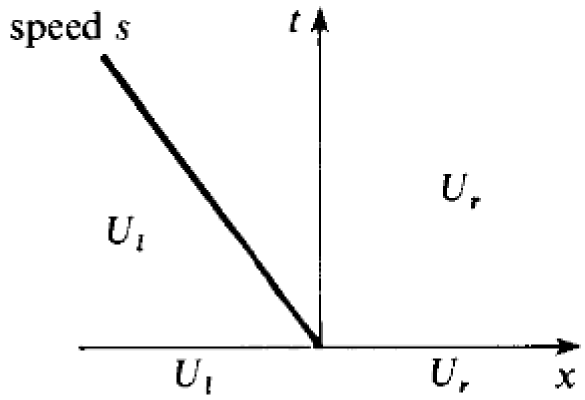


Figure 3.1: Back Shock

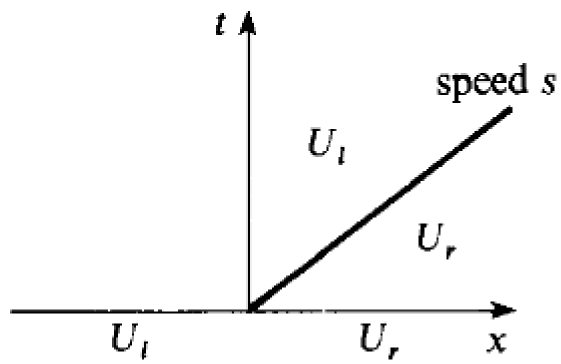


Figure 3.2: Front Shock

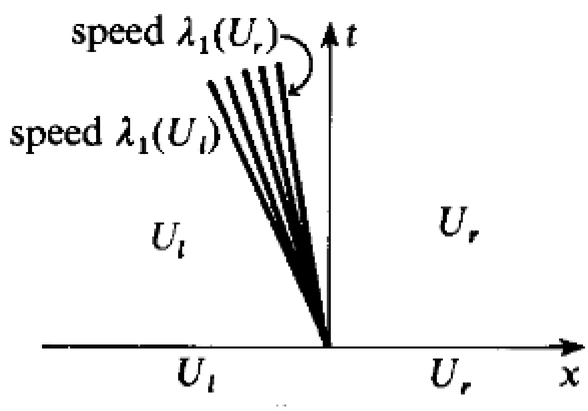


Figure 3.3: Back Rarefaction

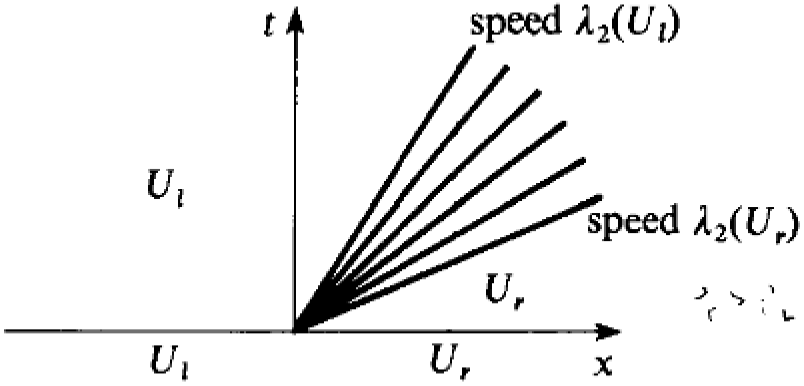


Figure 3.4: Front Rarefaction

Contact discontinuities: These solutions are not smooth as its name suggests, here a discontinuity remains.

From the latter simple definitions we can observe that smoothness is a basic property to know the kind of waves we are working with. So we define the next concepts.

Definition 2.

Genuinely nonlinear: k-characteristic family of waves that in a region $D \subset \mathbb{R}^n$ *satisfy* $\nabla\lambda_k \cdot r_k \neq 0$.

Linear degenerate: k-characteristic family of waves that in a region $D \subset \mathbb{R}^n$ *satisfy* $\nabla\lambda_k \cdot r_k = 0$.

Is important to remark that the above definitions come intuitively from an analogy to convexity in scalar equations, where $u_t + f(u)_x = 0$, $\lambda = f'(u)$, $r = 1$ and $\nabla\lambda \cdot r = f''(u)$.

Since in the Riemann problem we are working with conservation laws, we know that the measurable properties should hold even during any discontinuity hence there should exist some variables which remain invariant along it.

Definition 3.

k-Riemann Invariant: is a smooth function $w : N \to \mathbb{R}$ where $N \subset \mathbb{R}^n$ defines a neighborhood around our initial condition $\mathbf{U}_L \in \mathbb{R}^n$ such that if $u \in N$, $\langle r_k, \nabla_u w(u)\rangle = 0$

So the gradients of the k-Riemann invariant are orthogonal to r_k and given that there exists $n-1$ gradients ($\nabla_u w$) which are linearly independent in D, we construct $n-1$ Riemann invariants that are hypersurfaces that represent the interactions between the measure properties (due to the ∇_u) along r_k in a region D.[1]

3.1.3 P-System

A P-System is a set of two conservation laws, used to model isentropic (constant entropy) or polytropic gases. It has the next form in Lagrangian coordinates stated as a Riemann Problem:

$$\begin{cases} v_t - u_x = 0 & for \;\; x \in \mathbb{R} \\ u_t + P(v)_x = 0 & t > 0 \end{cases} \qquad U(x,0) = \begin{cases} \boldsymbol{U}_L = (v_l, u_l), & x < 0 \\ \boldsymbol{U}_r = (v_r, u_r), & x > 0 \end{cases}$$

Where $v = \frac{1}{\rho}$ and ρ represents the density, u the velocity and $P(v)$ a function of Pressure depending on the density, with the properties $P_x < 0$, $P_{xx} > 0$.

.

The two equations represent conservation of mass and momentum respectively, since the temperature is held constant to keep a constant entropy, there is no conservation of energy equation, since the energy must be added up to the system.

The P-System can be rewritten as (3.1) and its Jacobian $F(\boldsymbol{U})_U$ is:

$$J = \begin{pmatrix} 0 & -1 \\ p(v)_x & v \end{pmatrix}$$

with the eigenvalues $\lambda_1 = -\sqrt{-p(v)_x}$ and $\lambda_1 = \sqrt{-p(v)_x}$ which are real and distinct, therefore the is an Hyperbolic system.

[1] For a better understanding of subjects mentioned previously in this subsection, read Chapter 17 in [1]

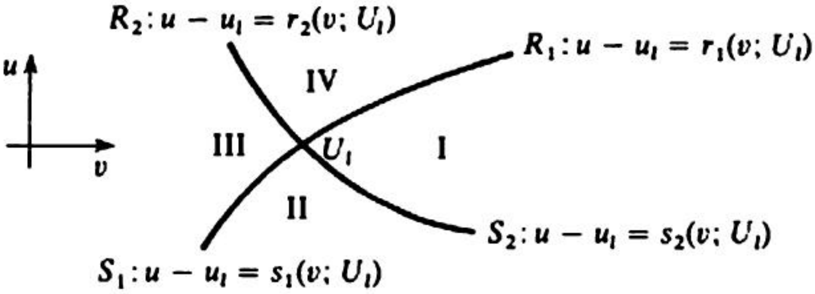


Figure 3.5: States Graph [1]

This problem can be solved with a class of functions that consists of constant states, separated by Shock waves and/or Rarefaction waves. In this system the latter equations are respectively:

$$\begin{array}{lll} S_1: & u - u_l = -\sqrt{(v - v_l)(p(v_l) - p(v))} \equiv s_1(v, \boldsymbol{U}_l), & v_l > v \\ S_2: & u - u_l = -\sqrt{(v - v_l)(p(v_l) - p(v))} \equiv s_2(v, \boldsymbol{U}_l), & v_l < v \\ R_1: & u - u_l = \int_{v_l}^{v} \sqrt{-p(y)_x} dy \equiv r_1(v, \boldsymbol{U}_l), & v_l < v \\ R_2: & u - u_l = \int_{v_l}^{v} \sqrt{-p(y)_x} dy \equiv r_2(v, \boldsymbol{U}_l), & v_l > v \end{array}$$

where the subindexes (1,2) correspond to the back wave and the front wave. See pictures (3.1), (3.3) and (3.2), (3.4) correspondingly.[2] With this equations we can depict the states graph as shown in Figure 3.5 to analyze our possible solutions.

Where depending on the region where $\boldsymbol{U}_r$ lies, is the way we will connect the constant states.

If we define

$$\begin{array}{l} S_i(\overline{\boldsymbol{U}}) = \{(v, u) \, : \, u = s_i(v, \overline{\boldsymbol{U}})\}, \quad i = 1, 2 \\ R_i(\overline{\boldsymbol{U}}) = \{(v, u) \, : \, u = r_i(v, \overline{\boldsymbol{U}})\} \end{array}$$

and

$$W_i = S_i(\overline{\boldsymbol{U}}) \cup R_i(\overline{\boldsymbol{U}}), \quad i = 1, 2$$

If we have a fixed $\boldsymbol{U}_l$we can consider the family of curves

[2]The procedure to reach and analyze this waves can be found in [1] Chapter 17 A

$$\mathcal{F} = \{W_2(\overline{\boldsymbol{U}}) \ : \ \overline{\boldsymbol{U}} \in W_1(\boldsymbol{U}_l)\}$$

to find the point $\overline{\boldsymbol{U}}$ such that the states can be connected as:

- Region I $\boldsymbol{U}_l \rightarrow^{R_1}\rightarrow \overline{\boldsymbol{U}} \rightarrow^{S_2}\rightarrow \boldsymbol{U_r}$
- Region II $\boldsymbol{U}_l \rightarrow^{S_1}\rightarrow \overline{\boldsymbol{U}} \rightarrow^{S_2}\rightarrow \boldsymbol{U_r}$
- Region III $\boldsymbol{U}_l \rightarrow^{S_1}\rightarrow \overline{\boldsymbol{U}} \rightarrow^{R_2}\rightarrow \boldsymbol{U_r}$
- Region IV $\boldsymbol{U}_l \rightarrow^{R_1}\rightarrow \overline{\boldsymbol{U}} \rightarrow^{R_2}\rightarrow \boldsymbol{U_r}$

The fourth Region does not allow us to connect all the possible states of $\boldsymbol{U}_r$ with $\boldsymbol{U}_l$ since there is a a point where vacuum is generated, i.e. $\rho < 0$. This needs another approach to solve it, which we will not discuss in here because our system has always strictly positive density of ions in ρ_l and ρ_r .

3.1.4 Euler Equations for Gas Dynamics

Since the Riemann problem is very useful for the understanding of hyperbolic partial differential equations like the Euler equations because all its properties, such as shocks and rarefaction waves, appear as characteristics in the solution, we are concern about this subject.

The problem of our concern can be regard with an analogy as a Gas Dynamics problem, where the motion of the ions resembles the motion of gases, which can be described using the Euler equations.

Therefore we present the Euler equations.

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + uu_x + p_x\rho = 0 \\ s_t + us_x = 0 \end{cases} \tag{3.3}$$

Where ρ is the density, u is the velocity, p is the pressure and s is the entropy.

The latter equations are the Mass, Momentum and Energy conservation laws.

Hence the Euler equations can be represented as (3.2) with

$$\mathbf{U} = \begin{pmatrix} \rho \\ u \\ s \end{pmatrix} \qquad \text{and} \qquad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} u \\ p_\rho/\rho \\ 0 \end{pmatrix}$$

Obtaining the following results[3].

	$k=1$	$k=2$	$k=3$
λ_k	$u-c$	u	$u+c$
r_k	$(\rho,-c,0)^t$	$(p_s,0,p_\rho)^t$	$(p,c,0)^t$
$\nabla\lambda_k \cdot r_k$	$c-pc_\rho$	0	$c+pc_\rho$
Riemann Invariants	$\{s, u+\frac{2}{\gamma-1}-c\}$	$\{u,p\}$	$\{s, u-\frac{2}{\gamma-1}c\}$

where c is called the *sound speed*, h the *enthalpy* and γ the *adiabatic gas constant* and they satisfy $c^2 = \frac{\gamma p}{\rho} = \frac{\partial h}{\partial \rho}$, $\gamma > 1$, $h = h(\rho, s)$ and $h_\rho = c/\rho$.

We should describe our problem as a Riemann problem due to the difference of density in ions across both sides of the cell membrane. The Euler equations describe our passive transport (Ion channels) and an added source term will represent the active transport (Ion pumps).

[3]See the Appendix for an insight of the computations

3.2 Mathematical Model

3.2.1 A model for ionic channels: Hydrodynamic equation

The purpose of this section is to present a model for the ion channels as seen in [4]. In this paper the authors describe the flux of particle across the membrane assuming open channels and without ionic pumps. The model is analogous to the ones used in semiconductor devices, but the carriers moving are ions instead of electrons. This model consist on four coupled equations, which describe the interaction of mass, momentum, energy and the electric potential that give rise to the ion movement inside the channel. We now present the equations and also the scaling of the Hydrodynamic system in one-dimensional case. The goal of scaling the equations is to give a glance how this problems are easier to be treated numerically in order to do a later comparison and analysis of the results against the well know electronic case.

As seen in [4] this can give us a deeper insight of the problem since we can find out the behavior of the ions while they are crossing the ion channel, specially the temperature, fact on which we are interested because a rise in the temperature is awaited modifying the energy. Being that the we want to insert to the model the Ion pumps (as added sources), the energy is also affected owing it is an active transport.

We expect having a good understanding of this process can help us to develop a model for the ion transport in the cell, where the Euler equations conserve the overall energy of the system.

Now, we will start by putting the system in its original way, i.e. non scaled:

1. Conservation of Particles.
$$n_t + (nv)_x = 0 \tag{3.4}$$
2. Conservation of Momentum.
$$p_t + (pv + nk_BT)_x = enE - (p/\tau_p) \tag{3.5}$$
3. Conservation of Energy.
$$\omega_t + (v\omega + nvk_BT)_x = envE - \frac{\omega - \frac{3}{2}nk_BT_0}{\tau_\omega} + (\kappa nT_x)_x \tag{3.6}$$

4. Poisson Equation.

$$-\lambda\phi_{xx} = e(n + n_D) \tag{3.7}$$

This Hydrodynamic theory is a combination of the Poisson and Euler field Equations of electrostatics and fluid dynamics, therefore they describe the diffusive and convective flow of mass, heat and charge taking into account the density of the charges, temperature changes and electrical potential gradients, properties and variables present in the Ion transport in the Ion channels.

As we know the Ion channels are porous proteins inserted across cell membranes that translocate ions selectively from one side of the membrane to the other. Ionic movement has long been modeled in two traditions, that of diffusion theory and rate theory, where diffusion theory in particular, and both theories in general, usually assume isothermal systems, i.e. systems with constant temperature everywhere.

The movement of holes and electrons in semiconductors has been modeled by diffusion theory for nearly 50 years, assuming constant temperature in most cases, and for at least 45 years it has been clear that local temperature changes occur and produce important phenomena, such "hot electron" phenomena.

Rate theory can describe flux over determined potential barriers. This is also done by a combination of drift and diffusion obeying the Poisson and Nemst-Planck equations simultaneously, called the PNP theory, in order to describe the permeation of ions across channels. PNP theory assumes i) frequent collisions of ions with their surroundings and ii) the collisions will not significantly change the temperature, therefore this theory cannot deal with biological transport when energy is directly involved, for example, active transport. Semiconductor device theory also has its analog of the PNP model, termed the self-consistent drift-diffusion model.

We investigate the role of energy exchange within the channel in ion permeation by studying the natural extension of PNP theory, which is called the hydrodynamic model in the literature of the Boltzmann transport equation. The justification for this is properly explained in [4]

The hydrodynamic model began to be utilized in the mid-1980s. Computations with this model are considerably more involved than those of simpler models such as the PNP equations because the model allows a wide range of behavior it includes possible shock waves and propagating disturbances,

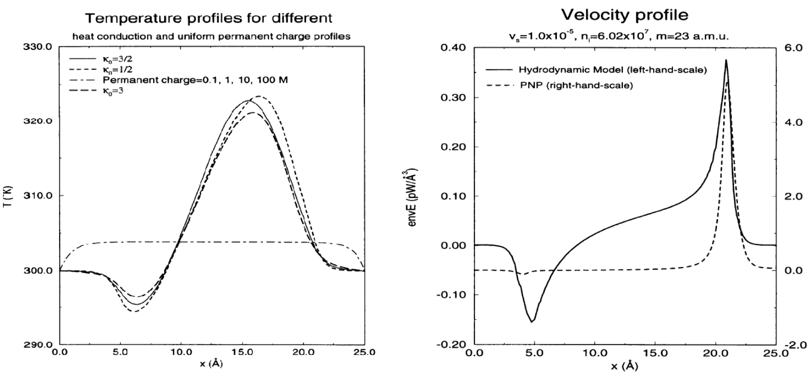


Figure 3.6: Results obteined from [4]

indeed much of the behavior of fluids. We should also remember that this behaviors is what we are looking for to introduce into the model the Ion pump, which is planned to be described as a static shock.

From our system the equation (3.4) says that the concentration of particles changes with time solely as the result of drift (flow). The equation (3.5) states the conservation of momentum, and the last equation (3.6) is the conservation of energy. The collisions in the last two equations are approximated as relaxations to values of the equilibrium state. Establishing the analogy to the Ion transport in the Ion channels the third term in (3.5) is equal to the change of momentum that is due to the pressure gradient: the mechanical force, contributed from ion-ion interactions and the thermal motion of ions. On the right-hand sides, the first term is the electrical force and the second term is the frictional force (due to ion-channel and ion-water interaction). The electrical force arises from all the charges in the system, viz., 1) the charge applied to the baths that sustains the (externally applied) transmembrane potential, 2) the permanent charge on the channel protein, 3) the mobile charge (ions) in the channel's pore, and 4) induced (i.e., polarization) charge of the several dielectrics. In (3.6) the third term on the left-hand side equals the (local accumulation of) work done by the mechanical force. On the right-hand side the first term is the input of electrical energy into mobile ions by the electrical force, the second term is energy loss because of the frictional force, and the last term is the heat flux into the system. In the appendix is shown the scalling of this model, this is made because is simplifies the numerical simulations.
In the Figure () its shown some of the results obteined in [4], where we can see the effect of the temperature should be considered inside the transport of ions in the channel, and how it affects the velocity.

In the left graph of Figure (3.6) we can see the temperature profiles for different heat conduction when 100-mV transmembrane potential is applied. The solid curve is the case when the thermoconductivity coefficient of the ions is $\kappa = 3/2$ (the ideal value), the short-dashed curve is for $\kappa = 1/2$, and the long-dashed curve is for $\kappa = 3$. The 67% reduction and the doubling of heat conduction make a negligible difference in the temperature profiles, which implies that the ion permeation and heat generation are electrically dominated processes. Results with uniform permanent charges of 0.1, 1, 10, and 100 M are shown by the dotted-dashed curve when $\kappa = 3/2$. The temperature rise is greatly suppressed without the acceleration of ions by the variations of the electric field. This again confirms that heat generation is an electrically dominated process.

In the right graph of Figure (3.6) shows the local electrical energy input (solid curve) when 100-mV transmembrane potential is applied. The first term in the energy equation, the local input of electrical energy, is shown to demonstrate that the temperature rise of sodium ions is due to the exchange of the sodium-ion thermal energy and the electrical energy of the applied electric potential. The result from PNP[4] is plotted as a dashed curve according to the scale on the right-hand side.

[4]See [4]

3.2.2 Description of the Model

In this section we will present a mathematical model for the transport in the ion channel as seen previously adding now an ion pump. Lets consider a one dimensional case, and let the ion pump be on the x axis, such that the interface between the outer cell and the inner cell is exactly on $x = 0$, moreover, let $x < 0$ (region 1) have a density ρ_l and $x > 0$ (region 2) a density ρ_r. The action of the ion pump is to take one ion from region 1 and put it in region 2, so, it is like if we have a point-wise source exactly an $x = 0$. First we will start by describing the P-system with source terms as in [10], at the beginning it will be assumed that the problem has a solution, and describe a method to solve analytically the Riemann problem, and what to expect in each of the four regions. The following part its a more rigorous proof of the existence of solutions using a self similar viscosity approach.

3.2.3 P-System

In this section we will analyze our model proposition to have a clearer idea by using a P-System. The system is as follows:

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + (P(v))_x &= 0 \end{aligned}$$

with

$$P(v) = \frac{k}{v^\gamma} \tag{3.8}$$

$v = \frac{1}{\rho}$, $1 < \gamma$, and k a positive constant. This is an example of a P-system.

Previously we have deduced the 1 and 2 shockwave and rarefaction curves. The shock curves are given by

$$S1: \qquad u - u_l = -\sqrt{(v - v_l)(P(v_l) - P(v))} \qquad v < v_l$$

$$S2: \qquad u - u_l = -\sqrt{(v - v_l)(P(v_l) - P(v))} \qquad v_l < v$$

and the rarefaction curves are given by:

$$R1 \qquad u - u_l = \int_{v_l}^{v} \sqrt{-\frac{dP}{dv}} dy \qquad v < v_l$$

$$R2 \qquad u - u_l = \int_{v_l}^{v} \sqrt{-\frac{dP}{dv}} dy \qquad v_l < v$$

Lets compute explicitly the integral above:

$$\int_{v_l}^{v} \sqrt{-\frac{dP(y)}{dy}} dy = \int_{v_l}^{v} \sqrt{-\left(-\frac{\gamma k}{y^{\gamma+1}}\right)} dy =$$

$$\int_{v_l}^{v} \sqrt{\frac{\gamma k}{y^{\gamma+1}}} dy = \sqrt{\gamma k} \int_{v_l}^{v} \sqrt{\frac{1}{y^{\gamma+1}}} dy =$$

$$\sqrt{\gamma k} \int_{v_l}^{v} (y)^{-\frac{1}{2}(\gamma+1)} \, dy = \frac{\sqrt{\gamma k}}{\left(\frac{1}{2} - \frac{\gamma}{2}\right)} \, (y)^{-\frac{1}{2}\gamma + \frac{1}{2}} \, |_{v_l}^{v}$$

$$= \frac{2\sqrt{k\gamma}}{(1-\gamma)} \left[v^{\frac{1}{2} - \frac{\gamma}{2}} - v_l^{\frac{1}{2} - \frac{\gamma}{2}} \right]$$

So, in summary we have the shock and rarefaction curves:

$$S1: \qquad u - u_l = -\sqrt{(v - v_l)(P(v_l) - P(v))} \qquad v < v_l \tag{3.9}$$

$$S2: \qquad u - u_l = -\sqrt{(v - v_l)(P(v_l) - P(v))} \qquad v_l < v \tag{3.10}$$

$$R1: \qquad u - u_l = \frac{2\sqrt{k\gamma}}{(1-\gamma)} \left[v^{\frac{1}{2} - \frac{\gamma}{2}} - v_l^{\frac{1}{2} - \frac{\gamma}{2}} \right] \qquad v < v_l \tag{3.11}$$

$$R2: \qquad u - u_l = \frac{2\sqrt{k\gamma}}{(1-\gamma)} \left[v^{\frac{1}{2} - \frac{\gamma}{2}} - v_l^{\frac{1}{2} - \frac{\gamma}{2}} \right] \qquad v_l < v \tag{3.12}$$

Now we have to analyze our specific problem, i.e. the same P-system with the pointwise source terms (representing the ion pump), and see what happens with the jump conditions, and how they are related to the two other intermediate states.

We consider the system

$$v_t - u_x = \alpha_1 (H(x))_x \tag{3.13}$$
$$u_t + p(v)_x = \alpha_2 (H(x))_x \tag{3.14}$$

coupled with the initial data (Riemann Problem)

$$U(u(x,0), v(x,0)) = \begin{cases} U_l & x < 0 \\ U_r & x > 0 \end{cases} \tag{3.15}$$

with H the Heaviside function, α_i , $i = 1, 2$ positive reals. This can be written in a conservative form a

$$\begin{aligned} v_t + (-u_x &- \alpha_1 H)_x = 0 \\ u_t + (p(v) &- \alpha_2 H)_x = 0 \end{aligned}$$

So in this form, the jump conditions are as follows

$$s\,[v] = [-u - \alpha_1 H] \tag{3.16}$$

$$s\,[u] = [p(v) - \alpha_2 H] \tag{3.17}$$

Where the brackets mean difference relative to the discontinuity. By the nature of the physical process been studied, we need to force a discontinuity, i.e. a steady shock wave owing the Ion pumps as stated before are located at $x = 0$. We do this by saying that is a shock with velocity $s = 0$ this gives

$$[-u - \alpha_1 H] = 0$$

$$[p(v) - \alpha_2 H] = 0$$

Expanding the first we get

$$-u_l - (-u - \alpha_1) = 0$$

so

$$u - u_l = -\alpha_1$$

Expanding the second one

$$p(v_l) - p(v) + \alpha_2 = 0$$

So we have the relations

$$u - u_l = -\alpha_1 \tag{3.18}$$

$$\frac{k}{v_l^\gamma} - \frac{k}{v^\gamma} = -\alpha_2 \tag{3.19}$$

We are expecting two new states, that we will call $U_- = (v_-, u_-)^t$and $U_+ = (v_+, u_+)^t$, connected by the contact discontinuity. Moreover if we have a left state U_l and a right state U_r, then the left state will connect to the minus state, then this to the plus state, and after the plus state with the right state, schematically:

$$U_l \to U_- \to U_+ \to U_r$$

Where the first connection can be done by a $S1$ or a $R1$, and the last connection can be done by $S2$ and $R2$. All this possibilities depend on the region where U_r lies. See previous chapters. Lets analyze by regions:

3.2.3.1 Region I

In this region the state U_l is connected to $U_-$by a $R1$, then the contact discontinuity, and then $U_+$with U_r by a $S2$. And we have the next system of equations

$$\begin{cases} u_- - u_l = \frac{2\sqrt{k\gamma}}{(1-\gamma)}\left[v_-^{\frac{1}{2}-\frac{\gamma}{2}} - v_l^{\frac{1}{2}-\frac{\gamma}{2}}\right] \\ u_+ - u_- = -\alpha_1 \\ \frac{k}{v_-^\gamma} - \frac{k}{v_+^\gamma} = -\alpha_2 \\ u_r - u_+ = -\sqrt{(v_r - v_+)(\frac{k}{v_+^\gamma} - \frac{k}{v_r^\gamma})} \end{cases} \tag{3.20}$$

Summing the first and last equation and using the secondone we get

$$u_- - u_l + u_r - u_+ = \frac{2\sqrt{k\gamma}}{(1-\gamma)}\left[v_-^{\frac{1}{2}-\frac{\gamma}{2}} - v_l^{\frac{1}{2}-\frac{\gamma}{2}}\right] - \sqrt{(v_r - v_+)\left(\frac{k}{v_+^\gamma} - \frac{k}{v_r^\gamma}\right)}$$

$$\alpha_1 + u_r - u_l = \frac{2\sqrt{k\gamma}}{(1-\gamma)}\left[v_-^{\frac{1}{2}-\frac{\gamma}{2}} - v_l^{\frac{1}{2}-\frac{\gamma}{2}}\right] - \sqrt{(v_r - v_+)\left(\frac{k}{v_+^\gamma} - \frac{k}{v_r^\gamma}\right)}$$

from the third equation we find that

$$\frac{k}{v_+^\gamma} = \frac{k}{v_-^\gamma} + \alpha_2$$

and

$$v_+ = \left(\frac{kv_-^\gamma}{k + v_-^\gamma \alpha_2}\right)^{\frac{1}{\gamma}}$$

substituting this we get an expression of v_- in terms of known parameters $\left(\beta = \frac{2\sqrt{k\gamma}}{(1-\gamma)}\right)$

$$\alpha_1+u_r-u_l = \beta\left[v_-^{\frac{1}{2}-\frac{\gamma}{2}} - v_l^{\frac{1}{2}-\frac{\gamma}{2}}\right] - \sqrt{\left(v_r - \left(\frac{kv_-^\gamma}{k+v_-^\gamma\alpha_2}\right)^{\frac{1}{\gamma}}\right)\left(\frac{k}{v_-^\gamma}+\alpha_2-\frac{k}{v_r^\gamma}\right)} \tag{3.21}$$

The solution of this equation (remember that we know u_l, u_r, k and γ), if exists, gives the value(s) of v_i and from the second equation of (3.20) we find v_+, then using the other two equations we find $u_+$and u_-. In the case that we have more that one solution of v_i we have to check if it satisfy the whole set of equations.

3.2.3.2 Region II

In this region the state U_l is connected to U_- by a $S1$, then the contact discontinuity, and then U_+ with U_r by a $S2$. And we have the next system of equations

$$\begin{cases} u_- - u_l = -\sqrt{(v_- - v_l)(\frac{k}{v_l^\gamma} - \frac{k}{v_-^\gamma})} \\ u_+ - u_- = -\alpha_1 \\ \frac{k}{v_-^\gamma} - \frac{k}{v_+^\gamma} = -\alpha_2 \\ u_r - u_+ = -\sqrt{(v_r - v_+)(\frac{k}{v_+^\gamma} - \frac{k}{v_r^\gamma})} \end{cases} \tag{3.22}$$

Summing the first and last equation and using the second one we get

$$u_- - u_l + u_r - u_+ = -\sqrt{(v_- - v_l)\left(\frac{k}{v_l^\gamma} - \frac{k}{v_-^\gamma}\right)} - \sqrt{(v_r - v_+)\left(\frac{k}{v_+^\gamma} - \frac{k}{v_r^\gamma}\right)}$$

$$\alpha_1 + u_r - u_l = -\sqrt{(v_- - v_l)\left(\frac{k}{v_l^\gamma} - \frac{k}{v_-^\gamma}\right)} - \sqrt{(v_r - v_+)\left(\frac{k}{v_+^\gamma} - \frac{k}{v_r^\gamma}\right)}$$

from the third equation we find that

$$\frac{k}{v_+^\gamma} = \frac{k}{v_-^\gamma} + \alpha_2$$

and

$$v_+ = \left(\frac{kv_-^\gamma}{k + v_-^\gamma \alpha_2}\right)^{\frac{1}{\gamma}}$$

substituting this we get an expression of v_- in terms of known parameters

$$\alpha_1 + u_r - u_l = -\sqrt{(v_- - v_l)(\frac{k}{v_l^\gamma} - \frac{k}{v_-^\gamma})} - \sqrt{\left(v_r - \left(\frac{kv_-^\gamma}{k + v_-^\gamma \alpha_2}\right)^{\frac{1}{\gamma}}\right)\left(\frac{k}{v_-^\gamma} + \alpha_2 - \frac{k}{v_r^\gamma}\right)} \tag{3.23}$$

The solution of this equation, if exists, gives the value(s) of v_i and from the second equation of (3.22) we find v_+, then using the other two equations we find u_+ and u_- . In the case that we have more that one solution of $v_i$we have to check if it satisfy the whole set of equations.

3.2.3.3 Region III

In this region the state U_l is connected to U_- by a $S1$, then the contact discontinuity, and then U_+ with U_r by a $R2$. And we have the next system of equations

$$\begin{cases} u_- - u_l = -\sqrt{(v_- - v_l)(\frac{k}{v_l^\gamma} - \frac{k}{v_-^\gamma})} \\ u_+ - u_- = -\alpha_1 \\ \frac{k}{v_-^\gamma} - \frac{k}{v_+^\gamma} = -\alpha_2 \\ u_r - u_+ = \frac{2\sqrt{k\gamma}}{(1-\gamma)}\left[v_+^{\frac{1}{2}-\frac{\gamma}{2}} - v_r^{\frac{1}{2}-\frac{\gamma}{2}}\right] \end{cases} \tag{3.24}$$

Summing the first and last equation and using the second one we get

$$u_- - u_l + u_r - u_+ = -\sqrt{(v_- - v_l)\left(\frac{k}{v_l^\gamma} - \frac{k}{v_-^\gamma}\right)} + \beta\left[v_+^{\frac{1}{2}-\frac{\gamma}{2}} - v_r^{\frac{1}{2}-\frac{\gamma}{2}}\right]$$

$$\alpha_1 + u_r - u_l = -\sqrt{(v_- - v_l)\left(\frac{k}{v_l^\gamma} - \frac{k}{v_-^\gamma}\right)} (+ \beta\left[v_+^{\frac{1}{2}-\frac{\gamma}{2}} - v_r^{\frac{1}{2}-\frac{\gamma}{2}}\right]$$

from the third equation we find that

$$\frac{k}{v_+^\gamma} = \frac{k}{v_-^\gamma} + \alpha_2$$

and

$$v_+ = \left(\frac{kv_-^\gamma}{k + v_-^\gamma \alpha_2}\right)^{\frac{1}{\gamma}}$$

so

$$v_+^{(\frac{1}{2}-\frac{\gamma}{2})} = \left(\frac{kv_-^\gamma}{k + v_-^\gamma \alpha_2}\right)^{\frac{1}{\gamma}(\frac{1}{2}-\frac{\gamma}{2})}$$

and substituting in the equation for v_- we get

$$\alpha_1 + u_r - u_l = -\sqrt{(v_- - v_l)\left(\frac{k}{v_l^\gamma} - \frac{k}{v_-^\gamma}\right)} + \beta\left[\left(\frac{kv_-^\gamma}{k + v_-^\gamma \alpha_2}\right) - v_r^{\frac{1}{2}-\frac{\gamma}{2}}\right] \quad (3.25)$$

The solution of this equation, if exists, gives the value(s) of v_i and from the second equation of (3.24) we find v_+, then using the other two equations we find $u_+$and u_-. In the case that we have more that one solution of $v_i$we have to check if it satisfy the whole set of equations.

3.2.3.4 Region IV

In this region the state U_l is connected to U_- by a $R1$, then the contact discontinuity, and then U_+ with U_r by a $R2$. And we have the next system of equations

$$\begin{cases} u_- - u_l = \frac{2\sqrt{k\gamma}}{(1-\gamma)}\left[v_-^{\frac{1}{2}-\frac{\gamma}{2}} - v_l^{\frac{1}{2}-\frac{\gamma}{2}}\right] \\ u_+ - u_- = -\alpha_1 \\ \frac{k}{v_-^\gamma} - \frac{k}{v_+^\gamma} = -\alpha_2 \\ u_r - u_+ = \frac{2\sqrt{k\gamma}}{(1-\gamma)}\left[v_+^{\frac{1}{2}-\frac{\gamma}{2}} - v_r^{\frac{1}{2}-\frac{\gamma}{2}}\right] \end{cases} \quad (3.26)$$

Summing the first and last equation and using the second one we get

$$u_- - u_l + u_r - u_+ = \beta\left[v_-^{\frac{1}{2}-\frac{\gamma}{2}} - v_l^{\frac{1}{2}-\frac{\gamma}{2}}\right] + \beta\left[v_+^{\frac{1}{2}-\frac{\gamma}{2}} - v_r^{\frac{1}{2}-\frac{\gamma}{2}}\right]$$

$$\alpha_1 + u_r - u_l = \beta\left[v_-^{\frac{1}{2}-\frac{\gamma}{2}} - v_l^{\frac{1}{2}-\frac{\gamma}{2}}\right] + \beta\left[v_+^{\frac{1}{2}-\frac{\gamma}{2}} - v_r^{\frac{1}{2}-\frac{\gamma}{2}}\right]$$

from the third equation we find that, as before

$$v_+^{(\frac{1}{2}-\frac{\gamma}{2})} = \left(\frac{kv_-^\gamma}{k + v_-^\gamma \alpha_2}\right)^{\frac{1}{\gamma}(\frac{1}{2}-\frac{\gamma}{2})}$$

so substituting we get

$$\alpha_1 + u_r - u_l = \beta\left[v_-^{\frac{1}{2}-\frac{\gamma}{2}} - v_l^{\frac{1}{2}-\frac{\gamma}{2}}\right] + \beta\left[\left(\frac{kv_-^{\gamma}}{k+v_-^{\gamma}\alpha_2}\right) - v_r^{\frac{1}{2}-\frac{\gamma}{2}}\right]$$

The solution of this equation, if exists, gives the value(s) of v_i and from the second equation of (3.26) we find v_+, then using the other two equations we find $u_+$and u_-. In the case that we have more that one solution of $v_i$we have to check if it satisfy the whole set of equations.

So at the end if the solutions of the previous problems exist and are unique, we have the states U_- and U_+ that connect the known states U_l and U_+ , so with this we know the solution of our Riemann problem.

3.2.3.5 Geometrical solution of the Riemann problem for $p(v) = \frac{1}{v}$

In the following section we will present the analytical results for the case for the pressure law with $\gamma = 1$ and $k = 1$ as the equation (3.8) as presented in [10]. So

$$p(v) = \frac{1}{v} \tag{3.27}$$

The jump condition for a steady shock i.e. $s = 0$, the pressure gives

$$[p(v)] = \alpha_2 \tag{3.28}$$

we conclude that

$$\frac{1}{v^+} = \frac{1}{v^-} + \alpha_2, \tag{3.29}$$

or

$$v_0^+ = \frac{v_0^-}{1+\alpha_2 v_0^-}. \tag{3.30}$$

where α_2 is such that $1 + \alpha_2 v_0^+ > 0$ and $v^- = \lim_{x\to 0^-} v(x)$ and $v^+ = \lim_{x\to 0^+} v(x)$. From (3.18) we have

$$u_0^+ = u_0^- + \alpha_1 \tag{3.31}$$

where $u^- = \lim_{x\to 0^-} u(x)$ and $u^+ = \lim_{x\to 0^+} u(x)$.
We look for a (unique) solution of the Riemann Problem that verifies the interface conditions (3.28) and (3.18). We consider the $(u, v) - \textit{space}$ and a

given state $U_l = (u_l, v_l)$ as in the Figure (3.5).

We have two intermediate state $U_0^- = (u_0^-, v_0^-)$ and $U_0^+ = (u_0^+, v_0^+)$ connected by a contact discontinuity. The intermediate states (U_0^-, U_0^+) are given by the solutions of the following system

$$\begin{aligned} u_0^+ &= u_0^- + \alpha_1 && (3.32)\\ v_0^+ &= \frac{v_0^-}{1+\alpha_2 v_0^-} && (3.33)\\ u_0^- &= u_l + \theta(v_0^-, v_l) && (3.34)\\ u_0^+ &= u_r - \theta(v_0^+, v_r) && (3.35) \end{aligned}$$

where

$$\theta(v,w) = \begin{cases} \int_v^w \sqrt{-p'(s)}ds & v < w \\ -\sqrt{(v-w)^2/vw} & v > w \end{cases} \qquad (3.36)$$

we eliminate the unknowns (u_0^+, v_0^+) (or (u_0^-, v_0^-)) we get

$$\begin{aligned} u_0^- &= u_l + \theta(v_0^-, v_l) && (3.37)\\ u_0^- + \alpha_1 &= u_r - \theta(v_0^+, v_r) && (3.38) \end{aligned}$$

The equations that conect the states in this region are:

Region I

$$\begin{aligned} u_0^- &= u_l + \int_{v_l}^{v_0^-} \sqrt{-p'(s)}ds = u_l + \ln \frac{v_0^-}{v_l} \\ & \quad v_0^- > v_l && (3.39)\\ u_0^- &= u_r - \alpha_1 + \sqrt{(v_r - (v_0^-/(1+\alpha_2 v_0^-)))^2/(v_0^-/(1+\alpha_2 v_0^-))v_r} \\ & \quad v_0^- < v_r/(1-\alpha_2 v_r) && (3.40) \end{aligned}$$

Remark 1. *An easy calculation shows $v_0^- > v_l$ and $v_0^- < v_r/(1-\alpha_1 u_r)$ imply*

$$0 < u_r - u_l < \ln \frac{v_m}{v_l} + \gamma_2$$

where $v_m = v_r/(1-\alpha_2 v_r)$. We also remark that if $(1-\alpha_2 v_r) < 0$ the system (-) does not admit solution for positive (and the physical reasonable values of v_0^+).

Region II

$$
\begin{aligned}
u_0^- &= u_l + \sqrt{(v_0^- - v_r)^2/v_0^- v_r} \\
&\quad v_0^- < v_l \qquad (3.41) \\
u_0^- &= u_r - \alpha_1 + \sqrt{((v_0^-/(1+\alpha_2 v_0^-)) - v_r)^2/(v_0^-/(1+\alpha_2 v_0^-))u_r} \\
&\quad v_0^- < v_r/(1-\alpha_2 v_r) \qquad (3.42)
\end{aligned}
$$

Region III

$$
\begin{aligned}
u_0^- &= u_l + \sqrt{(v_0^- - v_r)^2/v_0^- v_r} \\
&\quad v_0^- > v_l \qquad (3.43) \\
u_0^- &= u_r - \alpha_1 + \int_{\frac{u_0^-}{1+\alpha_2 v_0^-}}^{v_r} \sqrt{-p'(s)}ds = u_l + \ln \frac{v_r}{\frac{v_0^-}{1+\alpha_2 v_0^-}} \\
&\quad v_0^- > v_r/(1-\alpha_2 v_r) \qquad (3.44)
\end{aligned}
$$

Region IV

$$
\begin{aligned}
u_0^- &= u_l + \int_{v_l}^{v_0^-} \sqrt{-p'(s)}ds = u_l + \ln \frac{v_0^-}{v_l} \\
&\quad v_0^- > v_l \qquad (3.45) \\
v_0^- &= v_r - \chi_1 + \int_{\frac{v_0^-}{1+\alpha_2 v_0^-}}^{v_r} \sqrt{-p'(s)}ds = u_l + \ln \frac{v_r}{\frac{v_0^-}{1+\alpha_2 v_0^-}} \\
&\quad v_0^- > v_r/(1-\alpha_2 v_r) \qquad (3.46)
\end{aligned}
$$

3.2.3.6 Self-similar viscosity approach. [11]

As mentioned before we know that under the assumption $p_v(v) < 0$ the system (3.13-3.14) is strictly hyperbolic and it admits characteristic velocities $\lambda_1 = -\sqrt{-p_v(v)}$ and $\lambda_2 = +\sqrt{-p_v(v)}$.

We introduce in (3.14) a special form of the so called *self-similar viscosity* and we obtain:

$$
\begin{aligned}
v_t - w_x &= 0 \qquad (3.47) \\
u_t + p(v)_x &= \varepsilon t \left(\frac{1}{u} w_x\right)_x + \alpha_2 (H(x))_x. \qquad (3.48)
\end{aligned}
$$

where $H = H(x)$ is the Heavyside function and we introduce the change of variable $w \equiv u + \alpha_1 H$. It is not difficult to prove that (3.47)-(3.48) is self-similar, i.e it preserve the invariance under dilatation of coordinates. $(x, t) \to (ax, at)$, $a > 0$ and the solutions of the Riemann problem have form $\left(u\left(\frac{x}{t}\right), v\left(\frac{x}{t}\right)\right)$. This suggest that it is possible write our system as a function of a single variable $\xi = x/t$ such that $-\infty < \xi < +\infty$

$$-\xi v' - w' = 0 \tag{3.49}$$

$$-\xi u' + p(v)' = \varepsilon\left(\frac{1}{u}w'\right)' + \alpha_2 H'(\xi) \tag{3.50}$$

coupled with the following boundary conditions[5]

$$\begin{aligned} u(\pm\infty) = u_\pm \\ v(\pm\infty) = v_\pm, \end{aligned} \tag{3.51}$$

where ε is a positive given constant.

Properties of the solution

From the equations (3.49) and (3.50) and their structure we can divide our problem into two parts the first for $\xi < 0$ and the second for $\xi > 0$ separated by the singular point $\xi = 0$. The properties of the solutions at this point will be essential to understand the rule of the localized source (or well) in this kind of problems.

3.2.3.7 Weak solutions

Let $(w(\xi), v(\xi))$ be such that $(v, p(v)) \in (L^2_{loc}(\mathbb{R}))^3$ and assume $w(\xi) \in H^1_{loc}(\mathbb{R})$, that implies $\frac{1}{v}w' \in L^2_{loc}$. It follows that the couple (w, v) is weak solution of (3.49-3.51) if for all $\phi \in C^1_0(\mathbb{R})$, that verifies the boundary conditions (3.51):

$$\int (\zeta v - w)\phi' d\zeta + \int v\phi d\zeta = 0 \tag{3.52}$$

$$\int (\zeta w - (p(v) - \alpha_2 H) + \varepsilon\frac{1}{v}w')\phi' d\zeta + \int w\phi d\zeta = 0 \tag{3.53}$$

In the following theorem we derive some properties of the solution in particular on the singular point $\xi = 0$.

Theorem 1. *Let (u, v) be a solution of (3.49-3.51) then*

[5]The ' denotes derivative respect to ξ

i) *ξv and w are continuous on $\mathbb{R}$*

ii) *(u,v) verifies the*

$$[\xi v(\xi) + w(\xi)]_a^b - \int_a^b v(\zeta)d\zeta = 0 \tag{3.54}$$

$$[\xi u(\xi) - (p(v(\xi)) - \alpha_2 H(\xi)) + \varepsilon w'(\xi)]_a^b$$
$$- \int_a^b u(\zeta)d\zeta = 0 \tag{3.55}$$

iii) *$lim_{\xi \to \pm 0} \left[\xi w(\xi) - (p(v(\xi)) - \alpha_2 H(\xi)) + \varepsilon \frac{1}{v} w'(\xi)\right]$ exist and are equal, and then $\left[\xi w(\xi) - (p(v(\xi)) - \alpha_2 H(\xi)) + \varepsilon \frac{1}{v} w'(\xi)\right]$ is continuous in $\mathbb{R} - \{0\}$*

Proof. **i)** We observe that w is continuous function. By using the weak formulation (3.52) we deduce that ξv has the same regularity has w and then i) is proved.

ii) We prove (3.54). Following **[5]** we use, as test function in (3.52)

$$\psi_n(\xi) = \begin{cases} 0, \ -\infty < \xi \le a - 1/n \\ n\left[\xi - (a - 1/n)\right], a - 1/n \le \xi \le a \\ 1, \ a \le \xi \le b \\ -n\left[\xi - (b + 1/n)\right], \ b \le \xi \le b + 1/n \\ 0, \ b + 1/n \le \xi < +\infty \end{cases} \tag{3.56}$$

for a given $a, b \in \mathbb{R}$. (3.56) is not a continuous function and we use the C_0^1 regularization of $\psi_n(\xi)$ in our weak formulation

$$n \int_{a-1/n}^{a} (\zeta v - w) - n \int_{b}^{b+1/n} (\zeta v - w) + \int_{a-1/n}^{b+1/n} v \phi_n d(\zeta) = 0$$

Taking into account the limit $n \to 0$ of the equation above and using the Lebesgue Differentiation Theorem and Dominated Convergence Theorem we obtain (3.54). In analogous way we prove (3.55).

iii) This follows assuming $a \to 0^-$ and $b \to 0^+$ in (3.55).

□

Remark 2. *We note that $v(\xi)$ and $w'(\xi)$ are continuous in $\mathbb{R} - \{0\}$. And $w(\xi)$ is continous in $\mathbb{R}$.*

Theorem 2. *Let (u, v) be a solution of (3.49-3.51) such that*

$$0 < \delta \leq v(\xi) \leq \Delta \tag{3.57}$$

then we define

$$0 < a_0 \leq p(v) = \frac{1}{v} \leq A_0 \tag{3.58}$$

where the constants a_0, A_0 depend on δ and Δ. We set $\lambda_\pm = \sqrt{a_0}$ and $\Lambda_\pm = \sqrt{A_0}$.
Then for $0 < \xi < \alpha_+ < \lambda_+$

$$|w'(\xi)| \leq \frac{1}{v(\alpha_+)}|u'(\alpha_+)|\frac{1}{a_o}\left(\frac{\xi}{\alpha_+}\right)^{\frac{\lambda_+^2-\alpha_+^2}{\varepsilon A_0}} \tag{3.59}$$

for $\Delta_+ < \alpha_+ < \xi$

$$|u'(\xi)| \leq \frac{1}{v(\alpha_+)}|u'(\alpha_+)|\frac{1}{a_0}\exp\left\{-\frac{\alpha_+^2 - \Delta_+^2}{2\varepsilon A_0}\left(\left(\frac{\xi}{\alpha_+}\right)^2 - 1\right)\right\} \tag{3.60}$$

for $\xi < \alpha_- < \Lambda_-$

$$|w'(\xi)| \leq \frac{1}{v(\alpha_-)}|u'(\alpha_-)|\frac{1}{a_0}\left(\frac{\xi}{\alpha_-}\right)^{\frac{\lambda_-^2-\alpha_-^2}{\varepsilon A_0}} \tag{3.61}$$

for $\lambda_- < \alpha_- < \xi < 0$

$$|w'(\xi)| \leq \frac{1}{v(\alpha_-)}|u'(\alpha_-)|\frac{1}{a_0}\exp\left\{-\frac{\alpha_-^2 - \Delta_-^2}{2\varepsilon A_0}\left(\left(\frac{\xi}{\alpha_-}\right)^2 - 1\right)\right\} \tag{3.62}$$

Proof. First we observe that, for all $\xi \neq 0$, (w, v) satisfy

$$\varepsilon\left(\frac{\beta}{v}w'\right)' + \frac{\xi^2 - \frac{1}{v^2}}{\xi}w' = 0, \tag{3.63}$$

and integrate it with respect to ξ in the intervals (α_-, ξ) and (ξ, α_+) Thus we get for $u'(\xi)$:

$$w'(\xi) = \begin{cases} \frac{1/v(\alpha_+)u'(\alpha_+)}{1/v(\xi)}\exp\left\{-\frac{1}{\varepsilon}\int_\xi^{\alpha_+}\frac{\zeta^2-\frac{1}{v(\zeta)^2}}{\zeta/v(\zeta)}d\zeta\right\} & \xi > 0 \\ \frac{1/v(\alpha_-)u'(\alpha_-)}{1/v(\xi)}\exp\left\{-\frac{1}{\varepsilon}\int_\xi^{\alpha_-}\frac{\zeta^2-\frac{1}{v(\zeta)^2}}{\zeta/v(\zeta)}d\zeta\right\} & \xi < 0 \end{cases} \tag{3.64}$$

Consider now (3.59) and $0 < \xi < \alpha_+ < \lambda_+$. By using (3.57) for all $0 < \zeta < \alpha_+$

$$\frac{\zeta^2 - \frac{1}{v(\zeta)^2}}{\zeta} \leq \frac{\zeta^2 - \frac{1}{\Delta^2}}{\zeta} \leq -\frac{(\lambda_+^2 - \alpha_+^2)}{\zeta}$$

then

$$\exp\left\{-\frac{1}{\varepsilon}\int_{\xi}^{\alpha_+} \frac{\zeta^2 - \frac{1}{v(\zeta)^2}}{\zeta/v(\zeta)} d\zeta\right\} \leq \exp\left\{-\frac{(\lambda_+^2 - \alpha_+^2)}{\varepsilon A_0}\int_{\xi}^{\alpha_+} \frac{1}{\zeta} d\zeta\right\} = \left(\frac{\xi}{\alpha_+}\right)^{\frac{\lambda_+^2 - \alpha_+^2}{\varepsilon A_0}}$$

and in view of (3.64) we get (3.59). In the same way we prove (3.61). Now we consider (3.60) and then $0 < \Lambda_+ < \alpha_- < \xi$. We have

$$\frac{\zeta^2 - \frac{1}{v(\zeta)^2}}{\zeta} \geq \zeta - \frac{A_0}{\zeta} \geq \frac{(\alpha_+^2 - \Lambda_+^2)}{\alpha_+^2}\zeta > 0$$

then

$$\begin{aligned}\exp\left\{-\frac{1}{\varepsilon}\int_{\alpha_+}^{\xi} \frac{\zeta^2 - \frac{1}{v(\zeta)^2}}{\zeta/v(\zeta)} d\zeta\right\} &\leq \exp\left\{-\frac{(\alpha_+^2 - \Lambda_+^2)}{\alpha_+^2 \varepsilon A_0}\int_{\alpha_+}^{\xi} \zeta d\zeta\right\} \\ &= \exp\left\{-\frac{\alpha_+^2 - \Lambda_+^2}{2\varepsilon A_0}\left(\left(\frac{\xi}{\alpha_+}\right)^2 - 1\right)\right\}\end{aligned}$$

(3.60) follows by using the above inequality and in the same way is it possible prove (3.62). □

This theorem implies that under the assumption (3.57) the solution of our problem has the following properties:

1. $|w'(\xi)| \to 0$ and $|v'(\xi)| \to 0$ as $\xi \to \pm\infty$

2. $|w'(\xi)| = O(|\xi|^\alpha)$ $\xi \to 0$ and $|v'(\xi)| = O(|\xi|^{\alpha-1})$ as $\xi \to 0^\pm$, for some $\alpha > 0$.

This meas that $w'(0^+) = w'(0^-) = 0$ and by using the theorem 1

$$w(0^+) = w(0^-) \tag{3.65}$$

which implies

$$\begin{aligned} u(0^+) &= u(0^-) - \alpha_1 \\ p(v(0^+)) &= p(v(0^-)) + \alpha_2 \end{aligned}$$

in particular,we assume (3.27) and the then

$$\begin{aligned} u(0^+) &= u(0^-) - \alpha_1 \\ \frac{1}{v(0^-)} &= \frac{1}{v(0^+)} - \alpha_2. \end{aligned} \tag{3.66}$$

We note that (3.66) has an interesting physical explanation. The source term $\alpha_2 H'(\xi) = \alpha_2\delta(\xi)$ add or take away a particle from our system and it modifies locally the pressure of the system that reorganizes itself in order to get the equilibrium between left and right pressure.
In the paper [10] the autors prove the existence of solution for (3.49-3.51) using the Leray-Schauder degree theory. We summarize the result in a simplied version, without details.

For this we consider the space $X = C^0(-\infty, 0) \cup (0, +\infty)$ of the function continuous for each $\xi \neq 0$ such that there exist the left and right limit $(w(0^+), v(0^+))$ and $(w(0^-), v(0^-))$. Moreover $|w(0^+) - w(0^-)|$ and $|v(0^+) - v(0^-)|$ are bounded.

In this space we also define the norm $\|v\|_{L^\infty(-\infty,\infty)} < \infty$.
Let be Ω a subset of X

$$\Omega = \{v \in X : \delta \leq v(\xi) \leq \Delta\}. \tag{3.67}$$

and it verifies (3.65) and (3.66).

We construct two maps namely T such that $T : V(\xi) \to v(\xi)$ for all $\xi \in (-\infty, \infty)$. These operators are defined by solving the following problem

$$-\xi v'(\xi) - w'(\xi) = 0 \tag{3.68}$$

$$-\xi w' - \frac{1}{V(\xi)} v'(\xi) = \varepsilon\left(\frac{1}{V(\xi)} w'(\xi)\right)' \tag{3.69}$$

$$\begin{aligned} w(\pm\infty) = w_\pm(\mu) &:= w_- + \mu(w_\pm - w_-) \\ v(\pm\infty) = v_\pm(\mu) &:= v_- + \mu(v_\pm - v_-) \end{aligned} \tag{3.70}$$

respectively for $\xi < 0$ and $\xi > 0$ and for all $\mu \in [0, 1]$. T_1 and T_2 are connected at each step of the iterative process by using (3.65) and (3.66) where $V(\xi)$ is the solution of the linearized problem at the previous step of the iterative process.

3.2.3.8 *A-Priori* estimates

The goal of this section is prove the existence of the solutions for (3.49), (3.50) and (3.51) by using the iterative process defined in the previous section.

Consider the system (3.49) and (3.50) coupled with (3.51). We know that w, and then also v are strictly monotone in the intervals $(-\infty, 0) \cup (0, +\infty)$ or is a constant in one of these. The same holds for v and also for $p(v)$.
We distinguish five cases:

- C_1: v is increasing on $(-\infty, +\infty)$, $p(v)$ is decreasing on $(-\infty, +\infty)$, u is increasing on $(-\infty, 0)$ and it is decreasing on $(0, +\infty)$

- C_2: v is decreasing on $(-\infty, +\infty)$, $p(v)$ is increasing on $(-\infty, +\infty)$, u is decreasing on $(-\infty, 0)$ and it is increasing on $(0, +\infty)$

- C_3: v is increasing on $(-\infty, 0)$ and it is decreasing on $(0, +\infty)$, $p(v)$ is decreasing on $(-\infty, 0)$ and it is increasing on $(0, +\infty)$, u increasing on $(-\infty, +\infty)$

- C_4: v is decreasing on $(-\infty, 0)$ and it is increasing on $(0, +\infty)$, $p(v)$ is increasing on $(-\infty, 0)$ and it is decreasing on $(0, +\infty)$, u decreasing on $(-\infty, +\infty)$

- C_5: (w, v) (and consequently $p(v)$) are constant on $(-\infty, 0)$ or $(0, +\infty)$ and they are increasing or decreasing in the other one.

Remark 3. *In the region $\xi < 0$ and $\xi > 0$ the functions $v'(\xi)$ and $w'(\xi)$ are well defined, moreover at the point $\xi = 0$ v' and u' become singular according with the relations (3.65) and (3.66).*

Our goal in this section is derive L^∞ estimate for u and v, both if $\alpha_i > 0$ and if $\alpha_i < 0$, for $i = 1, 2$. We remark that w and v are not a continuous functions, but it jumps at $\xi = 0$ because of (3.65) and (3.66). It is clear that the discontinuity of v depends on α_2 and on the form of $p(v)$ in a non liner way, to prove the results given in this section we will use the monotonicity of p with respect to v and viceversa.
We start considering the last case C_5.

Theorem 3. *(w, v) are in one of the classes $C_1 - C_5$ and*

$$\underline{w} \le w(\xi) \le \overline{w} \tag{3.71}$$

$$0 \le \underline{p} \le p(v(\xi)) \le \overline{p} \tag{3.72}$$

$$0 \le \underline{v} \le v(\xi) \le \overline{v} \tag{3.73}$$

where the constants $\overline{v}$, $\overline{p}$, $\overline{w}$, $\underline{v}$, $\underline{p}$, $\underline{w}$ depend on v_+, v_- and γ (but do not on ε and μ.)

Proof. We will present only a sketch (w, v) of the proof of one of the cases, class C_5. Concerning $p(v)$, we consider v is constant in the interval $(-\infty, 0)$. It is easy to show that:

- if $\alpha_2 > 0$ and $p(v)$ is decreasing on $(0, +\infty)$

$$\min\{p_-, p_+\} \leq p(v(\xi)) \leq \max\{p_- + \alpha_2, p_+ + \alpha_2\}$$

- if $\alpha_2 < 0$ and $p(v)$ is increasing on $(0, +\infty)$

$$\min\{p_- - \alpha_2, p_+ - \alpha_2\} \leq p(v(\xi)) \leq \max\{p_-, p_+\}$$

- if $\alpha_2 > 0$ and $p(v)$ is increasing on $(0, +\infty)$ or $\alpha_2 < 0$ and v is decreasing on $(0, +\infty)$

$$\min\{p_-, p_+\} \leq p(v(\xi)) \leq \max\{p_-, p_+\},$$

where $p_\pm = \frac{1}{v_\pm}$. All of these cases are summarized in the relation (3.72), (3.73) follows from (3.72) by using the monotonicity of p. We work in a similar way if (u, v) are constant in the interval $(0, +\infty)$. By using the same steps we prove (3.71). The proof of the other cases follows in a similar way (see for example [10, 11])

□

3.2.3.9 Existence of solution

In this section following we will prove the existence of solutions of our problem. For each $\mu \in [0, 1]$, $U \in \overline{\Omega}$ and $v \in X$ we define the map $\mathcal{F} : [0, 1] \times \overline{\Omega} \to X$ that carries (μ, V) to the first component v_μ of the solution (v, w) to the problem (3.68) and (3.70). For this we define the applications $T : \overline{\Omega} \to X$, $S : \overline{\Omega} \to X$ that carry $V \in \overline{\Omega}$ in the solution (v, w) of our problem. And $(v_- + \mu T(V), w_- + \mu S(V))$ is the solution of (3.69).

The first step is the analysis of the linear problem that define in same sense the iterative process

The linearized problem

Consider the linearized problem

$$-\xi v' - w' = 0 \tag{3.74}$$

$$-\xi w' - a(\xi)v' = \varepsilon(k(\xi)w')' + \alpha_2 H(\xi)' \tag{3.75}$$

for all $\xi \in (-\infty, 0) \bigcup (0, +\infty)$ and coupled to the boundary conditions:

$$w(-\infty) = 0, \quad w(+\infty) = w_+ - w_- \tag{3.76}$$

$$v(-\infty) = 0, \quad v(+\infty) = v_+ - v_-. \tag{3.77}$$

In this contest $a(\xi) = -\frac{1}{V(\xi)^2}$ and $k(\xi) = \frac{1}{V(\xi)}$ are continuous functions for all $\xi \neq 0$ and they verify:

$$0 < a_0 \leq a(\xi) \leq A_0 \tag{3.78}$$

$$0 < k_0 \leq k(\xi) \leq K_0. \tag{3.79}$$

for all $\xi \in (-\infty, +\infty)$. In the intervals $(-\infty, 0)$ and $(0, +\infty)$ the solution (w, v) of (3.74) and (3.77) can be calculated explicitly by solving the ordinary differential equation

$$\varepsilon(k(\xi)w')' + \frac{\xi^2 - a(\xi)}{\xi} w' = 0, \tag{3.80}$$

whereas the solution at the interface point $\xi = 0$ will be defined by using the jump conditions (3.65) and (3.66). Integrating (3.80) we get

$$w'(\xi) = \begin{cases} c_+ I_+(\xi) \\ c_- I_-(\xi) \end{cases} \tag{3.81}$$

and, by using (3.75)

$$v'(\xi) = \begin{cases} \frac{1}{\xi} c_+ I_+(\xi) \\ \frac{1}{\xi} c_- I_-(\xi) \end{cases} \tag{3.82}$$

where

$$I_\pm = \exp\left(\frac{1}{\varepsilon} \int_{\pm 1}^{\xi} \frac{\varsigma^2 - a(\varsigma)}{\varsigma k(\varsigma)} d\varsigma \right). \tag{3.83}$$

$c_\pm$ are constant that will be fixed according to (3.65) and (3.66).

Theorem 4. *There exist constants α, β, η, κ and C_ε such that*

$$\frac{1}{C_\varepsilon} |\xi|^{\eta/\varepsilon} \leq I_\pm \leq \frac{1}{C_\varepsilon} |\xi|^{\alpha/\varepsilon} \qquad 0 < |\xi| \leq 1 \tag{3.84}$$

$$\frac{1}{C_\varepsilon} e^{-\xi^2 \kappa/\varepsilon} \leq I_\pm \leq \frac{1}{C_\varepsilon} e^{-\xi^2 \beta/\varepsilon} \qquad |\xi| \geq 1 \tag{3.85}$$

Proof. First we estimate $I_+ = \exp\left(\frac{1}{\varepsilon}\int_\xi^1 \frac{\varsigma^2 - a(\varsigma)}{\varsigma k(\varsigma)} d\varsigma\right)$ in the case $0 < \xi \leq 1$:

$$\begin{aligned}\frac{1}{\varepsilon}\int_\xi^1 \frac{\varsigma^2 - a(\varsigma)}{\varsigma k(\varsigma)} d\varsigma &= \frac{1}{\varepsilon}\int_\xi^1 \frac{\varsigma}{k(\varsigma)} d\varsigma - \int_\xi^1 \frac{a(\varsigma)}{\varsigma k(\varsigma)} d\varsigma \leq \\ &\leq \frac{1}{\varepsilon}\left(\frac{1}{2k_0} + \frac{a_0}{K_0}\ln\xi\right)\end{aligned}$$

finally we conclude:

$$I_+ \leq e^{\frac{1}{2\varepsilon k_0}} \xi^{\frac{a_0}{\varepsilon k_0}} \tag{3.86}$$

Now we consider the low bound for I_+

$$\begin{aligned}\frac{1}{\varepsilon}\int_\xi^1 \frac{\varsigma^2 - a(\varsigma)}{\varsigma k(\varsigma)} d\varsigma &= \frac{1}{\varepsilon}\int_\xi^1 \frac{\varsigma}{k(\varsigma)} d\varsigma - \int_\xi^1 \frac{a(\varsigma)}{\varsigma k(\varsigma)} d\varsigma \geq \\ &\geq \frac{1}{\varepsilon} - \int_\xi^1 \frac{a(\varsigma)}{\varsigma k(\varsigma)} d\varsigma \geq \frac{1}{\varepsilon}\frac{a_0}{K_0}\ln\xi\end{aligned}$$

and then

$$I_+ \geq \xi^{\frac{a_0}{\varepsilon k_0}} \tag{3.87}$$

The same holds for I_- in the interval $-1 \leq \xi < 0$

$$\begin{aligned}\frac{1}{\varepsilon}\int_\xi^1 \frac{\varsigma^2 - a(\varsigma)}{\varsigma k(\varsigma)} d\varsigma &= \frac{1}{\varepsilon}\int_\xi^1 \frac{\varsigma}{k(\varsigma)} d\varsigma - \int_\xi^1 \frac{a(\varsigma)}{\varsigma k(\varsigma)} d\varsigma \leq \\ &\leq -\frac{1}{\varepsilon}\int_\xi^{-1} \frac{a(\varsigma)}{\varsigma k(\varsigma)} d\varsigma \leq \frac{1}{\varepsilon}\frac{A_0}{k_0}\ln|\xi|\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\varepsilon}\int_\xi^1 \frac{\varsigma^2 - a(\varsigma)}{\varsigma k(\varsigma)} d\varsigma &= \frac{1}{\varepsilon}\int_\xi^1 \frac{\varsigma}{k(\varsigma)} d\varsigma - \int_\xi^1 \frac{a(\varsigma)}{\varsigma k(\varsigma)} d\varsigma \geq \\ &\geq \frac{1}{\varepsilon}\left(\frac{1}{K_0} - \frac{\xi^2}{k_0}\right) + \frac{1}{\varepsilon}\left(\frac{a_0}{K_0}\ln|\xi|\right) \\ &\geq \frac{1}{\varepsilon}\left(\frac{a_0}{K_0}\ln|\xi| - \frac{\xi^2}{k_0}\right)\end{aligned}$$

This prove (3.84) in a similar way we prove (3.85) □

Now we can integrate (3.81) and (3.82) and we obtain respectively:

$$w(\xi) = \begin{cases} (w_+ - w_-) - c_+ \int_\xi^\infty I_+(\xi), & \xi > 0 \\ c_- \int_{-\infty}^\xi I_-(\xi), & \xi < 0 \end{cases} \tag{3.88}$$

$$v(\xi) = \begin{cases} (v_+ - v_-) + c_+ \int_\xi^\infty \frac{I_+(\zeta)}{\zeta}, & \xi > 0 \\ c_- \int_{-\infty}^\xi \frac{I_-(\zeta)}{-\zeta}, & \xi < 0 \end{cases} \tag{3.89}$$

The constants c_+ and c_- can be determined by using (3.65) and (3.66)

$$\begin{cases} w(0^+) = w(0^-) \\ v_0^+ = \frac{v_0^-}{1+\alpha_2 v_0^-} \end{cases} \tag{3.90}$$

In the following section we will analyze the solutions of the system above compatible of our system and how they depends on the sources.

Determination of the constants c_+ and c_-

We rewrite system (3.90) by using (3.88)-(3.89)

$$\begin{cases} (w_+ - w_-) - c_+ \int_\xi^\infty I_+(\zeta) d\zeta = c_- \int_{-\infty}^\xi I_-(\zeta) d\zeta \\ \left((v_+ - v_-) + c_+ \int_\xi^\infty \frac{I_+(\zeta)}{\zeta} d\zeta\right)\left(1 + \alpha_2 c_- \int_{-\infty}^\xi \frac{I_-(\zeta)}{-\zeta} d\zeta\right) = c_- \int_{-\infty}^\xi \frac{I_-(\zeta)}{-\zeta} d\zeta \end{cases} \tag{3.91}$$

or introducing A, B, C and D:

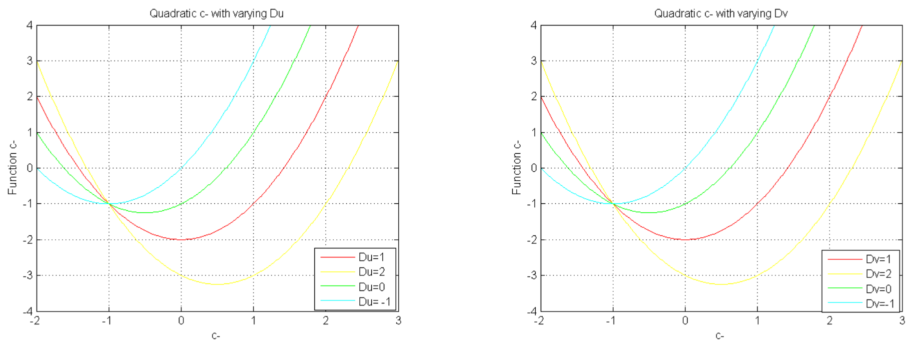
$$\begin{cases} \Delta w - c_+ A = c_- B + \alpha_1 \\ \Delta v + c_+ C = c_- D - \alpha_2 (\Delta v + (c_+ C))(c_- D) \end{cases} \tag{3.92}$$

$$\begin{aligned} A &= \int_\xi^\infty I_+(\zeta) d\zeta \\ B &= \int_{-\infty}^\xi I_-(\zeta) d\zeta \\ C &= \int_\xi^\infty \frac{I_+(\zeta)}{\zeta} d\zeta \\ D &= \int_{-\infty}^\xi \frac{I_-(\zeta)}{-\zeta} d\zeta \\ \Delta w &= w_+ - w_- \\ \Delta v &= v_+ - v_- \end{aligned}$$

Because of Theorem 4 it is easy to prove the following Corollary

Corollary 1. *Under the assumption of Lemma 4 there exist the two constants $\underline{A}$ and $\overline{A}$ such that*

$$0 < \underline{A} \le A \le \overline{A} < +\infty \tag{3.93}$$



(a) Varying $\Delta u = u_+ - u_-$ and $\Delta v = 1$. (b) Varying $\Delta v = v_+ - v_-$ and $\Delta v = 1$.

Figure 3.7: Plot of (3.99) with $\alpha_2 = A = B = C = D = 1$.

then same hold for B, C and D:

$$0 < \underline{B} \le B \le \overline{B} < +\infty \tag{3.94}$$

$$0 < \underline{C} \le C \le \overline{C} < +\infty \tag{3.95}$$

$$0 < \underline{D} \le D \le \overline{D} < +\infty \tag{3.96}$$

By using the first relation in (3.92) we get

$$c_+ = \frac{\Delta w}{A} - \frac{c_- B}{A} \tag{3.97}$$

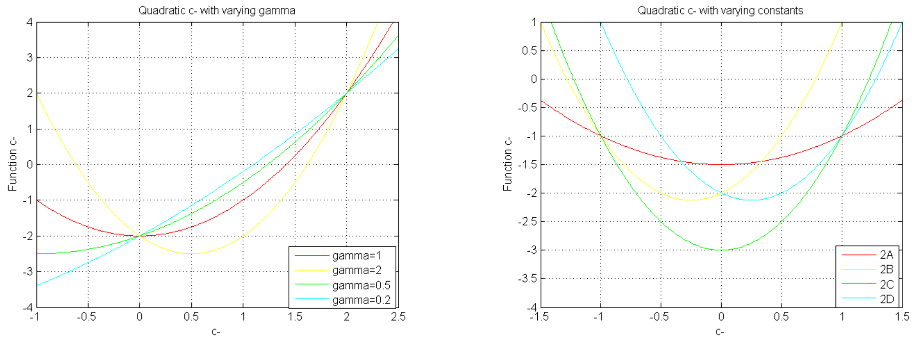
and substituting in the second equation

$$\Delta v + C\left(\frac{\Delta w}{A} - \frac{c_- B}{A}\right) = c_- D - \alpha_2 \left(\Delta v + C\left(\frac{\Delta w}{A} - \frac{c_- B}{A}\right)\right)(c_- D) \tag{3.98}$$

after easy computation we get

$$\alpha_2 \frac{BCD}{A} c_-^2 + c_- \left(\frac{BC}{A} + D - \alpha_2\left(D\Delta v + \frac{CD\Delta w}{A}\right)\right) - \left(\Delta v + \frac{C\Delta w}{A}\right) \tag{3.99}$$

In Figure (3.7) we see a plot of the function (3.99) with all constants fixed and only varying the differences between the initial values. In Figure (3.8) we plotted the same function but by varying γ and then doubling each constant and fixing all the others. As in [11], we can see, from the figures, that with the value of this parameters there is always a solution for the systema, since the parabolas cross always the $y = 0$ line.



(a) Varying $\gamma = \alpha_2$ with $A = B = C = D = 1$.

(b) Varying constants A, B, C, D with $\gamma = 1$.

Figure 3.8: Plot of (3.99) with $\Delta u = \Delta v = 1$.

The discriminant of this equation is:

$$\Delta_{\alpha_2} = \left(\frac{BC}{A} + D - \alpha_2\left(D\Delta v + \frac{CD\Delta w}{A}\right)\right)^2 + 4\alpha_2\left(\Delta v + \frac{C\Delta w}{A}\right)\left(\frac{BCD}{A}\right) \tag{3.100}$$

We distinguish the following cases:

1. $\alpha_2 > 0$, $\Delta v + \frac{C\Delta w}{A} > 0$ $\frac{BC}{A} + D - \alpha_2\left(D\Delta v + \frac{CD\Delta w}{A}\right) > 0$

In this case (3.99) admits one and only one positive solution

$$c_- = \left(-\left(\frac{BC}{A} + D - \alpha_2\left(D\Delta v + \frac{CD\Delta w}{A}\right)\right) \pm \sqrt{\Delta_{\alpha_2}}\right)\frac{A}{2\alpha_2 BCD} \tag{3.101}$$

$$= -f \pm \frac{h}{\alpha_2}\sqrt{\Delta_{\alpha_2}} = -f \pm \frac{h}{\alpha_2}\sqrt{f^2 + \alpha_2 g}. \tag{3.102}$$

Where $h, g > 0$, first we assume f to be positive (is sufficient to use for example a large value of γ), and then the positive solution corresponds to the sign $+$ in (3.101),i.e:

$$c_- = \left(-\left(\frac{BC}{A} + D - \alpha_2\left(D\Delta v + \frac{CD\Delta w}{A}\right)\right) + \sqrt{\Delta_{\alpha_2}}\right)\frac{A}{2\alpha_2 BCD} \tag{3.103}$$

Now by using (3.97)

$$c_+ = \frac{\Delta w}{A} - \frac{c_- B}{A},$$

we look for the value of the constants in particular α_2, Δv and Δw such that c_+ is positive, for example it is sufficient to consider Δw large enough.

3.2.3.10 Solution of the Riemann problem

Theorem 5. *Let $\varepsilon > 0$ then there exist a solution of our boundary problem for all $\xi \in (-\infty, +\infty)$.*

Proof. The operators $T, S : \overline{\Omega} \to X$ map $V \in \overline{\Omega}$ to the solution of the linearized problem with boundary conditions (3.76)-(3.77). We define the operator $F(\mu, w) = w_- + \mu T(w)$ such that $F : [0,1]\overline{\Omega} \to X$ and we look for a solution of $v = v_- + T(v)$ by using the Leray-Schauder degree theory. The function v and the related $w = S(v)$ is the solution $w, v)$ of (3.49) and (3.50). Fist we prove that $T(\overline{\Omega})$ is precompact and continuous in X.
$T(\overline{\Omega})$ *is precompact in X*
To do this we will apply the *Ascoli-Arzela compactness criterion.* We fix a sequence $V_n \in \overline{\Omega}$ and define $v_{1,n} = T(V_n)$ and $v_{2,n} = T(V_n)$ respectively for $\xi < 0$ and $\xi < 0$. The results of the previous section show that $\{v_{1,n}\}$ and $\{v_{2,n}\}$ are uniformly bounded and uniformly equicontinuous and they verify the jump condition (3.65) and (3.66). Because of the *Ascoli-Arzela compactness criterion* there exist two subsequence such that $\{v_{1,n_k}\} \to v_1$ and $\{v_{2,n_k}\} \to v_2$, where v_1 and v_2 also verify the jump conditions. The results of the previous section show that $\{v_{1,n}\}$ and $\{v_{2,n}\}$ are uniformly bounded and uniformly equicontinuous and they verify the jump condition (3.65) and (3.66). Because of the *Ascoli-Arzela compactness criterion* there exist two subsequence such that $\{v_{1,n_k}\} \to v_1$ and $\{v_{2,n_k}\} \to v_2$, where v_1 and v_2 also verify the jump conditions.
$T(\overline{\Omega})$ *are continuous in X*
As usually we consider a sequence $V_n \in \overline{\Omega}$ and the functions $v_{1,n} = T(V_n)$ and $v_{2,n} = T(V_n)$ respectively for $\xi < 0$ and $\xi < 0$. They are connected by

$$v_n(\xi) = \begin{cases} (v_+ - v_-) + {c_+}^n \int_\xi^\infty \frac{{I_+}^n(\zeta)}{\zeta}, & \xi > 0 \\ c_- \int_{-\infty}^\xi \frac{I^n_-(\zeta)}{-\zeta}, & \xi < 0 \end{cases} \tag{3.104}$$

where $I^n_\pm$ are

$$I_{\pm^n} = \exp\left(\frac{1}{\varepsilon}\int_{\pm 1}^{\xi} \frac{\zeta^2 - a(V_n(\zeta))}{\zeta k(V_n(\zeta))} d\zeta\right). \tag{3.105}$$

and $c^n_\pm$ are the solutions of (3.91). □

Now, there is a subsequence $\{v_n\}$ and $v \in X$ such that $v_n \to v$ in X. So using $\{V_{n_k}\}$ we can pass to the limit and obtain $v = T(V)$, and since all limiting point in X is of the form of this last equality then we deduce that $T(V_n) \to T(V) \in X$ this means that T is continuous. With this we can say that μT is compact and we can compute the Larey-Schuder degree of the map. So by the definition of Ω and previous results the solution v lies in the interior of the set, so the degree is 1

$$d(I - \mu T, \Omega, u_-) = d(I, \Omega, u_-) = 1$$

so we have a fix point and the problem admits at least one solution for each $\mu \in [0.1]$.
Now, knowing this we can pass to the limit of $\epsilon \to 0$, and letting the viscous solutions satisfy the a-priori bounds, hence by doing the previous limit and by Helly's selection principle we can get

$$w_\epsilon \to w$$

$$v_\epsilon \to v$$

Solutions of the original Riemann problem.

3.3 Gas Dynamics with Added Sources

In this section we will give a more complicated model for the ion transport. Afterwards we will follow the same steps as in the case of the P-system model, i.e. we will present the four regions analysis and finally we will present a self similar approach for the description of other important properties of the system. Recalling the equations used to model gas dynamics , and inserting a point-wise source of ions in the interface position, we propose our model (which includes the action of the ion Channels given by the Euler equations and the Ion Pumps represented by the added sources in the equations)

1. Conservation of Mass.

$$\rho_t + (\rho u)_x = \alpha_1 \delta(x) \tag{3.106}$$

2. Conservation of Momentum.

$$(\rho u)_t + (p + \rho u^2)_x = \alpha_2 \delta(x) \tag{3.107}$$

3. Conservation of Energy.

$$\left(\rho(\frac{u^2}{2} + e)\right)_t + \left(\rho u(\frac{u^2}{2} + e) + pu\right)_x = \alpha_3 \delta(x) \tag{3.108}$$

Where ρ is the density of mass, u is the velocity of the gas, e the energy density, p is the pressure, α_i, $i = 1, 2, 3$ are constants (rates of transfer of mass, momentum and energy density). The function $\delta(x)$ is the Delta Dirac function, and is given by:

$$\delta(x) = H_x \tag{3.109}$$

With H the Heaviside function. We assume that the gas is ideal gas and *polytropic*. An important and fundamental part of the model are the initial conditions, let $\mathbf{U} = \left(\rho, \rho u, \rho(\frac{u^2}{2} + e)\right)^T$, so the conditions are:

$$\mathbf{U}(x, 0) = \begin{cases} \mathbf{U}_l = \left(\rho_l, \rho_l u_l, \rho_l(\frac{u_l^2}{2} + e_l)\right)^T & \text{if } x \leq 0, \\ \mathbf{U}_r = \left(\rho_r, \rho_r u_r, \rho_r(\frac{u_r^2}{2} + e_r)\right)^T & \text{if } 0 < x. \end{cases}$$

And this completes the model. Notice that this can be seen as a conservative system of the form:

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0$$

The system has the form:

$$\begin{cases} \rho_t + (\rho u - \alpha_1 H)_x = 0 \\ (\rho u)_t + (p + \rho u^2 - \alpha_2 H)_x = 0 \\ \Big(\rho(\frac{u^2}{2} + e)\Big)_t + \Big(\rho u(\frac{u^2}{2} + e) + pu - \alpha_3 H\Big)_x = 0 \end{cases} \tag{3.110}$$

3.3.1 Analyzing the two Equation Model

In the previous section we have presented general idea by doing the analysis for a P-system, now we will present an analogous procedure with our model equations. For sake of simplicity we will start with only two equations, the conservation of mass, and the conservation of momentum. Our system is the following

$$(\rho)_t + (\rho u)_x = \alpha_1 H_x$$

$$(\rho u)_t + \left(P + \rho u^2\right)_x = \alpha_2 H_x$$

First we need to derive the shock and rarefaction curves of the homogeneous system. From the Rankine-Hugoniot condition (for the homogeneous system) we know that

$$\begin{aligned} s\,[\rho] &= [\rho u] \\ s[\rho u] &= [P + \rho u^2] \end{aligned} \tag{3.111}$$

where [ ] denotes change in the discontinuity, we can eliminate s and find explicit the shock curves

$$u - u_l = -\frac{\sqrt{(2\rho_l \rho u_l)^2 - 4\rho\rho_l\left[-(\rho_l - \rho)(P_l - P + \rho_l u_l^2) + (\rho_l u_l)^2\right]}}{2\rho_l \rho} \qquad \rho_l < \rho \tag{3.112}$$

$$u - u_l = -\frac{\sqrt{(2\rho_l \rho u_l)^2 - 4\rho\rho_l\left[-(\rho_l - \rho)(P_l - P + \rho_l u_l^2) + (\rho_l u_l)^2\right]}}{2\rho_l \rho} \qquad \rho < \rho_l \tag{3.113}$$

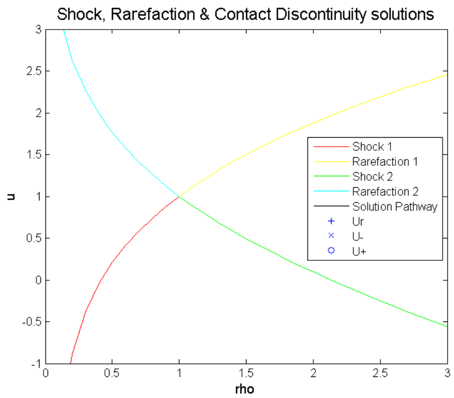


Figure 3.9: Shock and Rarefaction curves.

where, as before, the subscript l its related to the left state.

Now by considering the Riemann Invariants of the gas dynamics system [1], $\left\{u - \frac{2}{\gamma-1}c, u + \frac{2}{\gamma-1}c\right\}$ where the first is due to the first and second eigenvalue respectively. So, we have that this Riemann Invariants are constant in a 1-2 rarefaction, so the rarefaction curves should be given by

$$u_l - \frac{2}{\gamma-1}c_l = u_r - \frac{2}{\gamma-1}c_r \qquad \rho_l < \rho \tag{3.114}$$

$$u_l + \frac{2}{\gamma-1}c_l = u_r + \frac{2}{\gamma-1}c_r \qquad \rho < \rho_l \tag{3.115}$$

Lets recall that In our problem we are using $P = k\rho^\gamma$ with $1 < \gamma$, and $c = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{k\gamma}\sqrt{\rho^{\gamma-1}} = \sqrt{k\gamma}\rho^{\frac{\gamma}{2}-\frac{1}{2}}$. The Figure 3.9 shows the four curves.

So our goal know is to find the two intermediate states, U_- and U_+, in terms of known parameters, as in the previous section. Now using the jump conditions of the non-homogeneous system we get

$$s\,[\rho] = [\rho u - \alpha_1 H]$$

$$s[\rho u] = [P + \rho u^2 - \alpha_2 H]$$

In this step we have to force a non-moving shock, so we set $s = 0$, we get the expected contact discontinuities relations

$$\rho_+ u_+ = \rho_- u_- + \alpha_1 \tag{3.116}$$

$$\rho_+ u_+^2 = P_- - P_+ + \rho_- u_-^2 + \alpha_2 \tag{3.117}$$

Now we have to solve the Riemann problem knowing the values of (u_l, ρ_l) and (u_r, ρ_r), so if we have a contact discontinuity at $x = 0$ with a source term at that point we expect two new intermediate states (u_-, ρ_-) and (u_+, ρ_+) connected by the previous relations. So we have to connect $l - states$ to $-states$ with a shock or rarefaction curves, and $+states$ to $r - states$ by shock or rare, and the $-states$ and $+states$ by contact discontinuity. So lets consider the case that we connect $l - states$ to $-states$ with a shock and $+states$ to $r - states$ by rare, as an example. So we have to solve the system

$$u - u_l = -\frac{\sqrt{(2\rho_l \rho u_l)^2 - 4\rho\rho_l \left[-(\rho_l - \rho)(k\rho_l^\gamma - k\rho^\gamma + \rho_l u_l^2) + (\rho_l u_l)^2\right]}}{2\rho_l \rho} \tag{3.118}$$

$$\rho_+ u_+ = \rho_- u_- + \alpha_1 \tag{3.119}$$

$$\rho_+ u_+^2 = P_- - P_+ + \rho_- u_-^2 + \alpha_2 \tag{3.120}$$

$$u \pm \frac{2}{\gamma - 1} c = u_r \pm \frac{2}{\gamma - 1} c_r \tag{3.121}$$

Where the $\pm$ depends if we are connecting the states by a $R1$ or $R2$. The first equation connects U_l to U_-, the second and third connect U_- to U_+, and the last equation U_+ to U_r. Substituting P and c we get

$$u - u_l = -\frac{\sqrt{(2\rho_l \rho u_l)^2 - 4\rho\rho_l \left[-(\rho_l - \rho)(k\rho_l^\gamma - k\rho^\gamma + \rho_l u_l^2) + (\rho_l u_l)^2\right]}}{2\rho_l \rho}$$

$$\rho_+ u_+ = \rho_- u_- + \alpha_1$$

$$\rho_+ u_+^2 = k\rho_-^\gamma - k\rho_+^\gamma + \rho_- u_-^2 + \alpha_2$$

$$u \pm \frac{2}{\gamma - 1}\sqrt{k\gamma}\rho^{\frac{\gamma}{2} - \frac{1}{2}} = u_l \pm \frac{2}{\gamma - 1}\sqrt{k\gamma}\rho_l^{\frac{\gamma}{2} - \frac{1}{2}}$$

The solution for the intermediate states is given by this system of equations. We have four unknowns and four equations, so we can solve (assuming that the known parameters are such that the system has solution, maybe not unique) the system. Analytically its very complicated to arrive at a solution, because of the value of γ, that is strictly bigger that 1.

So our next step is to make some simplifications to arrive at some results and study what will happend of certain values of the parameters. We will assume for a moment that α_i for $i = 1, 2$ are so small, that the system will behave like it where a gas, so we can use the approximation and results of the Gas Dynamics equations (more specifically as an almost homogeneous system), so we will assume that the pressures and velocities through the discontinuity are constant. This assumption is physically meaningful because if we have a different pressure across it, the forces will be different and the shock will eventually move, thing that we don't want. So the simplify system is

$$u - u_l = -\frac{\sqrt{(2\rho_l \rho u_l)^2 - 4\rho\rho_l \left[-\left(\rho_l - \rho\right)\left(k\rho_l^\gamma - k\rho^\gamma + \rho_l u_l^2\right) + (\rho_l u_l)^2\right]}}{2\rho_l \rho}$$

$$\rho_+ u_- = \rho_- u_- + \alpha_1 \tag{3.122}$$

$$\rho_+ u_-^2 = \rho_- u_-^2 + \alpha_2 \tag{3.123}$$

$$u \pm \frac{2}{\gamma - 1} c = u_r \pm \frac{2}{\gamma - 1} c_r \tag{3.124}$$

The second and third equation be combined, and they result is

$$u_- = \frac{\alpha_2}{\alpha_1} \tag{3.125}$$

With this result we can start the analysis of each region.

3.3.1.1 Region I

In this region the state U_l is connected to $U_-$by a $R1$, then the contact discontinuity, and then $U_+$with U_r by a $S2$. And we have the next system of equations

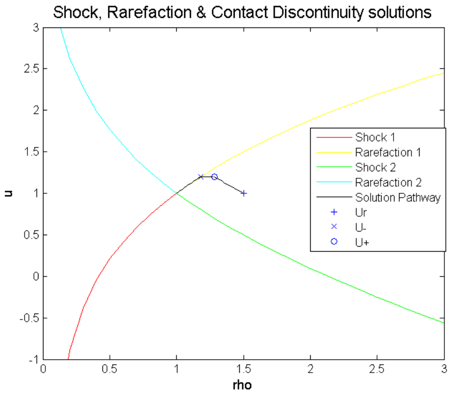


Figure 3.10: Solution Region I.

$$\begin{cases} u_- - \frac{2}{\gamma-1}c_- = u_l - \frac{2}{\gamma-1}c_l \\ \rho_+ u_-^2 = \rho_- u_-^2 + \alpha_2 \\ \rho_+ u_-^2 = \rho_- u_-^2 + \alpha_2 \\ u_r - u_+ = -\frac{\sqrt{(2\rho_+\rho_r u_+)^2 - 4\rho_r\rho_+\left[-(\rho_+ - \rho_r)\left(k\rho_+^\gamma - k\rho_r^\gamma + \rho_+ u_+^2\right) + (\rho_+ u_+)^2\right]}}{2\rho_l\rho_r} \end{cases} \tag{3.126}$$

So if we put $u_- = \frac{\alpha_2}{\alpha_1}$, we can solve this system numerically. In Figure 3.10 we show a result for the given values of: $u_l = 1$, $\rho_l = 1$, $\rho_r = 1.5$, $u_r = 1$, $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $k = 1$, $\gamma = 1.4$.

We see in the picture that values of α_i cannot be such that the ratio is out of region I.

3.3.1.2 Region II

In this region the state U_l is connected to $U_-$by a $S1$, then the contact discontinuity, and then $U_+$with U_r by a $S2$. And we have the next system of equations:

$$\begin{cases} u_- - u_l = -\frac{\sqrt{(2\rho_l\rho_- u_l)^2 - 4\rho_-\rho_l\left[-(\rho_l - \rho_-)\left(k\rho_l^\gamma - k\rho_-^\gamma + \rho_l u_l^2\right) + (\rho_l u_l)^2\right]}}{2\rho_l\rho_-} \\ \rho_+ u_-^2 = \rho_- u_-^2 + \alpha_2 \\ \rho_+ u_-^2 = \rho_- u_-^2 + \alpha_2 \\ u_r - u_+ = -\frac{\sqrt{(2\rho_+\rho_r u_+)^2 - 4\rho_r\rho_+\left[-(\rho_+ - \rho_r)\left(k\rho_+^\gamma - k\rho_r^\gamma + \rho_+ u_+^2\right) + (\rho_+ u_+)^2\right]}}{2\rho_l\rho_r} \end{cases} \tag{3.127}$$

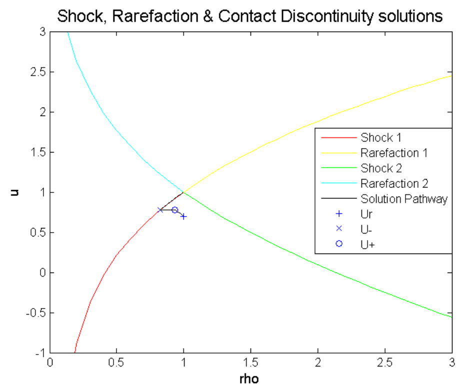


Figure 3.11: Solution Region II.

So if we put $u_- = \frac{\alpha_2}{\alpha_1}$, we can solve this system numerically. In Figure 3.11 we show a result for the given values of: $u_l = 1$, $\rho_l = 1$, $\rho_r = 1$, $u_r = 0.7$, $\alpha_1 = 0.9$, $\alpha_2 = 0.7$, $k = 1$, $\gamma = 1.4$.

We see in the picture that values of α_i cannot be such that the ratio is out of region II.

3.3.1.3 Region III

In this region the state U_l is connected to $U_-$by a $S1$, then the contact discontinuity, and then $U_+$with U_r by a $R2$. And we have the next system of equations

$$\begin{cases} u_- - u_l = -\frac{\sqrt{(2\rho_l\rho_- u_l)^2 - 4\rho_-\rho_l\left[-(\rho_l-\rho_-)\left(k\rho_l^\gamma - k\rho_-^\gamma + \rho_l u_l^2\right) + (\rho_l u_l)^2\right]}}{2\rho_l\rho_-} \\ \rho_+ u_-^2 = \rho_- u_-^2 + \alpha_2 \\ \rho_+ u_-^2 = \rho_- u_-^2 + \alpha_2 \\ u + \frac{2}{\gamma-1}c = u_r - \frac{2}{\gamma-1}c_r \end{cases} \tag{3.128}$$

So if we put $u_- = \frac{\alpha_2}{\alpha_1}$, we can solve this system numerically. In Figure 3.12we show a result for the given values of: $u_l = 1$, $\rho_l = 1$, $\rho_r = 0.3$, $u_r = 1.1$, $\alpha_1 = 0.3$, $\alpha_2 = 0.21$, $k = 1$, $\gamma = 1.4$.

We see in the picture that values of α_i cannot be such that the ratio is out of region III.

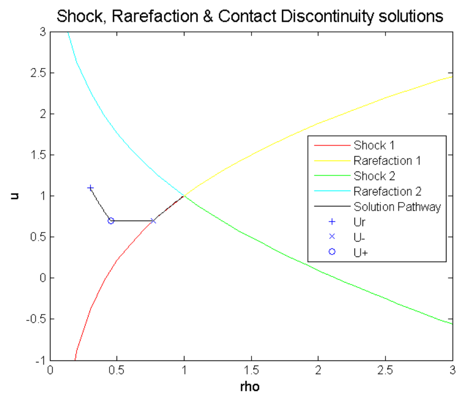


Figure 3.12: Solution Region III.

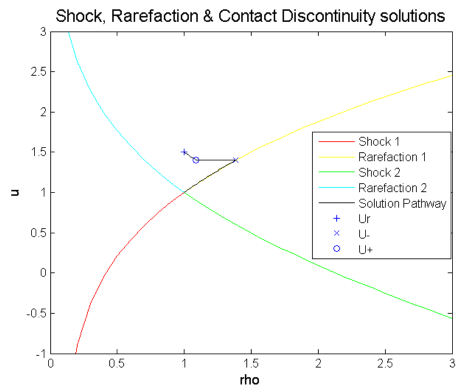


Figure 3.13: Solution Region IV.

3.3.1.4 Region IV

In this region the state U_l is connected to $U_-$by a $R1$, then the contact discontinuity, and then $U_+$with U_r by a $R2$. And we have the next system of equations

$$\begin{cases} u_- - \frac{2}{\gamma-1}c_- = u_l - \frac{2}{\gamma-1}c_l \\ \rho_+ u_-^2 = \rho_- u_-^2 + \alpha_2 \\ \rho_+ u_-^2 = \rho_- u_-^2 + \alpha_2 \\ u + \frac{2}{\gamma-1}c = u_r - \frac{2}{\gamma-1}c_r \end{cases} \tag{3.129}$$

So if we put $u_- = \frac{\alpha_2}{\alpha_1}$, we can solve this system numerically. In Figure 3.13 we show a result for the given values of: $u_l = 1$, $\rho_l = 1$, $\rho_r = 1$, $u_r = 1.5$, $\alpha_1 = 1$, $\alpha_2 = 1.4$, $k = 1$, $\gamma = 1.4$.

We see in the picture that values of α_i cannot be such that the ratio is out of region IV.

3.3.2 Simulations of the two equation model

Here we will present some simulations of the system presented in the previous part. It will be showed how the solutions ρ and u evolve in space and time for some specific values of the initial data, the sources and constants in the equations.

The simulations where made using a code written in MATLAB and the main idea is as follows: the first step is to solve numerically one of the systems (3.126), (3.127), (3.128) or (3.129) given some initial data (U_l and U_r) and some values for the parameters γ and k, with this the intermediate states where found (U_- and U_+). The second step was to compute numerically the speeds of the shocks using (3.111) and the speed of the rarefaction using the eigenvalues of gas dynamics. Then, having this information, its posible to use the idea in Figures (3.1) and (3.4), for example, to now the behavior of the solution in space and time, and then plot all this information. An example of the main code used can be found in the appendix.

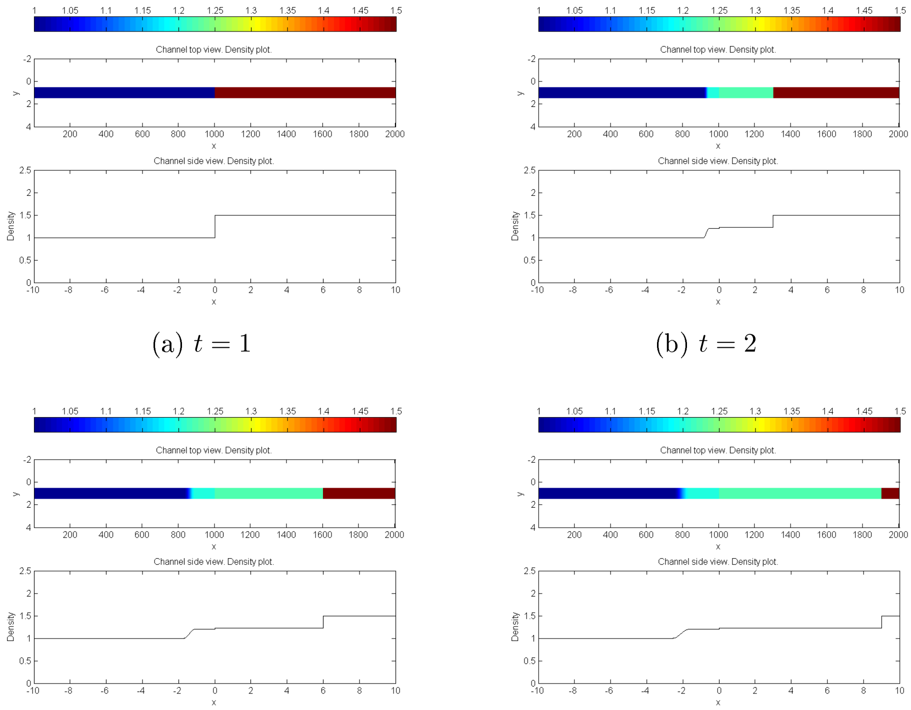
Now lets go to the simulations, first lets look region I.

3.3.2.1 Region I

In this region we wanted to solve numerically the equations (3.126). We can solve this for given values of: $u_l = 1$, $\rho_l = 1$, $\rho_r = 1.5$, $u_r = 1$, $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $k = 1$, $\gamma = 1.4$, in Figure (3.14) we can see the result for $\rho(x,t)$ for four time steps. The figure represents an ion pump. Colored graph shows the top view of the Chanel, colours related to the density value..

Notice that, as expected from the previous section, for $t > 0$ we can see a contact discontinuity at $x = 0$, the values of the left and right values of the $\rho_- = 1.18$ and $\rho_+ = 1.281$ respectively. It can be seen that the to initial values are connected by a rarefaction moving to the left and a shock moving to the right. The speed of the shock is higher than the speed of the rarefaction wave.

Now the evolution for u is similar, we will present only evolution for last time step ($t = 4$). See Figure (3.15). Notice that here the velocity at $x = 0$ is the same, so the velocity is continues at this point. Of course here we



(a) $t = 1$ (b) $t = 2$

(c) $t = 3$ (d) $t = 4$

Figure 3.14: ρ evolution in space and time. Solution of the system (3.126). Values of parameters are: $u_l = 1$, $\rho_l = 1$, $\rho_r = 1.5$, $u_r = 1$, $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $k = 1$, $\gamma = 1.4$.

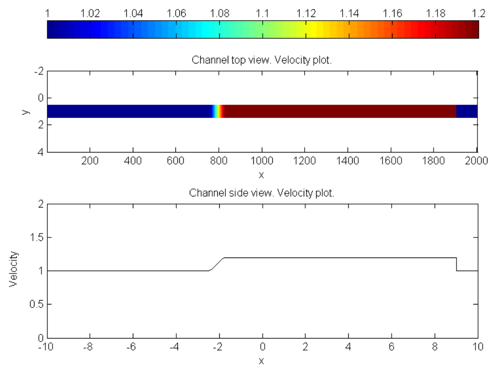


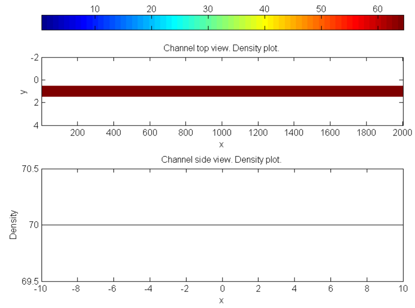
Figure 3.15: u evolution in space and time ($t = 4$). Solution of the system for $u_l = 1$, $\rho_l = 1$, $\rho_r = 1.5$, $u_r = 1$, $\alpha_1 = 0.5$, $\alpha_2 = 0.6$, $k = 1$, $\gamma = 1.4$.

have the same behavior as in the case of the density, one rarefaction moving to the left and one shock moving to the right. We can see that the effect of the sources is to generate two intermediate states $U_- = (1.18, 1.2)$ and $U_+ = (1.281, 1.2)$

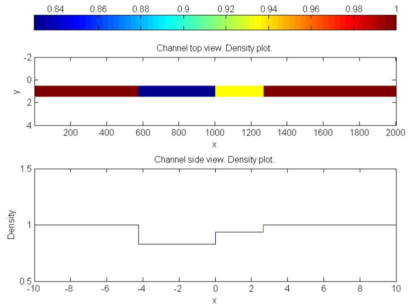
3.3.2.2 Region II

Now lets look what happends in the next region. It can be solved numerically the system (3.127). As before we present a plot of the evolution of the density in space and time for some time steps, and the picture of the velocity in the last step. The solution for the values of : $u_l = 1$, $\rho_l = 1$, $\rho_r = 1$, $u_r = 0.7$, $\alpha_1 = 0.9$, $\alpha_2 = 0.7$, $k = 1$, $\gamma = 1.4$ is in Figure (3.16). Notice that in this region we have the appearance of three shock waves, one moving to the left, with the higher speed, one moving to the right, with the lower, and a standing shock, this is the contact discontinuity at $x = 0$. The values of the left and right densities are $\rho_- = 0.83$ and $\rho_+ = 0.936$ respectively.

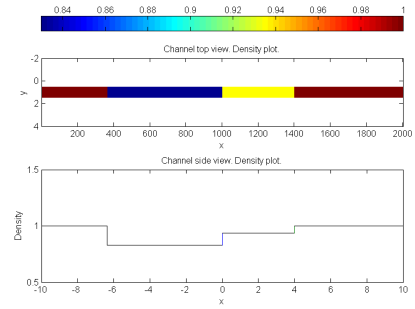
Now for the velocity we have the same behavior, two shocks, but not a contact discontinuity, the value of the intermediate state is $u = 0.77$. The result, for the last time step can be seen in Figure (3.17), notice that the velocities of the shocks are the same as the ones of the density. The effect of the sources is to generate two intermediate states $U_- = (0.83, 0.77)$ and $U_+ = (0.936, 0.77)$.



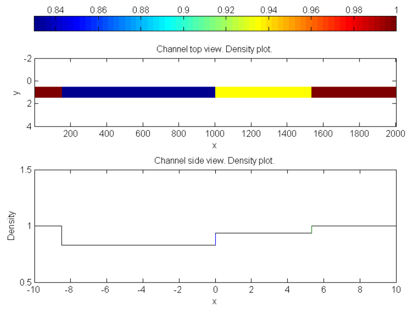
(a) $t = 1$



(b) $t = 2$



(c) $t = 3$



(d) $t = 4$

Figure 3.16: ρ evolution in space and time. Solving the system (3.127) with values of parameters are: $u_l = 1$, $\rho_l = 1$, $\rho_r = 1$, $u_r = 0.7$, $\alpha_1 = 0.9$, $\alpha_2 = 0.7$, $k = 1$, $\gamma = 1.4$

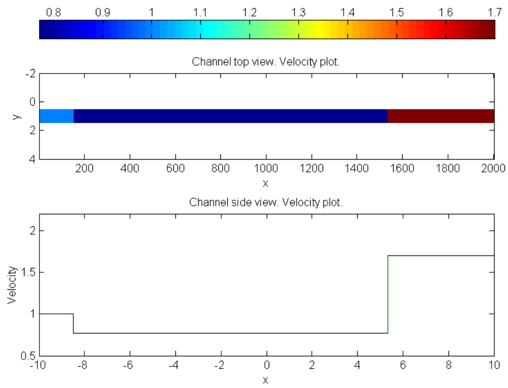
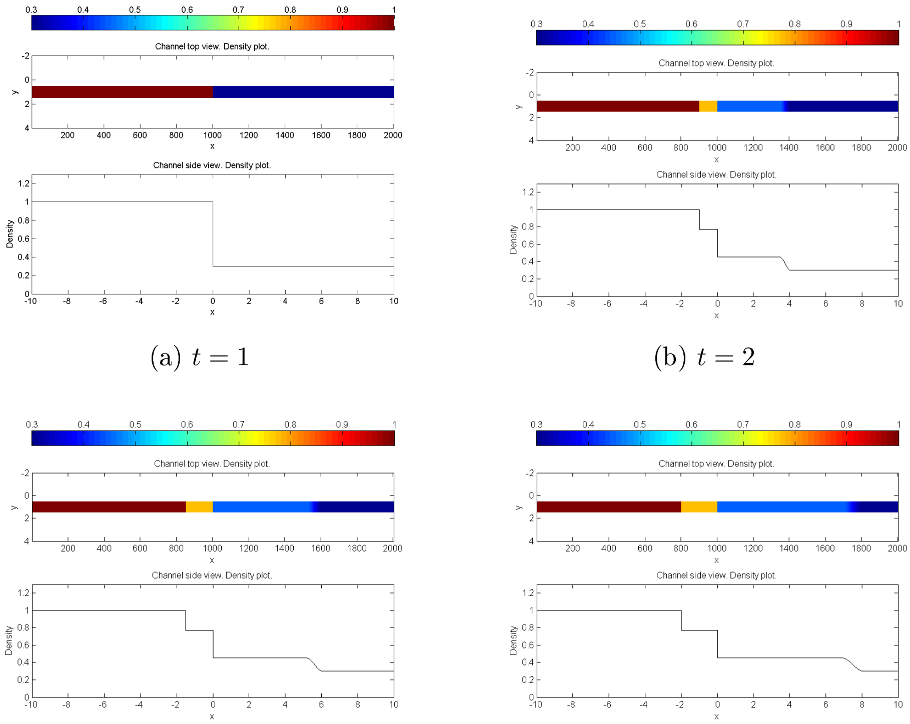


Figure 3.17: u evolution in space and time ($t = 4$). Solution of the system for $u_l = 1$, $\rho_l = 1$, $\rho_r = 1$, $u_r = 0.7$, $\alpha_1 = 0.9$, $\alpha_2 = 0.7$, $k = 1$, $\gamma = 1.4$



(a) $t = 1$ (b) $t = 2$

(c) $t = 3$ (d) $t = 4$

Figure 3.18: ρ evolution in space and time. Solving the system (3.128) with values of parameters are: $u_l = 1$, $\rho_l = 1$, $\rho_r = 0.3$, $u_r = 1.1$, $\alpha_1 = 0.3$, $\alpha_2 = 0.21$, $k = 1$, $\gamma = 1.4$

3.3.2.3 Region III

For this region we want to solve numerically the system (3.128). Solving with the parameter values $u_l = 1$, $\rho_l = 1$, $\rho_r = 0.3$, $u_r = 1.1$, $\alpha_1 = 0.3$, $\alpha_2 = 0.21$, $k = 1$, $\gamma = 1.4$ we get the solutions. As before the solution for the density looks like Figure (3.18) for some time steps. In this case it appear one shock moving to the left, one rarefaction moving to the right. The rarefaction speed is higher than the speed of the shock. Two intermediate states appear, $\rho_- = 0.771$ and $\rho_+ = 0.453$ separated by a contact discontinuity.

For the velocity its presented only the last time step, Figure (3.19). The behavior is the same, one shock moving to the left, and one rarefaction moving faster to the right. The effect of the sources is to generate two intermediate states $U_- = (0.771, 0.7)$ and $U_+ = (0.453, 0.7)$.

Now the last region.

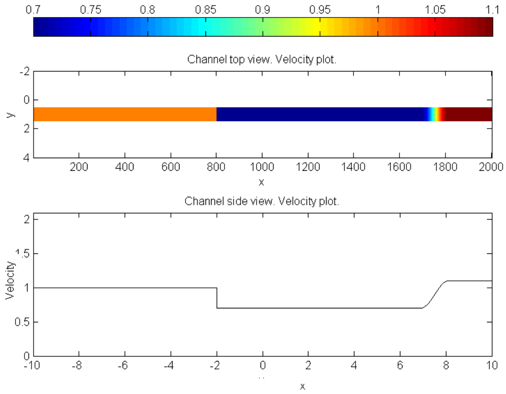


Figure 3.19: u evolution in space and time ($t = 4$). Solution of the system for $u_l = 1$, $\rho_l = 1$, $\rho_r = 0.3$, $u_r = 1.1$, $\alpha_1 = 0.3$, $\alpha_2 = 0.21$, $k = 1$, $\gamma = 1.4$

3.3.2.4 Region IV

Our last example is one of region IV, it has to be solved the system (3.129). Numerically it can be solved by setting $u_l = 1$, $\rho_l = 1$, $\rho_r = 1$, $u_r = 1.5$, $\alpha_1 = 1$, $\alpha_2 = 1.4$, $k = 1$, $\gamma = 1.4$. The result is shown in Figure (3.20).

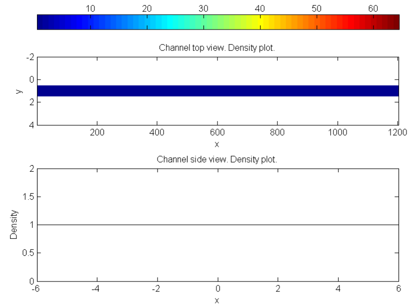
Here we can easily see two rarefaction waves moving one to the left and the other to the right, moving approximately with the same speed, its important to remark that it seems that the wave of the right is diffusing more rapidly that the one of the left. Also notice the appearance of a contact discontinuity at $x = 0$, the values of the densities at this point are $\rho_- = 1.387$ and $\rho_+ = 1.087$.
For the velocity, in the last time step, we have Figure (3.21). There are the same two rarefactions as in the density, but only one intermediate state. The effect of the sources is to generate two intermediate states $U_- = (1.387, 1.4)$ and $U_+ = (1.087, 1.4)$.

3.3.3 Analyzing the Model

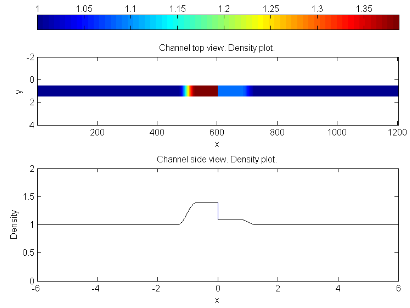
Recalling the equations (3.141), (3.142) and (3.148)

$$\begin{cases} \rho_t + (\rho u)_x = \alpha_1 H_x \\ (\rho u)_t + (p + \rho u^2)_x = \alpha_2 H_x \\ \Big(\rho(\frac{u^2}{2} + e)\Big)_t + \Big(\rho u(\frac{u^2}{2} + e) + pu\Big)_x = \alpha_3 H_x \end{cases} \tag{3.130}$$

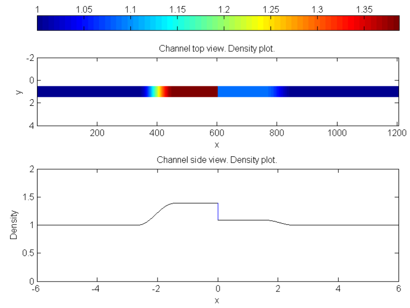
(3.131)



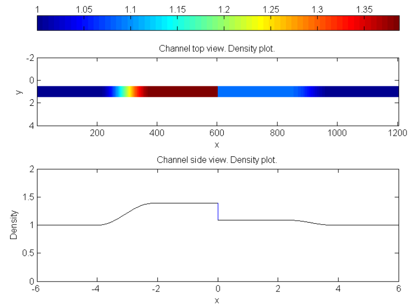
(a) $t = 1$



(b) $t = 2$



(c) $t = 3$



(d) $t = 4$

Figure 3.20: ρ evolution in space and time. Solving the system (3.129) with values of parameters are: $u_l = 1$, $\rho_l = 1$, $\rho_r = 1$, $u_r = 1.5$, $\alpha_1 = 1$, $\alpha_2 = 1.4$, $k = 1$, $\gamma = 1.4$.

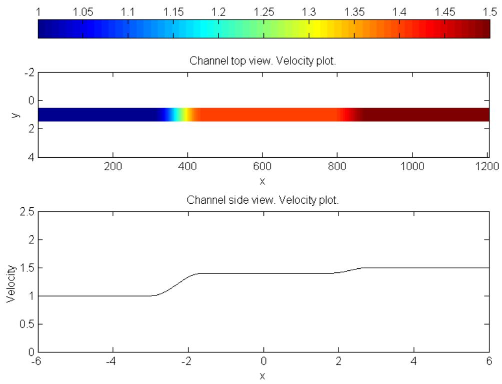


Figure 3.21: u evolution in space and time ($t = 4$). Solution of the system for $u_l = 1$, $\rho_l = 1$, $\rho_r = 0.3$, $u_r = 1.1$, $\alpha_1 = 0.3$, $\alpha_2 = 0.21$, $k = 1$, $\gamma = 1.4$

From the latter system in Conservative Form (i.e. without source terms), we know its shock and simple wave curves for the Riemann Problem.
The *Shock Curves* are given by

$$1.Shock\,Curves \;\; = \;\; \begin{cases} \frac{P_R}{P_L} = e^{-x} \\ \frac{\rho_R}{\rho_L} = \frac{e^x+\beta}{1+\beta e^{-x}} \\ \frac{u_R-u_L}{c_L} = \frac{2\sqrt{\tau}}{\gamma-1}\frac{1-e^{-x}}{\sqrt{1+\beta e^{-x}}} \end{cases}$$

$$3.Shock\,Curves \;\; = \;\; \begin{cases} \frac{P_R}{P_L} = e^{x} \\ \frac{\rho_R}{\rho_L} = \frac{1+\beta e^x}{e^x+\beta} \\ \frac{u_R-u_L}{c_L} = \frac{2\sqrt{\tau}}{\gamma-1}\frac{e^{-x}-1}{\sqrt{1+\beta e^{x}}} \end{cases}$$

The Simple Waves are:

$$1.Simple\,Wave\,Curves \;\; = \;\; \begin{cases} \frac{P_R}{P_L} = e^{-x} \\ \frac{\rho_R}{\rho_L} = e^{-\frac{x}{\gamma}} \\ \frac{u_R-u_L}{c_L} = \frac{2}{\gamma-1}(1-e^{-\tau x}) \end{cases} \qquad for \quad x \geq 0$$

$$3.Simple\,Wave\,Curves \;\; = \;\; \begin{cases} \frac{P_R}{P_L} = e^{x} \\ \frac{\rho_R}{\rho_L} = e^{\frac{x}{\gamma}} \\ \frac{u_R-u_L}{c_L} = \frac{2}{\gamma-1}(e^{\tau x}-1) \end{cases} \qquad for \quad x \geq 0$$

Where $\beta = \frac{\gamma+1}{\gamma-1}$, $\tau = \frac{\gamma-1}{2\gamma}$ and $c^2 = \frac{\gamma P}{\rho}$.

In order to solve the Riemann Problem in our case with the added sources we assume to expect a Steady Shock Curve as an effect due to the external forces, hence we introduce the following shock relations by the Rankine-Hugoniot condition

$$Shock\ Relations \;\; = \;\; \begin{cases} s[\rho] = [\rho u + \alpha_1 H_x] \\ s[\rho u] = [P + \rho u^2 + \alpha_2 H_x] \\ s[\frac{1}{2}\rho u^2 + \frac{1}{2}\rho e] = [\frac{1}{2}\rho u^3 + \rho u e + P u - +\alpha H_x] \end{cases}$$

Where the $[\cdot]$ represents the relative difference in the discontinuity.

As we said this are supposed to be steady shocks, therefore $s = 0$ getting:

$$\begin{cases} [\rho u] = -\alpha_1 H_x \\ [P + \rho u^2] = -\alpha_2 H_x \\ [\frac{1}{2}\rho u^3 + \rho u e + P u] = -\alpha_3 H_x \end{cases}$$

We are expecting two new states, that we will call $U_- = (v_-, u_-)^t$and $U_+ = (v_+, u_+)^t$, connected by the contact discontinuity. Moreover if we have a left state U_l and a right state U_r, then the left state will connect to the minus state, then this to the plus state, and after the plus state with the right state, schematically

$$U_l \to U_- \to U_+ \to U_r$$

Where the first connection can be done by a $S1$ or a $R1$, and the last connection can be done by $S2$ and $R2$. All this possibilities depend on the region where U_r lies.

3.3.3.1 Analyzing the Effect of the Steady Shock

As we know, in the case of the Ion transport across the cell, the ions densities are unequal. In order to generate some analytical results to have an steady shock we assume that our system behaves approximately like a gas dynamic problem, thus we expect equal pressures, i.e. $P_+ = P_L = P$, as well as the same velocities in the discontinuity $u_+ = u_- = u$. Which in the Conservation case result to be the Riemann Invariants for a Contact Discontinuity in the Euler Equations.

Now we expect to find a relation from the new Steady Shock Equations. First since $u_+ = u_- = u$. we multiply the first shock condition by u and substitute into the second and as $P_+ = P_L = P$ we get.

$$u_\pm = \frac{\alpha_2}{\alpha_1}$$

We know from the P-system that vacuum appears in the Riemann Problem when u is increasing, which is consistent with the latter relation, where the “mass source” is inversely proportional to the velocity.

Expanding (3.117) we know

$$\begin{aligned}
\frac{1}{2}\rho_+u_+^3 + \rho_+u_+e_+ + P_+u_+ - \frac{1}{2}\rho_-u_-^3 - \rho_-u_-e_- - P_-x_-u_- &= -\alpha_3 H_x \\
\frac{1}{2}\rho_+u_+^2 + \rho_+e_+ + P_+ - \frac{1}{2}\rho_-u_-^2 - \rho_-e_- - P_-x_- &= \frac{-\alpha_3}{u} H_x
\end{aligned}$$

Knowing the fact from the Second Thermodynamic Law and the shape of the pressure we know $e = \frac{c^2}{\gamma(\gamma-1)} = \frac{\gamma P}{\rho\gamma(\gamma-1)} = \frac{P}{\rho(\gamma-1)}$plugging it into the latter equation

$$\begin{aligned}
\frac{1}{2}\rho_+u_+^2 + \rho_+\frac{P_+}{\rho_+(\gamma-1)} + P_+ - \frac{1}{2}\rho_-u_-^2 - \rho_-\frac{P_-}{\rho_-(\gamma-1)_-} - P_-x_- &= \frac{-\alpha_3\alpha_1}{\alpha_2} H_x \\
\frac{1}{2}(\rho_+u_+^2 + P_+ - \rho_-u_-^2 - P_-x_-) &= \frac{-\alpha_3\alpha_1}{\alpha_2} H_x \\
-\frac{\alpha_2}{2} H_x &= \frac{-\alpha_3\alpha_1}{\alpha_2} H_x
\end{aligned}$$

The result is the relation

$$\alpha_3 = \frac{1}{2}\frac{\alpha_2^2}{\alpha_1}$$

We recall that α_1, α_2 and α_3 represent the sources of mass, momentum and energy, respectively. Thus the latter relation represents the kinetic energy into the system by the action of the sources.

In our problem this term is similar to a kinetic energy (rate of change of the energy) so this could be the one delivered by the ATP molecules in the ion pump. Moreover with the previous relations we can find

$$u = 2\alpha_3\alpha_1$$

on which we can conclude that is possible to have an increment of mass in the system having steady densities as long as we don't add energy to it.

3.3.4 Self similar Viscosity Approach for the Riemann Problem in Isentropic Gas Dynamics with added source

The main idea of this section is to present the first steps needed to proof the existance of solutions for the Euler equations with sources, following similar procedure as in [11] and [5].We consider the one dimensional system describing the isentropic motions of inviscid gases

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (p + \rho u^2)_x = \alpha_2 H_x \end{cases} \qquad x \in \mathbb{R}, \quad t > 0$$

where ρ, u, and $p(\rho)$ represent respectively the density, velocity and pressure.

We assume that the pressure function $p(\rho)$ satisfy

$$p'(\rho) > 0 \qquad for \qquad \rho > 0$$

$$p(\rho) \to \infty \;\text{ as } \rho \to \infty \qquad \text{and} \qquad p(\rho) \to 0 \quad \text{as} \quad \rho \to 0$$

with the initial data

$$(\rho(x,0), u(x,0)) = \begin{cases} (\rho_-, u_-), & x < 0 \\ (\rho_+, u_+), & x > 0 \end{cases}$$

This system can be described in terms of the variable $\xi = \frac{x}{t}$, so it can be rewritten as

$$(P) \equiv \begin{cases} -\xi\rho' + (\rho u)' = 0 \\ -\xi(\rho u)' + (p + \rho u^2 - \alpha_2 H)' = 0 \end{cases}$$

The goal through this subsection is to find, justify and analyze a solution for the latter system with a self similar viscosity approach, ie. the system

$$(P_\epsilon) \equiv \begin{cases} -\xi\rho' + (\rho u)' = 0 \\ -\xi(\rho u)' + (p + \rho u^2 - \alpha_2 H)' = \epsilon u'' \end{cases} \qquad \epsilon \in (0,1), \qquad (3.132)$$

with

$$\begin{cases} \rho(\pm\infty) = \rho\pm \\ u(\pm\infty) = u\pm \end{cases}$$

3.3.4.1 Weak Solution

Now we try to find a weak solution for (3.132) . Multiplying the first equation by the test function $\psi \in C_c^1(\mathbb{R})$ we have.

$$\int (-\xi\rho')\psi d\xi + \int (\rho u)'\psi d\xi = 0$$

After an integration by parts becomes the first term becomes

$$\begin{aligned} \int (-\xi\rho')\psi d\xi &= \int (\xi\rho - \int \rho d\xi)\psi' d\xi \\ &= \int (\xi\rho)\psi' d\xi + \int \rho\psi d\xi \end{aligned}$$

And the second one

$$\int (\rho u)'\psi d\xi = -\int (\rho u)\psi' d\xi$$

Hence (3.132.1) has a weak form as

$$\int (\xi - u)\rho\psi' d\xi + \int (\rho u)'\psi d\xi = 0$$

Following the same steps for (3.132.2)

$$\int -\xi(\rho u)'\psi d\xi + \int (p + \rho u^2 - \alpha_2 H)'\psi d\xi - \int \epsilon u''\psi d\xi = 0$$

solving for each term

$$\begin{aligned} \int -\xi(\rho u)'\psi d\xi &= \int (\xi\rho u\psi' - \psi' \int \rho u d\xi) d\xi \\ &= \int \xi\rho u\psi' d\xi + \int \rho u\psi d\xi \end{aligned}$$

$$\int (p + \rho u^2 - \alpha_2 H)'\psi d\xi = -\int (p + \rho u^2 - \alpha_2 H)\psi' d\xi$$

$$-\int \epsilon u''\psi d\xi = \epsilon \int u'\psi' d\xi$$

Therefore the definition in the weak sense of (3.132) is

$$\begin{aligned} \int (\xi - u)\rho\psi' d\xi + \int (\rho u)'\psi d\xi &= 0 \qquad (3.133) \\ \int [(\xi - u)\rho u - p + \epsilon u' + \alpha_2 H]\psi' d\xi + \int \rho u\psi d\xi &= 0 \end{aligned}$$

so the functions (ρ, u) are solutions of (3.132) if they satisfy the latter weak definition of the system, with $\psi \in C_c^1$, $\rho > 0$, where $\rho \in L_{loc}^\infty(\mathbb{R})$ and $u \in W_{loc}^1(\mathbb{R})$.

From the Weak formulation it can be seen that as $\rho \in L^{\infty}_{loc}(\mathbb{R})$ implies that $\rho u \in L^1_{loc}(\mathbb{R})$ which are the weak derivatives of $(\xi - u)\rho$ and $(\xi - u)\rho u - p + \epsilon u' + \alpha_2 H$ respectively which belong to the space $W^1_{loc}(\mathbb{R})$ and hence continuous a.e. $C(\mathbb{R} - \{0\})$.

Theorem 6. *Let (ρ, u) be a solution of (3.132). Then (i) for $a, b \in \mathbb{R}$,*

$$\begin{array}{c} [(\xi - u)\rho]_a^b + \int_a^b \rho d\xi = 0 \\ [(\xi - u)\rho u - p + \epsilon u' + \alpha_2 H]_a^b + \int_a^b \rho u d\xi = 0 \end{array} \tag{3.134}$$

(ii)u, $(\xi - u)\rho$ and $-p + \epsilon u' + \alpha_2 H$ are continuous a.e. on $\mathbb{R} - \{0\}$. If $p \in C^n(\mathbb{R}^+)$for $n \geq 0$, then ρ and u are $C^{n+1}(\mathbb{R}^+)$for all ξ such that $\xi \neq u(\xi)$.

Proof. We already saw that u, $(\xi - u)\rho$ and $-p + \epsilon u' + \alpha_2 H$ are continuous on $\mathbb{R} - \{0\}$. Fix $a, b \in \mathbb{R}$ with $a < b$ and consider

$$\psi_n(\xi) \begin{cases} 0 & -\infty < \xi \leq a - 1/n \\ n(\xi - a) + 1 & a - 1/n \leq \xi \leq a \\ 1 & a \leq \xi \leq b \\ -n(\xi - b) + 1 & b \leq \xi \leq b + 1/n \\ 0 & b + 1/n \leq \xi < +\infty \end{cases}$$

As $\psi_n \notin C^1_c(\mathbb{R})$ it cannot be directly used as a test function. However, since ψ_n is Lipschitz continuous, it can be approximated by $C^1_n(\mathbb{R})$ functions. Let the sequence $\psi^k_n \notin C^1_c(\mathbb{R})$ converge to ψ_n as $k \to \infty$. If we put ψ^k_n in the place of ψ in (3.133.2), then we get

$$\int [(\xi - u)\rho u - p + \epsilon u' + \alpha_2 H](\psi^k_n)' d\xi + \int \rho u \psi^k_n d\xi = 0$$

Taking the limit $k \to \infty$, we obtain

$$\begin{array}{c} n \int_{a-1/n}^{a} [(\xi - u)\rho u - p + \epsilon u' + \alpha_2 H] - n \int_b^{b+1/n} [(\xi - u)\rho u - p + \epsilon u' + \alpha_2 H] \\ + \int_{a-1/n}^{b+1/n} \rho u \psi_n d\xi = 0 \end{array} \tag{3.135}$$

The *Lebesgue Differentiation Theorem* states that for almost every point, the value of an integrable function is the limit of infinitesimal averages taken about the point, i.e. if a real or complex valued function f mapping a measurable set A to the Lebesgue Integral $\int_A f dy$ where dy denotes the n-dimensional Lebesgue measure, then the derivative of this integral at x is defined to be $\lim \frac{1}{|B|} = \int_B f dy$ when $B \to 0$ being B a ball centered at x.

The *Lebesgue Convergence Theorem* provides sufficient conditions under which two limit processes commute, namely Lebesgue integration and almost everywhere convergence of a sequence of functions. The statement says, assume that the sequence $\{f_n\}$ converges pointwise to a function f and is dominated by some integrable function g i.e. $|f_n(x)| \leq g(x)$,where in our case $|\rho u \psi_n| \leq |\rho u| \in L^1_{loc}$. Then the limiting function f is integrable and $\lim \int_S f_n d\mu = \int_S f d\mu$as $n \to 0$.

Due to the a.e. property of the Theorems we can apply the former to the first two terms and the latter to last one in (3.135) getting then (3.134.2). A similar statement show (3.134.1) .

Now taking (ii), owing to the weak formulation of the system (3.133) $(\xi - u)\rho$ is continuous, hence ρ is continuous at ξ if $\xi \neq u(\xi)$. From (3.134.2)

$$\begin{array}{c} \epsilon u'(\xi) = \int_a^\xi \rho(\varsigma)u(\varsigma)d\varsigma - [(\xi - u(\xi))\rho(\xi)u(\xi) - p(\rho(\xi)) + \alpha_2 H(\xi)] \\ +[(\xi - u(a))\rho(a)u(a) - p(\rho(a)) + \epsilon u'(a) + \alpha_2 H(a)] \end{array} \tag{3.136}$$

we have to remember that the Heavyside function $H(\xi) = 0$ for values of $\xi < 0$.

If p is $C^0(\mathbb{R})$, u is C^1at $\xi \neq u(\xi)$ and $\xi \neq 0$. It can also be said from (3.134.1),

$$(\xi - u(\xi))\rho(\xi) + = \int \rho(\varsigma)d\varsigma + (a - u(a))\rho(a)$$

Applying the derivative respect to ξ to the last equation we are able to find an expression for ρ' continuously dependant on u'. Therefore if p is $C^n(\mathbb{R})$ we can consider $(\rho, u) \in C^{n+1}(\mathbb{R})$as long as $\xi \neq u(\xi)$ and $\xi \neq 0$.

□

The demonstration that the singular point of a solution (ρ, u) in (76) is unique is demonstrated in [5] *Lemma 4* where we find out that $u(\xi) = \xi$ is a unique point in our problem (P_ϵ) and moreover two important relations are shown for some $\tau > 0$ small in $(s, s + \tau)$ we have $u(\xi) < \xi$ and in the case that $(s - \tau, s)$ then $\xi < u(\xi)$. The proof is the same since the conservation of momentum equation remains sourceless in our case. The only restriction to keep in consideration here is that the jump due to the source α_2 located at 0 should be small enough to keep the relations previously shown.

3.3.5 Monotonicity Properties

The monotonicity of solutions plays a key role in our problem. It can be easily verified that, at a point of smoothness, the solution (ρ, u) of (P_ϵ) satisfies

$$\begin{cases} (u-\xi)\rho' + \rho u' = 0 \\ (u-\xi)\rho u' + p(\rho)' - \alpha_2 H' = \epsilon u'' \end{cases}$$

Before starting with the monotonicity properties lets do some important remark: the singularity in this equations appear when the term $(u - \xi)$ is zero, i.e. in the fix point of u, different from case of P-system that it appear in $\xi = 0$. Next we want to analyze the behavior of (ρ, u) in a neighborhood of the singular point $\xi = s$, $\xi = 0$, and $\xi = \pm\infty$. From the latter we know

$$\rho' = \frac{\rho u'}{(\xi - u)} \tag{3.137}$$

$$\epsilon u'' = \frac{\{(\xi-u)^2 - p'(\rho)\}}{\xi - u}\rho u' - \alpha_2 H'$$

where H' is equal to the Dirac delta function δ which is 0 everywhere except in 0 . Since we know that due to the source we will have a steady jump in 0 for our solution, we can neglect this point, thus the latter equation can be written in a differential form

$$\frac{d}{d\xi}\left[u'(\xi)\exp\left\{\frac{1}{\epsilon}\int^{\xi}\frac{\{(\varsigma-u)^2 - p'(\rho)\}\rho}{\varsigma - u}d\varsigma\right\}\right] = 0 \tag{3.138}$$

keeping in mind that if $p \in C^0(\mathbb{R})$ then $u \in C^1(\mathbb{R})$as long as $\xi \neq u(\xi)$, $\xi \neq 0$ after doing an integration through ξ we get

$$u'(\xi) = \begin{cases} u'(\alpha_+)\exp\left\{-\frac{1}{\epsilon}\int_{\alpha+}^{\xi}\frac{\{(\varsigma-u)^2-p'(\rho)\}\rho}{\varsigma-u}d\varsigma\right\}, & s < \xi \\ u'(\alpha_0)\exp\left\{-\frac{1}{\epsilon}\int_{\alpha_0}^{\xi}\frac{\{(\varsigma-u)^2-p'(\rho)\}\rho}{\varsigma-u}d\varsigma\right\}, & 0 < \xi < s \quad \text{for } 0 < s \\ u'(\alpha_-)\exp\left\{-\frac{1}{\epsilon}\int_{\alpha-}^{\xi}\frac{\{(\varsigma-u)^2-p'(\rho)\}\rho}{\varsigma-u}d\varsigma\right\}, & \xi > 0 \end{cases}$$

for any $\alpha_{\pm 0}$ such that $0 < s < \alpha_+$, $\alpha_- < 0$ and $0 < \alpha_0 < s$. In the case where the source $\alpha_2 H$ is located before the singularity s, i.e. $0 < s$. And

$$u'(\xi) = \begin{cases} u'(\alpha_+)\exp\left\{-\frac{1}{\epsilon}\int_{\alpha+}^{\xi}\frac{\{(\varsigma-u)^2-p'(\rho)\}\rho}{\varsigma-u}d\varsigma\right\}, & 0 < \xi \\ u'(\alpha_0)\exp\left\{-\frac{1}{\epsilon}\int_{\alpha_0}^{\xi}\frac{\{(\varsigma-u)^2-p'(\rho)\}\rho}{\varsigma-u}d\varsigma\right\}, & s < \xi < 0 \quad \text{for } s < 0 \\ u'(\alpha_-)\exp\left\{-\frac{1}{\epsilon}\int_{\alpha-}^{\xi}\frac{\{(\varsigma-u)^2-p'(\rho)\}\rho}{\varsigma-u}d\varsigma\right\}, & \xi > s \end{cases}$$

for any $\alpha_{\pm_0}$ such that $0 < \alpha_+$, $\alpha_- < s$ and $s < \alpha_0 < 0$ when the $s < 0$.

Since the exponential is always positive u is strictly monotone on the domains $\{(s,\infty),(0,s),(-\infty,0)\}$ for $0 < s$ and $\{(0,\infty),(s,0),(-\infty,s)\}$ for $s < 0$.

Is clear that thanks to (3.137) that ρ when $0 < s$ has the same monotonicity as u on (s,∞) while holds the opposite in $(0,s)$ and$(-\infty,0)$. When $s < 0$ the monotonicity of ρ is equal to u in $(-\infty,s)$ and opposite in $(s,0),(0,\infty)$. Since we are only interested in solutions of $\rho(\xi) > 0$ due to our hypothesis then we will continue only focusing in the case $u'(\xi)$ for $0 < s$.

The monotonicity of the positive solution $\rho \in L^\infty$ implies that

$$0 < k \leq \rho(\xi) \leq K < \infty, \quad \xi \in \mathbb{R}$$

where k and K depend only on $\rho_\pm$and $\rho(s_\pm) = \lim_{\xi\to s_\pm}\rho(\xi)$.Under the hypothesis made that $p'(\rho) > 0$ for $\rho > 0$ we can say that $p'(\rho)$ is bounded by

$$0 < a_0 \leq p'(\rho) \leq A_0, \qquad \xi \in \mathbb{R}$$

where a_0 and A_0 may depend on k and K.

Theorem 7. *Let (ρ,u) be a solution of (P_ϵ) with a unique singular point $s \in \mathbb{R}$ where $0 < s$. (i) There exists three constants $\alpha_- < 0 < s$,$0 < \alpha_0 < s$, $0 < s < \alpha_+$, depending on a_0 and k, such that*

$$\begin{aligned} |u'(\xi)| &\leq |u'(\alpha_+)| \left\|\tfrac{\xi-s}{\alpha_+-s}\right\|^{\frac{\alpha}{\epsilon}}, && 0 < s < \xi < \alpha_+ \\ |u'(\xi)| &\leq |u'(\alpha_0)| \left\|\tfrac{\xi-s}{\alpha_0-s}\right\|^{\frac{\alpha}{\epsilon}}, && 0 < \alpha_0 < \xi < s \\ |u'(\xi)| &\leq |u'(\alpha_-)| \left\|\tfrac{\xi-s}{\alpha_--s}\right\|^{\frac{\alpha}{\epsilon}}, && \alpha_- < \xi < 0 < s \end{aligned}$$

(ii) There exists three constants $\beta_- < 0 < s$, $0 < \beta_0 < s$ and $0 < s < \beta_+$, depending on A_0, and a constant $\beta > 0$, depending on A_0 and k , such that

$$\begin{aligned} |u'(\xi)| &\leq |u'(\beta_+)| \exp\left\{-\tfrac{\beta}{\epsilon}\left(\left(\tfrac{\xi-s}{\beta_+-s}\right)^2 - 1\right)\right\}, && 0 < s < \xi < \beta_+ \\ |u'(\xi)| &\leq |u'(\beta_0)| \exp\left\{-\tfrac{\beta}{\epsilon}\left(\left(\tfrac{\xi-s}{\beta_0-s}\right)^2 - 1\right)\right\}, && 0 < \beta_0 < \xi < s \\ |u'(\xi)| &\leq |u'(\beta_-)| \exp\left\{-\tfrac{\beta}{\epsilon}\left(\left(\tfrac{\xi-s}{\beta_--s}\right)^2 - 1\right)\right\}, && \beta_- < \xi < 0 < s \end{aligned}$$

(iii) $u'(s) = 0$ *for the pressure* $p \in C^n(\mathbb{R}^+)$, $n \geq 1$, *the solution* (ρ, u) *has the regularity*

$$\rho \in C(\mathbb{R}) \cup C^{n+1}(\mathbb{R} - \{s\} - \{0\}) \qquad ; \qquad u \in C^1(\mathbb{R}) \cup C^{n+1}(\mathbb{R} - \{s\} - \{0\}) \tag{3.139}$$

Proof. $u(\xi) \to u_+$as $\xi \to \infty$ and $u'(s)$ is finite from (3.136). We have $u(\xi) < \xi$ on (s, ∞) when $0 < s$. Thus there is a positive constant b dependant on α_2 such that $-b(\xi - s) + s < u(\xi) < \xi$ always holds on (s, ∞) (see Figure). Let α_+ be a constant such that $s < \alpha_+ < s + \frac{\theta}{b+1}$ with $\theta = \sqrt{a_0}$. Then for all $\varsigma \in (s, \alpha_+)$.

$$\frac{(\varsigma - u)^2 - p'(\rho)\rho}{\varsigma - u} \leq \left\{(1+b)(\alpha_+ - s)^2 - \frac{\theta^2}{(1+b)}\right\} \frac{\rho}{\varsigma - s} \leq -\alpha \frac{1}{\varsigma - s} < 0$$

with

$$\alpha = \frac{(\theta^2 - (1+b)^2(\alpha_+ - s)^2)k}{1+b}$$

Then α is positive and from (3.138)

$$|u'(\xi)| \leq |u'(\alpha_+)| \exp\left\{\frac{\alpha}{\epsilon} \int_{\alpha_+}^{\xi} \frac{1}{\varsigma - s} d\varsigma\right\} = |u'(\alpha_+)| \left(\frac{\xi - s}{\alpha_+ - s}\right)^{\frac{\alpha}{\epsilon}}$$

for all $\xi \in (s, \alpha_+)$. The second and third statement of (i) can be proved similarly since $\xi < u(\xi)$ for $(-\infty, 0)$, $(0, s)$.

Now we prove for (ii). Fix $\beta_+ > s + max\{2(u_+ - u_-), 2\sqrt{2\Theta}\}$ with $\Theta = \sqrt{A_0}$. Then, for any $\xi \in (\beta_+, \infty)$,$\xi - u(\xi) \geq \frac{1}{2}(\xi - s)$ and

$$\frac{\{(\varsigma - s)^2 - p'(\rho)\}\rho}{\varsigma - u} \geq \left\{\frac{1}{2} - \frac{2\Theta^2}{(\varsigma - s)^2}\right\} \rho(\varsigma - s) \geq \frac{k}{4}(\varsigma - s) > 0$$

Set

$$\beta = \frac{(\beta_+ - s)^2}{2} \frac{k}{4}$$

Then β is positive and

$$\begin{aligned} |u'(\xi)| &\leq |u'(\beta_+)| \exp\left\{ -\tfrac{2\beta}{\epsilon(\beta_+ - s)^2} \int_{\beta_+}^{\xi} \varsigma - s d\varsigma \right\} \\ &= |u'(\beta_+)| \exp\left\{ -\tfrac{\beta}{\epsilon} \left(\left(\tfrac{\xi - s}{\beta_+ - s} \right)^2 - 1 \right) \right\} \end{aligned}$$

As in (i) the proof for the second and third statement in (ii) are similar.

Part (i) implies regularity for u' near the singular point $\xi = s$ and specially that $u'(s) = 0$. Since $-p(\rho) + \epsilon u' + \alpha_2 H'$ is continuous in $\mathbb{R} - \{0\}$, ρ is also continuous due to the proportionality between $p(\rho)$ and ρ. Hence the regularity of the solutions (ρ, u) is improved to (3.139).

□

3.3.6 *A-priori* Estimates

In this section we will consider the solutions of the following system in $-\infty < \xi < \infty$

$$\begin{cases} (u - \xi)\, \rho' + \rho u' = 0 \\ (u - \xi)\, \rho u' + p'(\rho) = \epsilon u'' + \alpha_2 H_x \end{cases} \tag{3.140}$$

with the boundary conditions:

$$\begin{aligned} \rho(\pm\infty) &= \rho_- + \mu(\rho_\pm - \rho_-) \qquad\qquad 0 \leq \mu \leq 1 \\ u(\pm\infty) &= u_- + \mu(u_\pm - u_-) \end{aligned}$$

Notice that the boundary conditions for ρ are all positive, so the family of solutions of this system have the same regularity and monotonicity derived before. The goal of this section is to prove the following estimations of the solutions are independent of μ and ϵ

$$0 < \delta < \rho(\xi) < M$$

$$|u(\xi)| < M$$

this two in $-\infty < \xi < \infty$, and

$$-b(\xi - s) + s < u(\xi) < a(\xi - s) + s$$

for $s < \xi < \infty$

$$a(\xi - s) + s < u(\xi) < -b(\xi - s) + s$$

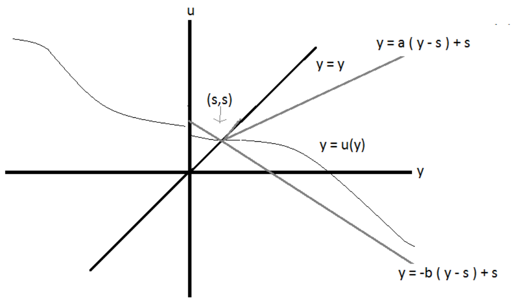


Figure 3.22: Jump in u. $0 < s$.

for $-\infty < \xi < s$. We have to prove also that $0 < a < 1$ and $0 < b$. Combining the last two we get

$$A|\xi - s| < |u - \xi| < B|\xi - s|$$

with $s \neq \xi$.

Before starting the justification of the previous estimates, we have to consider and classify all the possible behaviors of monotonicity of the solutions. It's known that

$$\rho' = \frac{\rho u'}{(\xi - u)}$$

so the monotonicity of ρ will be dependent of the monotonicity of u and the sign of $(\xi - u)$. As seen before *LEMMA 4* from *[5]* if $s < \xi$ then $\xi > u$ and the opposite relation holds when $\xi < s$. Considering this we have to different options, one is when $0 < s$, the result of monotonicities is then summarized in Table (3.1).

The other option is when we have the case of $s < 0$, so the regions will change and the results are summarized in Table (3.2).

We want to prove that the constants that bound the solutions are independent of μ and ϵ, we will prove this in the case that α_2 is such that the jump in the velocity is small enough not to over pass the value of u_+or u_-(i.e. the asymptote), see Figure (3.22) or Figure (3.23)[6]. In other words, α_2 such

[6]This two figures are only examples on how the solution of u will be, and to give an

Case	u			ρ		
$0 < s$	I	II	III	I	II	III
1	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
2	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
3	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$
4	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$
5	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$
6	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$
7	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow$
8	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow$

Table 3.1: Monotonicity for $0 < s$. $I = (-\infty, 0)$, $II = (0, s)$, $III = (s, \infty)$

Case	u			ρ		
$s < 0$	I	II	III	I	II	III
1	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
2	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
3	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow$
4	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow$
5	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$
6	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$
7	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$
8	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$

Table 3.2: Monotonicity for $s < 0$. $I = (-\infty, s)$, $II = (s, 0)$, $III = (0, \infty)$

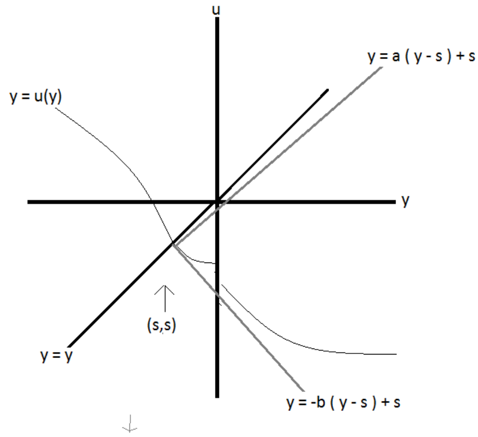


Figure 3.23: Jump in u. $s < 0$.

that the monotonicity of the solution doesn't change in a neighborhood of 0. So with this restriction we will give justification about the bounds for the solution only in the first four cases of tables (3.1) and (3.2).

Theorem 8. *Let (u, ρ) be a solution of the system (3.140). If the solution has the behavior of cases 1,2 or 3, in the tables (3.1) and (3.2), then there exists (M, δ) constants bounding the solution, that are independent of μ and ϵ. For solutions satisfying the behavior of case 4 in the same tables, there exist a constant M independent of μ and ϵ such that bounds the solution.*

Proof. First part, the existence of δ we can see that for cases 1 and 2 we can take $\delta = min\{\rho_-, \rho_+\}$ and $M = min\{\rho_-, \rho_+\}$. For case 3 we can take the same δ and for case 4 we can take the same M. Now we have to prove the lower bound for the case 3, from [5] Lemma 6[7] we can see that this bound exist and the value is $max\{\rho_-, \rho_+\} + \frac{1}{\xi - s}(u_- - u_+)$.

Now lets justify the existence of A. Let[8] $\tau \epsilon [s+1, s+2]$ then is satisfied $u'(\tau) = u(s+2) - u(s+1) > u_+ - u_-$, and lets consider the case where $s+1 < 0$ and $0 < s+2$, notice that $u'(\tau)$ is bounded even if τ is positive or negative (in this range of s), this because α_2 is such that the monotonicity

idea of how the geometry works, in the case of always decreasing monotonicity, but the logic and proofs are analogous even if monotonicities are different.

[7]We see in the proof that this is proven using only the density, and because in our problem we have a source only in the equation of momentum, the same proof apply.

[8]We are considering this interval because we will use this τ to make the proof, and in this range we can pass the discontinuity and find some α_2 in the proof, the other cases of s and τ are considered below.

doesn't change near zero. Lets assume that $\tau > 0$, so integrating the second equation of (3.140) from $\xi > s$ to τ we get

$$\rho(\xi)u^2(\xi) + p(\rho(\xi)) - \epsilon u'(\xi) - \rho(\tau)u^2(\tau) - p(\rho(\tau)) +$$

$$\epsilon u'(\tau) - \alpha_2 H(\xi) + \alpha_2 H(\tau) = -\int_\xi^\tau \xi^*(\rho u')d\xi^* =$$

$$\int_\xi^\tau \xi^* \left(\rho(s-u)\right)' d\xi^* - s\int_\xi^\tau \xi^* \rho' d\xi^* = \left[\xi\left(\rho(s-u)\right)\right]_\xi^\tau - \int_\xi^\tau \left(\rho(s-u)\right) d\xi^* - s\int_\xi^\tau \xi^* \rho' d\xi^*$$

$$\tau\left(\rho(\tau)(s-u(\tau))\right) - \xi\left(\rho(\xi)(s-u(\xi))\right) - \int_\xi^\tau \left(\rho(s-u)\right) d\xi^* - s\rho(\tau)u(\tau) + s\rho(\xi)u(\xi) \leq$$

$$\leq \tau\left(\rho(\tau)(s-u(\tau))\right) - -s\rho(\tau)u(\tau) + s\rho(\xi)u(\xi)$$

then if $\xi \to s$ and because $\tau > 0$, we have $H(\xi) = 0$ and $H(\tau) = 1$ so we find a condition that α_2 has to satisfy in order to have A as wanted (for this particular case of s and τ)

$$\rho(\xi)u^2(\xi) + p(\rho(\xi)) - \epsilon u'(\xi) - \rho(\tau)u^2(\tau) - p(\rho(\tau)) + \epsilon u'(\tau) + \alpha_2 \leq$$

$$\leq \tau\left(\rho(\tau)(s-u(\tau))\right) - s\rho(\tau)u(\tau) + s\rho(\xi)u(\xi)$$

moreover we get

$$p(\rho(s)) \leq max_{\rho_+ < q < \rho(s+1)}\{3qu^* + p(q)\} + (u_- - u_+) - \alpha_2 \equiv A$$

where $u^* = max\{|u_-|, |u_+|\}$, so A is independent of μ and ϵ. In the case just described if we put $\tau < 0$ the proof is the same but without α_2. For the case of $s > 0$ then $\tau > s$ and in this region there is no source so the proof is the same without α_2. The same happends if $s + 2 < 0$. Notice that if $\tau = 0$ we have a problem, because we cannot valuate the expressions here, but if for $\tau > 0$ and $\tau < 0$ is bounded, and α_2 doesn't change the monotonicity, then for the equality to zero it should be bounded, and this finishes all the cases. □

Now lets prove the bound for the velocity. For the cases 3 and 4 of the tables is enough to have $M = max\{u_-, u_+\}$[9]. Now from the shape of the solution we know that $|u(\xi)| \leq \{|u_\pm|, |u(s)|, |u(0^\pm)|\}$, where $0^\pm$ denotes the value of the velocity in the limit from the left, and from the right, and we know that this value is finite because α_2 is finite, so the only thing is to prove that the singular point $u(s) = s$ is bounded. In [5] we find that this value is bounded by $u(s) \leq u(\alpha) + \frac{\rho_-}{\rho_+ - 1}$, where $u(\alpha)$, $\alpha > s$, then is we have a discontinuity is obvious that $u(s)$ can be bounded by

$$u(s) \leq u(\alpha) + \frac{\rho_-}{\rho_+ - 1} + max\{|u(0^+)|, |u(0^-)|\}$$

and this is independent of μ and ϵ. The proof for the case 1 is similar. This completes the proof.

As seen in [5] the lower bound of case 4 can depend in ϵ.

Theorem 9. *Assume that $0 < \delta_\epsilon < \rho$, and that α_2 is such that the singular point is unique*[10]*, then there exist constants $0 \leq a \leq 1$, $0 \leq b$ dependins the initial data and constants δ and M in the previous theorem such that they satisfy the bounding of u.*

Proof. From the system (3.140) we can get

$$\left(p'(\rho(\xi)) - (u(\xi) - \xi)^2 - \frac{\alpha_2 H'}{\rho'(\xi)}\right)\rho'(\xi) = \epsilon u''(\xi)$$

.

Since we know that at the point where $\alpha_2 H'$ is valid, i.e. in 0 we will have a finite discontinuity. Hence we can say that for all the points in the domain different than 0 . [11]

$$\left(p'(\rho(\xi)) - (u(\xi) - \xi)^2\right)\rho'(\xi) = \epsilon u''(\xi)$$

Lets focus on some cases, first lets say that u is increasing on (s, ∞). Following exactly the same idea[12] as Lemma 7 of [5], we should choose α_2 such that

[9]Notice that this fact is true because we have that α_2 is small enough.

[10]This means that the jump caused by α_2 is small enough such the graph of $y = u(\xi)$ does not cross $y = \xi$ more than one time.

[11]Since the jump of u in 0 will shift the graph a little bit, such that it does not overpass the line $y = \xi$, the value of the constant a will just change a small amount but it will still be less than 1.

[12]And using the same definitions and names.

$$\epsilon u''(\xi) < 0$$

(moreover in this region $\rho' > 0$ so there is no problem with $\frac{1}{\rho'}$ in the expression) and then the value of a will be:

$$a \leq max\{\frac{1}{2}, \frac{2(u_+ - s) - \sqrt{a_0}}{2(u_+ - \sqrt{a_0})} < 1\}$$

Now the case of u decreasing. choose α_2 such that $\epsilon u''(\xi) > 0$ (moreover in this region $\rho' < 0$ so there is no problem with $\frac{1}{\rho'}$ in the expression). With this restriction on the source Lemma 7 of [5] shows that $a = 0$ and $0 < b$. [13]

□

[13]Notice that this proof is somehow technical, and rigorous, but with the restrictions given to α_2 the results are obvious from the geometrycal point of view, as seen in the Figures (3.1) and (3.2) and considering the strict monotonicities.

Conclusions

In this thesis we mean to develope a more robust model for the ion transport in the cell membrane based on the idea of [4]. We introduce the Ion pumps into the model, since they are an important active process for the ion transport. By doing this we expect to develope a model where the gathing for the Ion channels can also be included, allowing us to incorporate the biogical behaviour seen in the cells as the resting and action potential, developing a more complete model for this problem.

In the first chapter it has been presented all the theoretical background needed to understand the problem whereas in the second chapter a few applications were described; due to this work is meant to be the beggining of a deeper research, this can be regard as small set of goals for future works.

The third chapter introduces the mathematical background needed to do the model, the first approach to model the ion transport taking into account the ion pumps was done by analyzing the P-system with added sources and a forced steady shock in the origin. The existence of solutions for the Riemann problem in here was done with a specific pressure law, using a self-similar viscosity approach as in [11]. A more general model was introduced adding sources and steady shock to the Euler equations, this show us the effect of the forced contact discontinuity, which introduce a kinetic energy to the system, important remark and justification to keep working in this model since it cames up naturally in the mathematics the energy from the ATP molecules consumed by the Ion pumps. Using the self-similar viscosity approach in this latter model with a source in the momentum equation it was possible to conclude that the solution for the density and the velocity are continues and monotonic in regions I,II and III of the domain (see section 3.3.5), except on the origin, in the case of not vanishing viscosity. Also we have done some numerical calculations (solutions of a simplified model) to sketch the appearance of the results showing the two intermediate states connecting the initial data.

Appendix

Equivalence of systems.

Here we will justify the equivalence of the systems (3.141) and (3.142). This will help to compute the eigenvalues and eigenvectors of the system in a more easy way.

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (p + \rho u^2)_x = 0 \\ \Big(\rho(\frac{u^2}{2} + e)\Big)_t + \Big(\rho u(\frac{u^2}{2} + e) + pu\Big)_x = 0 \end{cases} \tag{3.141}$$

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + uu_x + p_x\rho = 0 \\ s_t + us_x = 0 \end{cases} \tag{3.142}$$

The system (3.141) has the form $\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0$ so the eigenvalues and the eigenvectors are given by the Jacobian, due to the relations of the unknowns in the equations, the Jacobian is hard to compute.

In (3.142) the shape is $\mathbf{U}_t + \mathbf{F}(\mathbf{U})\mathbf{U}_x = 0$ (quasilinear), since here $\mathbf{F}$ is a matrix, eigenvalues and eigenvectors are given by it, which is easier to solve.

We shall proof that both systems are equivalents, then the eigenvalues that we got in (3.142) correspond also for (3.141).

Proof. First we verify that equations $(3.141)_1$ and $(3.142)_1$ are the same, then for simplicity in the following process we write it down as

$$\rho_t = -(\rho u)_x \tag{3.143}$$

Then we find the equivalence for the second equation on the systems. We expand $(3.141)_2$ and substitute after (3.143) obtaining thus $(3.141)_2$ is the same as $(3.141)_1$

$$
\begin{aligned}
(\rho u)_t + (p+\rho u^2)_x = (\rho)_t u + (u)_t \rho + p_x + \rho_x u^2 + 2\rho u u_x \\
= -(\rho u)_x u + u_t \rho + p_x + \rho_x u^2 + 2\rho u u_x \\
= -(\rho)_x u^2 - u_x u \rho + u_t \rho + p_x + \rho_x u^2 + 2\rho u u_x \\
u_t \rho + p_x + \rho u u_x = u_t + u u_x + p_x/\rho = 0
\end{aligned}
$$

The equation in $(3.141)_3$ is described in terms of energy while $(3.142)_3$ is expressed with the entropy, we need to relate this unknowns. The second law of thermodynamics asserts a relation between energy and entropy.

$$TdS = de + pdv = de + pd\left(\frac{1}{\rho}\right) \tag{3.144}$$

where T is temperature, $v = 1/\rho$ dS, de, dv are differentials of entropy, energy and density volume respectively.

The derivative of (3.144) respect to time t and space x give us.

$$e_t = TS_t + \frac{p}{\rho^2}\rho_t \tag{3.145}$$

$$e_x = TS_x + \frac{p}{\rho^2}\rho_x \tag{3.146}$$

$$\tag{3.147}$$

Expanding (3.141.3) and using (3.143)we have

$$
\begin{aligned}
&\left(\rho(\frac{u^2}{2}+e)\right)_t + \left(\rho u(\frac{u^2}{2}+e)+pu\right)_x = \\
&= \left(\rho_t(\frac{u^2}{2}+e)\right) + \rho\left(\frac{u^2}{2}+e\right)_t + (\rho u)_x(\frac{u^2}{2}+e) + (\rho u)(uu_x+e_x) + (pu)_x + (pu)_x \\
&= \rho(uu_t+e_t) + (\rho u)(uu_x+e_x) + (pu)_x
\end{aligned}
$$

Then using the relations of the derivatives of the energy and (3.142.2) we can write the latter equation as

$$\begin{aligned}\rho(uu_t + TS_t + \frac{p}{\rho^2}\rho_t) + (\rho u)\,(uu_x + TS_x + \frac{p}{\rho^2}\rho_x) + up_x + pu_x = \\ = \rho(uu_t + TS_t + \frac{p}{\rho^2}\rho_t) + (\rho u)\,(uu_x + TS_x + \frac{p}{\rho^2}\rho_x) + up_x + pu_x \\ = \rho u\left(u_t + uu_x + \frac{p_x}{\rho}\right) + \frac{p}{\rho}\left[-(u\rho)_x + (u\rho_x + u_x\rho)\right] + T\rho S_t + T\rho u S_x \\ T\rho(S_t + uS_x) = S_t + uS_x = 0\end{aligned}$$

□

As we have seen the equations are equivalent so the eigenvalues and eigenvectors hold for both systems. We want to introduce the system.

$$\begin{cases}\rho_t + (\rho u)_x = \alpha_1\delta(x) \\ (\rho u)_t + (p + \rho u^2)_x = \alpha_2\delta(x) \\ \left(\rho(\frac{u^2}{2} + e)\right)_t + \left(\rho u(\frac{u^2}{2} + e) + pu\right)_x = \alpha_3\delta(x)\end{cases} \tag{3.148}$$

where $H_x = \delta(x)$ is the delta-Dirac as the derivative in space of the Heaviside function, since we are trying to add a point-source to the system.
As (3.148) can be represented as (3.141) and due to the eigenvalues in that case can be computed with the Jacobian, the point-source term vanishes, let it then the same eigenvalues and eigenvectors as in (3.141) and (3.142).

Table Parameters

Knowing that systems (3.141) and (3.142) are equivalent, we can compute the eigenvalues and eigenvectors. From the system (3.142) we have the Jacobian.

$$\begin{pmatrix} u & \rho & 0 \\ p_\rho/\rho & u & p_s/\rho \\ 0 & 0 & u \end{pmatrix}$$

The characteristic equation for the eigenvalues is given by

$$(u - \lambda)[(u - \lambda)^2 - p_\rho] = 0$$

Hence with $c = \sqrt{p_\rho}$ the **eigenvalues** are: $\lambda_1 = u - c$,$\lambda_2 = u$ and $\lambda_3 = u + c$, with the corresponding **eigenvectors** $(\rho, -c, 0)^t$, $(p_s, 0, -p_\rho)^t$ and $(\rho, c, 0)^t$.

Riemann Invariants

Now we take the **Riemann invariants**. Since we are in $\mathbb{R}^3$ then we should have 2 Riemann invariants for each eigenvalue. From the definition we know that $\langle \mathbf{r}, \nabla_u w \rangle = 0$, so

- for λ_1:

  $\langle (\rho, -c, 0)^t, (w_\rho, w_u, w_s) \rangle = 0$ then $\rho w_\rho - c w_u = 0$.

  So the Riemann Invariants are dictated by

  $$w = s, u - h$$

- For λ_2 we have.

  $\langle (p_s, 0, -p_\rho)^t, (w_\rho, w_u, w_s) \rangle = 0$ then $p_s w_\rho - p_\rho w_s = 0$

  The Riemann Invariants in this case are:

  $$w = u, p$$

- And for λ_3 the results are:

  $\langle (\rho, c, 0)^t, (w_\rho, w_u, w_s) \rangle = 0$ then $rhow_\rho + c w_u = 0$

  with Riemann Invariants:

  $$w = s, u + h$$

  //

  Where $h(\rho, s)$ satisfies $h_\rho = c/\rho$ and is called *enthalpy*.

Linearly Degenerate and Genuinely nonlinear

To finish with the table data, we check for linearly degenerate ($\nabla \lambda_k \cdot r_k = 0$) and genuinely nonlinear ($\nabla \lambda_k \cdot r_k \neq 0$) conditions.

- For λ_1

  $$\nabla \lambda_k \cdot (\rho, -c, 0) = c - p c_\rho$$

- For λ_2

$$\nabla\lambda_k \cdot (p_s, 0, -p_\rho) = 0$$

- For λ_3

$$\nabla\lambda_k \cdot (\rho, c, 0) = c + pc_\rho$$

Hence with $c = \sqrt{p_\rho}$, the **eigenvalues** are: $\lambda_1 = u-c$,$\lambda_2 = u$ and $\lambda_3 = u+c$, with **eigenvectors** $(\rho, -c, 0)^t$, $(p_s, 0, -p_\rho)^t$ and $(\rho, c, 0)^t$respectively.

Scalling of the Hydrodynamical Model

Having justify the use of the Hydrodynamical model to describe our concerning problem now the goal is to scale the system, to do this we need some scaling factors for all the variables. For computational reasons, we will define the density of current $J = nv$. Now the scaling is as follows

Symbol	Meaning	Scaling Factor	Scale Transition*
n	Ion density	n_i given by $\max(n_D)$	$n_s = \frac{n}{n_i}$
n_D	Permanent density distribution	n_i	$n_{D_s} = \frac{n_D}{n_i}$
x	Space variable	L (channel length)	$x_s = \frac{x}{L}$
t	Time variable	t^* given by $t^* = \frac{k_B T_0}{q}$	$t_s = \frac{t}{t^*}$
T	Temperature variable	T_0	$T_s = \frac{T}{T_0}$
v	Velocity	$\frac{L}{t^*}$	$V_s = \frac{V}{L/t^*}$
Φ	Potential	$\frac{k_B T_0}{q}$	$\phi_s = \phi / \left(\frac{k_B T_0}{q}\right)$
p	Momentum variable	$mn_i \frac{L}{t^*}$	$p_s = p / \left(mn_i \frac{L}{t^*}\right)$

Table 3.3: Scaling Factors

Let's start with equation (3.5), substituting J and $E = -\Phi_x$ in the equation, it reads:

$$J_t + \left(\frac{J^2}{n}\right)_x + \left(\frac{k_B}{m} nT\right)_x = -\frac{e}{m} n\Phi_x - \left(\frac{J}{\tau_p}\right)$$

Symbol	Meaning	Value	
κ	Thermoconductivity	$3\mu_0 k_B^2 T_0/2e$	
μ_0	Mobility		
m	Mass		
e	Charge		
k_B	Boltzman constant		

Table 3.4: Variables

The next step is to substitute the scaling factors in the equation, e.g. we will put $x \to Lx_s$:

$$\frac{1}{t^*}\frac{k_B T_0 n_i t^*}{mL} J_{st} + \frac{k_B^2 T_0^2 n_i^2 t^{*2}}{m^2 L^3 n_i}\left(\frac{J_s^2}{n_s}\right)_x + \frac{k_B T_0 n_i}{mL}(n_s T_s)_x =$$
$$-\frac{en_i}{mLk_B T_0}\frac{k_B T_0}{e} n_s \Phi_{sx} - \frac{k_B T_0 n_i t^*}{mL}\left(\frac{J_s}{\tau_p}\right)$$

Now eliminating constants we arrive at:

$$J_{st} + \frac{k_B T_0 t^{*2}}{mL^2}\left(\frac{J_s^2}{n_s}\right)_x + (n_s T_s)_x = -\frac{e}{k_B T_0}\frac{k_B T_0}{e} n_s \Phi_{sx} - \left(\frac{t^* J_s}{\tau_p}\right)$$

Using the relation $k_0 T_0 = \frac{mL^2}{t^{*2}}$ and defining $\frac{1}{\tau} = \frac{t^*}{\tau_p}$ we get the adimensional form of (3.5):

$$J_{st} + \left(\frac{J_s^2}{n_s}\right)_x + (n_s T_s)_x = -n\Phi_{sx} - \left(\frac{J}{\tau}\right)$$

Returning to the original variable p and dropping the "s" we have the adimentional form of (3.5):

$$p_t + (pv + nT)_x = nE - \left(\frac{p}{\tau}\right) \tag{3.149}$$

Now we scale for the Conservation of Mass

$$n_t + (nv)_x = 0$$

Scaling (substituting for the scale factor and the scaled variable) we reach

$$\begin{aligned} n_t + (nv)_x &= \left(\frac{n_i}{t^*}\right)(n_s)_{t_s} + \left[\frac{(n_i L)}{t^s}(n_s v_s)\right]_{x_s}\left(\frac{1}{L}\right) \\ &= (n_s)_{t_s} + (n_s v_s)_{x_s} = 0 \end{aligned}$$

Dropping the subindex s our Dimensionless Conservation of Mass equation is

$$n_t + (nv)_x = 0 \tag{3.150}$$

Scaling the Poisson Equation we have.

$$-\lambda\phi_{xx} = e(n + n_D)$$

Substituting.

$$\begin{aligned} -\lambda\phi_{xx} &= e(n + n_D) \\ -\frac{\lambda}{L^2}\phi_{x_s x_s} &= e(n_i n_s + n_i n_{D_s}) \\ -\frac{\lambda}{L^2}\phi_{x_s x_s} &= e n_i(n_s + n_{D_s}) \\ -\lambda^*\phi_{x_s x_s} &= (n_s + n_{D_s}) \end{aligned}$$

With $\lambda^* = \frac{\lambda}{eL^2 n_i}$.

Dropping the subindex "s" we have

$$-\lambda^*\phi_{xx} = n + n_D \tag{3.151}$$

Continuing with scaling, we will take equation (3.6). As before we will rewrite the equations in terms of J, first in the definition of w

$$w = \frac{3}{2}nk_B T + \frac{1}{2}nmv^2 = \frac{3}{2}nk_B T + \frac{m}{2}\frac{J^2}{n}$$

and then we put everything in (3.6)

$$\begin{aligned} &\frac{3}{2}k_B(nT)_t + \frac{m}{2}(\frac{J^2}{n})_t + \frac{3}{2}k_B(JT)_x + \frac{m}{2}(\frac{J^3}{n^2})_x + k_B(JT)_x \\ = \quad &-eJ\Phi_x - \frac{3k_B}{2\tau_w}n(T - T_0) - \frac{m}{2\tau_w}(\frac{J^2}{n}) + (\kappa n T_x)_x \end{aligned}$$

Using the scaling factors define above

$$\begin{aligned} &\frac{3k_B n_i T_0}{2t^*}(n_s T_s)_t + \frac{m}{2}\frac{k_B^2 T_0^2 n_i^2 t^{*2}}{m^2 L^2 n_i t^*}(\frac{J_s^2}{n_s})_t + \frac{3k_B^2 T_0^2 n_i t^*}{2mL^2}(J_s T_s)_x + \\ &\frac{m}{2}\frac{k_B^3 T_0^3 n_i^3 t^{*3}}{m^3 L^4 n_i^2}(\frac{J_s^3}{n_s^2})_x + \frac{k_B^2 T_0^2 n_i t^*}{mL^2}(J_s T_s)_x = -e\frac{k_B^2 T_0^2 n_i t^*}{emL^2}J_s\Phi_{sx} - \\ &\frac{3k_B n_i T_0}{2\tau_w}n_s(T_s - 1) - \frac{m}{2\tau_w}\frac{k_B^2 T_0^2 n_i^2 t^{*2}}{m^2 L^2 n_i}(\frac{J_s^2}{sn}) + \frac{n_i T_0}{L^2}(\kappa n_s T_{sx})_x \end{aligned}$$

Using the relation $k_0 T_0 = \frac{mL^2}{t^{*2}}$ and eliminating terms we arrive at

$$\frac{3}{2}(n_sT_s)_t + \frac{1}{2}(\frac{J_s^2}{n_s})_t + \frac{3}{2}(J_sT_s)_x + \frac{1}{2}(\frac{J_s^3}{n_s^2})_x + (J_sT_s)_x =$$
$$-J_s\Phi_{sx} - \frac{3t^*}{2\tau_w}n_s(T_s-1) - \frac{t^*}{2\tau_w}(\frac{J_s^2}{n_s}) + \frac{t^*}{k_BL^2}(\kappa n_s T_{sx})_x$$

Defining $\frac{1}{\tau'} = \frac{t^*}{\tau_w}$ and $\alpha = \frac{t^*\kappa}{k_BL^2}$ we arrive at the adimentional equation

$$\frac{3}{2}(n_sT_s)_t + \frac{1}{2}(\frac{J_s^2}{n_s})_t + \frac{3}{2}(J_sT_s)_x + \frac{1}{2}(\frac{J_s^3}{n_s^2})_x + (J_sT_s)_x =$$
$$-J_s\Phi_x - \frac{3}{2\tau'}n_s(T_s-1) - \frac{1}{2\tau'}(\frac{J_s^2}{n_s}) + (\alpha n_s T_{sx})_x$$

Returing to the variable w_s, the adimentional version of w and dropping the "s" we get

$$w_t + (vw + nvT)_x = -nv\Phi_x - \frac{w - \frac{3}{2}n}{\tau'} + (\alpha n T_x)_x \qquad (3.152)$$

In summary the equations are

$$\begin{aligned}
n_t + (nv)_x &= 0 \\
p_t + (pv + nT)_x &= nE - \left(\frac{p}{\tau}\right) \\
w_t + (vw + nvT)_x &= -nv\Phi_x - \frac{w - \frac{3}{2}n}{\tau'} + (\alpha n T_x)_x \\
-\lambda^*\phi_{xx} &= n + n_D
\end{aligned}$$

Simulations Example Code

This part of the code was used to compute numerically the intermediate states for region I, and plot the solution path in phase space.

```
function [resp1,resp2,resp3,resp4] = SolucionSR(rL,uL,rR,uR,A1,A2,k,g,reg)
a1=0; a2=0;
if (reg==1)
f = @(x)uL+(2/(g-1))*( sqrt(P(k,x,g)*g./x) - sqrt( P(k,rL,g)*g./rL ) )-
```

```
(A2/A1);
rhomin=fzero(f,1);
plot(rhomin,A2/A1,'x');
% Value and plot of rho+
f = @(x)-uR+A2/A1-sqrt(B(rR,x,A2/A1,a1).^2-4*A(rR,x).*...
C(rR,x,A2/A1,a1,a2,k,g))./(2*x*rR);
rhomas=fzero(f,1);
plot(rhomas,A2/A1,'o');
resp1=rhomin;
resp2=A2/A1;
resp4=resp2;
resp3=rhomas;
%Plot of UR plot(rR,uR,'+');
%Plot of solution path in Region I
r2=rhomas:.01:rR;
U2=A2/A1-sqrt(B(r2,rhomas,A2/A1,a1).^2-4*A(r2,rhomas).*...
C(r2,rhomas,A2/A1,a1,a2,k,g))./(2*rhomas*r2); plot(r2,U2,'k')
r2=rL:.01:rhomin;
R2=uL+(2/(g-1))*( sqrt(P(k,r2,g)*g./r2) - sqrt( P(k,rL,g)*g./rL ) );
plot(r2,R2,'k');
if (rhomas<=rhomin)
r3=rhomas:.01:rhomin;
else
r3=rhomin:.01:rhomas;
end

plot(r3,r3*0+A2/A1,'k');
end
% Region I ends————
```

This part of the code was used to plot the solution in space and time for region I. Some functions are note presented here.

```
function sim = canalRS(s1,s2,s3,uL,uR,um,uM)
% domx is the value of the extremes of the interval.
%Domain in x.
domx=30;
% Values to adjust the graohic
y1=uL-0.5; y2=uR+0.5;
% With larger frames, first adjust the figure's size to fit the movie:
fig = figure('position',[100 100 850 600]);
```

```
%Do the plots and animation.
for n=1:25
% Values of x in time n.
tn=n;
x1n=-tn/s1;
x2n=-tn/s2;
x3n=tn/s3;

%Needed in the firs plot.
X1n=-domx:0.01:x1n;
Xrare=x1n:0.01:x2n;
Xint1=x2n:0.01:0;
Xint2=0:0.01:x3n;
X3n=x3n:0.01:domx;
cosen=coseno(Xrare,um,uL,x2n,x1n);
v1=vectorshock(X1n,uL);
v2=vectorshock(Xint1,um);
v3=vectorshock(Xint2,uM);
v4=vectorshock(X3n,uR);
V=[v1 cosen v2 v3 v4];

if n==1

% Plots the initial condition
xm=-domx:0.01:0;
xM=0:0.01:domx;
subplot(2,1,2);
plot(xm,0*xm+uL,'k',[0 0],[uL uR],'k',xM,0*xM+uR,'k');
title('Channel side view. Density plot.');
xlabel('x');
ylabel('Density');
axis([-domx domx y1 y2]);
X1prim=-domx:0.01:0; X2prim=0:0.01:domx;
V1prim=vectorshock(X1prim,uL);
V2prim=vectorshock(X2prim,uR);
Vprim=[V1prim V2prim];
subplot(2,1,1);
imagesc(Vprim);
title('Channel top view. Density plot.'); xlabel('x'); ylabel('y')
colorbar('location','NorthOutside')
colormap(jet);
```

```
ylim([-2 4]);
F(1)=getframe(fig);
else
% Plots in time n>0:
%————–Graphic 2————-
subplot(2,1,2);
plot(X1n,0*X1n+uL,'k',Xrare,cosen,'k',Xint1,Xint1*0+um,'k',[0 0],...
[um uM],'k',Xint2,Xint2*0+uM,'k',[x3n x3n],[uM uR],'k',X3n,X3n*0+uR,'k');
title('Channel side view. Velocity plot.'); xlabel('x'); ylabel('Density');
axis([-domx domx y1 y2]);
%——————————————
%————–Graphic 1————-
subplot(2,1,1);
imagesc(V);
title('Channel top view. Velocity plot.'); xlabel('x'); ylabel('y')
colorbar('location','NorthOutside')
colormap(jet);
ylim([-2 4]);
%——————————————
F(n) = getframe(fig);
end

end
[h, w, p] = size(F(1).cdata);
% use 1st frame to get dimensions hf = figure;
% resize figure based on frame's w x h, and place at (150, 150)
set(hf, 'position', [150 150 w h]);
axis off
movie(hf,F,1,2);
```

For the other regions the code is similar.

Existence of solutions of problem P

First we will start with the solutions of the problem with self similar viscosity.

Existence of a solution of P_ϵ

In this section we will give a complete explanation of the proof of existence of solutions for the problem (P). This system is similar to the one propose for the model of ion pumps, but with source terms equal to zero. This part can be used as starting point for proving the existence of solution of the model (with pointwise sources) and, of course, some definitions and steps can be considerably different from the ones here. All the following proofs and main ideas are taken from [5].

Lets remind some estimators and properties, if ρ and u are the solutions of the problem then the following are true

$$0 < \delta < \rho(\xi) < M \qquad \xi \in (-\infty, \infty) \tag{3.153}$$

$$|u(\xi)| < M \qquad \xi \in (-\infty, \infty) \tag{3.154}$$

$$-b(\xi - s) + s < u(\xi) < a(\xi - s) + s \qquad \xi \in (s, \infty) \tag{3.155}$$

$$a(\xi - s) + s < u(\xi) < -b(\xi - s) + s \qquad \xi \in (-\infty, s) \tag{3.156}$$

Now take

$$-b(\xi - s) + s < u(\xi) < a(\xi - s) + s \qquad \xi \in (s, \infty)$$

in this range $u(\xi) - \xi < 0$ and $\xi - s > 0$, so subtracting ξ and factorizing we get

$$-(b+1)(\xi - 1) < u(\xi) - \xi < -(1-a)(\xi - s)$$

$$-B(\xi - 1) < u(\xi) - \xi < -A(\xi - s)$$

multiplying by -1 we get

$$A(\xi - s) < -(u(\xi) - \xi) < B(\xi - s)$$

Now take

$$a(\xi - s) + s < u(\xi) < -b(\xi - s) + s \qquad \xi \in (-\infty, s)$$

in this range $u(\xi) - \xi > 0$ and $\xi - s < 0$, so subtracting ξ and factorizing we get (with the same definitions of A and B)

$$-A(\xi - s) < u(\xi) - \xi < -B(\xi - s)$$

So if we put

$$A|\xi - s| < |u(\xi) - \xi| < B|\xi - s| \qquad \xi \in (-\infty, \infty), \xi \neq s \tag{3.157}$$

the inequality reduces to the previous two inequalities above if we consider $\xi \in (-\infty, s)$ or $\xi \in (s, \infty)$. This estimators will be used in the following sections to prove the existence of solutions of the problem.

Let's recall the system (P_ϵ) in the variable ξ

$$\begin{cases} (u-\xi)\rho' + \rho u' = 0 \\ (u-\xi)\rho u' + p(\rho)' = \epsilon u'' \end{cases} \quad \text{with} \quad \begin{cases} \rho(\pm\infty) = \rho^\mu_\pm := \rho_- + \mu(\rho_\pm + \rho_-) \\ u(\pm\infty) = u^\mu_\pm := u_- + \mu(u_\pm + u_-) \end{cases} \tag{3.158}$$

We will construct the necessary tools to prove that our system has a solution. The ideas and proofs are similar to the ones find in [**?**]. The set X is defined as:

$$X = \{(P, V) \in C^0(\mathbb{R}) \times C^1(\mathbb{R}) : \|(P, V)\|_X < \infty\}$$

where the norm is

$$\|(P, V)\|_X = \sup_{-\infty<\xi<\infty} |P(\xi)| + \sup_{-\infty<\xi<\infty} |V(\xi)| + \sup_{-\infty<\xi<\infty} |V'(\xi)|$$

The set Y is defined as the $(P, V) \in X$ such that are bounded by

$$0 < \bar{\delta} < P(\xi) < \bar{M} \quad \xi \in \mathbb{R} \tag{3.159}$$

$$|V(\xi)| < \bar{M} \quad \xi \in \mathbb{R} \tag{3.160}$$

$$\bar{A} < 1 - V'(s) < \bar{B} \tag{3.161}$$

$$\bar{A}|\xi - s| < |\xi - V(\xi)| < \bar{B}|\xi - s| \quad \xi \neq s \tag{3.162}$$

with constants that satisfy $0 < \bar{\delta}(\epsilon) < \delta$, $0 < M < \bar{M}$ and $0 < \bar{A} < A < 1 < B < \bar{B}$. And the last set defined is Ω

$$\Omega = \{(P, V) \in Y : |V'(\xi)| < K\}$$

We will prove some important properties of the set Ω.

Theorem 10. *Let $Y \subset X$ be the set with bounds defined before. The the set*

$$\Omega = \{(P,V) \in Y : |V'(\xi)| < K\}$$

is bounded and open subset of X.

Proof. By definition the set is bounded. Now fix the point $(P_o, V_o) \in \Omega$, the goal is to find a positive real ν such that $||(P,V) - (P_o, V_o)||_X < \nu$ which may imply that $(P,V) \in \Omega$. Lets construct it. We have previously find bounds for ρ and u, so the inequalities

$$0 < \bar{\delta} < P(\xi) < \bar{M} \qquad \xi \in \mathbb{R} \tag{3.163}$$

$$|V(\xi)| < \bar{M} \quad \xi \in \mathbb{R}$$

follow immediately. Now lets continue with the bound of the derivative. Let s_o be the fix point of V_o. Since V_o is continuous and bounded and because of (3.157) there exists positive constants A_o and B_o such that

$$\bar{A}|\xi - s_o| < A_o|\xi - s_o| < |\xi - V(\xi)| < B_o|\xi - s_o| < \bar{B}|\xi - s_o| \quad \xi \neq s_o \tag{3.164}$$

applying the derivative respect of ξ we have

$$A_o \leq 1 - V_o'(s_o) \leq B_o$$

notice two important things, the first is that it has the equality included, this because it holds if and only if $\xi = s_o$ in the not-derived inequality, and second that the inequalities still hold even when derived, this because its evaluated in one point[14], remember that in this point all the values of each member of the inequality are the same.

It can be chosen $\overline{A}$, A_o, $\overline{B}$, B_o close enough such that

$$\nu_o \equiv min\{A_o - \overline{A}, \overline{B} - B_o\} < 1$$

and this is always possible because it can be set $1 > A_o = 1 - a$ and $1 < 1 + b = B_o$, see (3.157). Because

$$\overline{A} < A_o \leq 1 - V_o'(s_o) \leq B_o < \overline{B}$$

and by the definition of ν_o then

[14]If $g(x) < f(x)$ and in there exist a point x_o such that $g(x_o) = f(x_o)$ then its true that $g'(x_o) < f'(x_o)$, it may fail in any other point.

$$\overline{A} < \overline{A} + \frac{\nu_o}{2} < A_o$$

and

$$B_o < \overline{B} - \frac{\nu_o}{2} < \overline{B}$$

then, it should exist a $\kappa < 1$ such that when evaluating $1 - V_o'(\xi)$ in $(s_o - \kappa, s_o + \kappa)$ it satisfies

$$\overline{A} + \frac{\nu_o}{2} < 1 - V_o'(\xi) < \overline{B} - \frac{\nu_o}{2} \qquad \xi \in (s_o - \kappa, s_o + \kappa)$$

Define ν as

$$\nu \equiv \frac{1}{2}\nu_o\kappa\frac{\bar{A}}{\bar{B}} \leq \frac{1}{2}\nu_o\kappa < \frac{1}{2}\nu_o$$

every inequality holds because $\frac{\bar{A}}{\bar{B}} \leq 1$ and $\kappa < 1$, suppose that $||(P,V) - (P_o, V_o)||_X < \nu$ then $|V(s) - V(s_o)| < \nu$, and because V has a fixed point we have

$$\nu \geq |V(s) - V(s_o)| = |s - V(s_o)| \geq \overline{A}|s - s_o|$$

the last inequality is because of (3.164). So in summary

$$|s - s_o| \leq \frac{v}{\bar{A}} = \frac{\nu_o\kappa}{2\bar{B}}$$

this is true in particular in $s \in [s - \kappa, s_o + \kappa]$. Since $|V_o'(\xi) - V'(\xi)| < \frac{\nu_o}{2}$, with this, $\overline{A} + \frac{\nu_o}{2} < 1 - V_o'(\xi) < \overline{B} - \frac{\nu_o}{2}$ implies

$$\overline{A} < 1 - V_o'(\xi) < \overline{B} \qquad \xi \in (s_o - \kappa, s_o + \kappa)$$

this proves the bound of the derivative (3.161) in the region $(s_o - \kappa, s_o + \kappa)$, now is only missing the bound in the complement of this set. Integrating this last in (s, ξ)

$$\bar{A}|\xi - s| < |\xi - V(\xi)| < \bar{B}|\xi - s| \quad \xi \in (s_o - \kappa, s_o + \kappa)$$

and this proves (3.162) in the same set. Now in the complementary set $(s_o - \kappa, s_o + \kappa)^c$. Notice that

$$|\xi - V(\xi)| \leq |\xi - V_o(\xi)| + \nu$$

adding something positive and zero

$$|\xi - V(\xi)| \leq |\xi - V_o(\xi)| + \nu < B_o|\xi - s| + \bar{B}|\xi - s| - \bar{B}|\xi - s| + \nu$$

and because of (3.164)

$$|\xi - V(\xi)| \leq |\xi - V_o(\xi)| + \nu < B_o|\xi - s| + \bar{B}|\xi - s| - \bar{B}|\xi - s| + B_o|s - s_o| + \nu$$

factoring

$$|\xi - V(\xi)| \leq \bar{B}|\xi - s| - (\bar{B} - B_o)|\xi - s| + B_o|s - s_o| + \nu$$

The parameters written above are

$$\nu \leq \frac{\kappa\nu_o}{2\bar{B}}, \qquad |s - s_o| \leq \frac{\nu_o\kappa}{2\bar{B}}, \qquad \nu_o \leq \bar{B} - B_o, \qquad \bar{B} > 1$$

so we want to see what happends with

$$|\xi - V(\xi)| \leq \bar{B}|\xi - s| + \textit{something}$$

where *something* is $-(\bar{B} - B_o)|\xi - s| + B_o|s - s_o| + \nu$. Since ξ takes values in $|\xi - s| > \kappa - \frac{\kappa\nu_o}{2\bar{B}}$, we get:

$$-(\bar{B} - B_o)|\xi - s| + B_o|s - s_o| + \nu \leq -(\bar{B} - B_o)\left(\kappa - \frac{\kappa\nu_o}{2\bar{B}}\right) + B_o\frac{\nu_o\kappa}{2\bar{B}} + \frac{\kappa\nu_o}{2\bar{B}}$$

$$= -\kappa(\bar{B} - B_o)\left(1 - \frac{\nu_o}{2\bar{B}}\right) + B_o\frac{\nu_o\kappa}{2\bar{B}} + \frac{\kappa\nu_o}{2\bar{B}}$$

and $\left(1 - \frac{\nu_o}{2\bar{B}}\right) \leq 1$ then

$$-(\bar{B} - B_o)|\xi - s| + B_o|s - s_o| + \nu \leq -\kappa(\bar{B} - B_o) + B_o\frac{\nu_o\kappa}{2\bar{B}} + \frac{\kappa\nu_o}{2\bar{B}}$$

factoring

$$-(\bar{B} - B_o)|\xi - s| + B_o|s - s_o| + \nu \leq -\kappa(\bar{B} - B_o) + \frac{\kappa\nu_o}{2\bar{B}}(1 + \bar{B})$$

$$\leq -\kappa(\bar{B} - B_o) + \frac{\kappa(\bar{B} - B_o)}{2\bar{B}}(1 + \bar{B}) \leq -\kappa(\bar{B} - B_o)\left(1 + \frac{\kappa}{2\bar{B}}(1 + \bar{B})\right)$$

and because $0 \leq (\bar{B} - B_o)$ and $0 \leq \left(1 + \frac{\kappa}{2\bar{B}}(1 + \bar{B})\right)$ then

$$-(\bar{B} - B_o)|\xi - s| + B_o|s - s_o| + \nu \leq 0$$

so $something = -(\bar{B} - B_o)|\xi - s| + B_o|s - s_o| + \nu \leq 0$ and

$$|\xi - V(\xi)| \leq \bar{B}|\xi - s| + something \leq \bar{B}|\xi - s|$$

$$|\xi - V(\xi)| \leq \bar{B}|\xi - s|$$

and the right hand of (3.162) has been proved, the lefthand side is analogous. In conclusion, choosing ν as we did, we prove that the point (P, V) satisfies all the conditions (3.159),(3.160),(3.161) and (3.162) so the point is inside Ω, then its open and bounded.

□

Estimates of the Operator

In this section we check that the map F is well defined and establish uniform estimates of $(\rho, u) = F(\mu, (P, V))$ and their derivatives. The derived estimates may depend on $\rho\pm, u\pm$ and ϵ but are independent of the choice of $(\mu, (P, V)) \in [0, 1] \times Y$. Consider a mapping T which carries $(P, V) \in Y$ to a solution $T(P, V) := (\rho, u)$ of (3.158) with boundary conditions

$$\rho(\pm\infty) = \rho_{\pm} - \rho_{-} \qquad\qquad u(\pm\infty) = u_{\pm} - u_{-}$$

We can easily verified that $(\rho_{-}, u_{-}) + \mu T(P, V)$ is a solution of (3.158), and hence $F(\mu, (P, V)) = (\rho_{-}, u_{-}) + \mu T(P, V)$. The proof is simple

Proof. We check for the solutions at infinity

$$\begin{array}{l} \text{in } +\infty : \ \rho_{-} + \mu(\rho_{+} - \rho_{-}) \\ \text{in } -\infty : \ \rho_{-} + \mu(\rho_{-} - \rho_{-}) \end{array}$$

The same process verifies for $u_{\pm}$ where the resultant conditions at infinity are the same as in (3.158).

□

The bounds in (3.163) and assuming positive derivative of the pressure with positive density

$$p'(\rho) > 0 \quad \text{for} \quad \rho < 0$$

this implies the existence of two positive constants a_0 and A_0 which satisfy

$0 < a_0 < p'(P(\xi)) < A_0 < \infty$ for all $\xi \in \mathbb{R}$ and depend on $\bar{\delta}$ and $\bar{M}$.

From $(3.158)_2$, we get

$$\epsilon u'' + \tfrac{\{p'(P)-(V-\xi)^2\}P}{V-\xi} u' = 0, \qquad V(\xi) \neq \xi$$

Since the latter equation has a unique singularity at the fixed point s of V, u' is obtain, by separation of variables,

$$u'(\xi) = \begin{cases} c_+ \exp\left\{ -\frac{1}{\epsilon} \int_{\alpha_+}^{\xi} \frac{\{(\varsigma-V)^2-p'(P)\}P}{\varsigma-V} d\varsigma \right\} & =: c_+ I_+ \qquad s < \xi \\ c_- \exp\left\{ -\frac{1}{\epsilon} \int_{\alpha_-}^{\xi} \frac{\{(\varsigma-V)^2-p'(P)\}P}{\varsigma-V} d\varsigma \right\} & =: c_- I_- \qquad \xi < s \end{cases} \tag{3.165}$$

for any $\alpha_- < s < \alpha_+$. In turn ρ' is obtained by $(3.158)_1$

$$\rho'(\xi) = \frac{P(\xi)}{\xi - V} u' \tag{3.166}$$

Theorem 11. *Let $(P, V) \in Y$ and $V(s) = s$. Then there exist positive constants $\alpha, \alpha', \beta, \beta'$ and C_ϵ which depend only on $a_0, A_0, \bar{A}, \bar{B}, \bar{\delta}$ and $\bar{M}$ (C_ϵ may depend on ϵ) and satisfy.*

$$\begin{array}{ll} \frac{1}{c_\epsilon}|\xi - s|^{\frac{\alpha'}{\epsilon}} \leq I_\pm(\xi) \leq C_\epsilon |\xi - s|^{\frac{\alpha}{\epsilon}}, & |\xi - s| < 1 \\ \frac{1}{C_\epsilon} e^{-\frac{\beta'}{\epsilon}(\xi-s)^2} \leq I_\pm \leq C_\epsilon e^{-\frac{\beta}{\epsilon}(\xi-s)^2}, & |\xi - s| > 1 \end{array} \tag{3.167}$$

Proof. Let $s < \xi < s + 1$. Then

$$\int_{\xi}^{s+1} \frac{\{(\varsigma - V)^2 - p'(P)\}P}{\varsigma - V} d\varsigma = \int_{\xi}^{s+1} (\varsigma - V) P d\varsigma - \int_{\xi}^{s+1} \frac{p'(P)\}P}{\varsigma - V} d\varsigma$$

From (3.157) we remember $(\xi - V) \leq \bar{B}|\xi - s|$ where $\bar{B} = b + 1$ then as the upper bound of P is $\bar{M}$ and for $p'(P)$ and P we have respectively as lower bounds a_0 and $\bar{\delta}$ (notice that the lower bound is due to the minus sign in front of the integral) doing the change of variables for $(\xi - V)$ we have

$$\int_{\xi}^{s+1} \frac{\{(\varsigma - V)^2 - p'(P)\}P}{\varsigma - V} d\varsigma \leq \bar{M}\bar{B} \int_{0}^{1} \varsigma d\varsigma - \frac{a_0 \bar{\delta}}{\bar{B}} \int_{\xi - s}^{1} \frac{1}{\varsigma} d\varsigma = A + \alpha \log|\xi - s|$$

where $\alpha \equiv \frac{a_o\bar{\delta}}{\bar{B}} > 0$ and $A \equiv \frac{\bar{M}\bar{B}}{2}$.[15] Also

$$I_+(\xi) = \exp\left\{\frac{1}{\epsilon}\int_{\xi}^{s+1}\frac{\{(\varsigma-V)^2-p'(P)\}P}{\varsigma-V}d\varsigma\right\} \leq e^{\frac{A}{\epsilon}}e^{\frac{\alpha}{\epsilon}\log|\xi-s|} = C_\epsilon|\xi-s|^{\frac{\alpha}{\epsilon}}$$

Let $s+1<\xi$, then using the same logic as in the previous steps

$$-\int_{s+1}^{\xi}\frac{\{(\varsigma-V)^2-p'(P)\}P}{\varsigma-V}d\varsigma = -\int_{s+1}^{\xi}(\varsigma-V)Pd\varsigma + \int_{s+1}^{\xi}\frac{p'(P)\}P}{\varsigma-V}d\varsigma$$

$$\leq -\bar{\delta}\bar{A}\int_{1}^{\xi-s}\varsigma d\varsigma + \frac{a_0\bar{M}}{\bar{A}}\int_{1}^{\xi-s}\frac{1}{\varsigma}d\varsigma \leq -\beta(\xi-s)^2 + A$$

where $\beta = \frac{\bar{\delta}\bar{A}}{2} + 1$ and A is a positive constant which depends on β and $\frac{A_0\bar{M}}{A}$. Also

$$I_+(\xi) = \exp\left\{-\frac{1}{\epsilon}\int_{s+1}^{\xi}\frac{\{(\varsigma-V)^2-p'(P)\}P}{\varsigma-V}d\varsigma\right\} \leq e^{\frac{A}{\epsilon}}e^{-\beta(\xi-s)^2} = C_\epsilon e^{-\beta(\xi-s)^2}$$

The rest is followed by similar arguments.

□

According to the last Theorem since the exponential is bounded then u' and ρ' are integrable on $(-\infty,\infty)$ and thus (ρ,u) can be calculated by (3.165) and (3.166) just integrating them in their domains getting the formulas

$$\rho(\xi) = \begin{cases}(\rho_+-\rho_-) - c_+\int_\xi^\infty \frac{P(\xi)I_+(\xi)}{\varsigma-V(\varsigma)}d\varsigma, & s<\xi\\ c_-\int_{-\infty}^{\xi}\frac{P(\xi)I_-(\xi)}{\varsigma-V(\varsigma)}d\varsigma, & \xi<s\end{cases} \quad (3.168)$$

$$u(\xi) = \begin{cases}(u_+-u_-) - c_+\int_\xi^\infty I_+(\xi)d\varsigma, & s<\xi\\ c_-\int_{-\infty}^{\xi} I_-(\xi), & \xi<s\end{cases} \quad (3.169)$$

Since we have continuity on (ρ,u) at $\xi=s$ this gives

[15]The lower limit of the second Integral is adjusted just to allow it to be solvable by removing a point from it

$$c_+ \int_{\xi}^{\infty} \frac{P(\xi)I_+(\xi)}{\varsigma - V(\varsigma)} d\varsigma + c_- \int_{-\infty}^{\xi} \frac{P(\xi)I_-(\xi)}{\varsigma - V(\varsigma)} = (\rho_+ - \rho_-) \tag{3.170}$$

$$c_+ \int_{\xi}^{\infty} I_+(\xi) d\varsigma + c_- \int_{-\infty}^{\xi} I_-(\xi) d\varsigma = (u_+ - u_-) \tag{3.171}$$

The determinant of the latter system is

$$\int_{\xi}^{\infty} \frac{P(\xi)I_+(\xi)}{\varsigma - V(\varsigma)} d\varsigma \int_{-\infty}^{\xi} I_-(\xi) d\varsigma - \int_{-\infty}^{\xi} \frac{P(\xi)I_-(\xi)}{\underbrace{\varsigma - V(\varsigma)}_{\text{negative due to domain}}} \int_{\xi}^{\infty} I_+(\xi) d\varsigma > 0$$

So there exists a unique solution (c_+, c_-) telling us that the operators T and F are well defined.

We now estimate $(\rho, u) = T(P, V)$, defined in (3.168) and (3.169). Since our objective is to get uniform bounds which are independent of the choice of $(P, V) \in Y$, we consider a generic constant K_ϵ which may depend on $a_0, A_0, \bar{A}, \bar{B}, \bar{\delta}$and $\bar{M}$. So we get

$$|c_+| + |c_-| < K_\epsilon$$

Now we estimate ρ , u and their derivatives to see the regularity properties of the operator. Since u', is given by (3.165) and as seen in the last theorem $I_\pm$ are bounded we have

$$\begin{aligned} |u'(\xi)| &= c_\pm I_\pm \leq K_\epsilon |\xi - s|^{\frac{\alpha}{\epsilon}} & |\xi - s| < 1, \\ |u'(\xi)| &= c_\pm I_\pm \leq K_\epsilon e^{-\beta(\xi - s)^2} & |\xi - s| > 1, \end{aligned}$$

Now for ρ' we have from (3.166) doing the same steps

$$\begin{aligned} |\rho'(\xi)| &= \tfrac{|P(\xi)|}{|\xi - V(\xi)|} |u'(\xi)| < K_\epsilon |\xi - s|^{\frac{\alpha}{\epsilon} - 1} & |\xi - s| < 1, \\ |\rho'(\xi)| &= \tfrac{|P(\xi)|}{|\xi - V(\xi)|} |u'(\xi)| < K_\epsilon e^{-\beta(\xi - s)^2} & |\xi - s| > 1, \end{aligned}$$

where the constant $|P(\xi)|$ is absorbed by K_ϵ. In the first equation the term $|\xi - V(\xi)|$ is replaced by $|\xi - s|$ which is subtracted in the exponent.

For the second equation we do the same change and as $|\xi - s| > 1$ then we can remove the term and the inequality holds.

We also have

$$u''(\xi) = \frac{1}{\epsilon}\frac{\{p'(P)-(\xi-V)^2\}P}{\xi-V}c_\pm I_\pm \qquad \xi \neq s$$

so

$$u''(\xi) = \frac{1}{\epsilon}\left(\frac{p'(P)}{\xi - V} - (\xi - V)\right)Pc_\pm I_\pm \leq \frac{1}{\epsilon}\left(\frac{A_0}{\xi - V} - (\xi - V)\right)\bar{M}c_\pm I_\pm$$

Remembering the inequalities from (3.155) and (3.156) we substitute the term $\xi - V$ noticing in which case we do not change the sign and which terms pick to keep a valid inequality, so we get

$$|u''(\xi)| \leq \frac{1}{\epsilon}\left(\frac{A_0}{(1-a)|\xi-s|} + (1+b)|\xi - s|\right)\bar{M}\underbrace{c_\pm I_\pm}_{=u'(\xi)} \leq K_\epsilon \underbrace{|\xi - s|^{\frac{\alpha}{\epsilon}-1}}_{<1} \leq K_\epsilon \qquad |\xi - s| < 1$$

$$|u''(\xi)| \leq \frac{1}{\epsilon}\left(\frac{A_0}{(1-a)|\xi-s|} + (1+b)|\xi - s|\right)\bar{M}c_\pm I_\pm \leq K_\epsilon \underbrace{e^{-\beta(\xi-s)^2}}_{max=1} \leq K_\epsilon \qquad |\xi - s| > 1$$

From these estimates we get equicontinuity of ρ, u and u' on any closed set which does not contain the singular point s. The latter results imply

$$|u'(\xi)| < K_\epsilon \qquad -\infty < \xi < \infty$$

and hence u is equicontinuos. Now we check from the boundedness of ρ and u , since they heavily depend on the exponential $I_\pm$from (3.167) we integrate it according to the defined know domains in (3.155) and (3.156).[16]

$$\begin{array}{lll}
\int_s^\xi I_+(\varsigma)d\varsigma < K_\epsilon(\xi - s)^{\frac{\alpha}{\epsilon}+1}; & \int_s^\xi \frac{PI_+}{|\varsigma-V|}d\varsigma < K_\epsilon(\xi - s)^{\frac{\alpha}{\epsilon}}; & s < \xi < s+1 \\
\int_\xi^s I_-(\varsigma)d\varsigma < K_\epsilon(s - \xi)^{\frac{\alpha}{\epsilon}+1}; & \int_\xi^s \frac{PI_-}{|\varsigma-V|}d\varsigma < K_\epsilon(s - \xi)^{\frac{\alpha}{\epsilon}}; & s - 1 < \xi < s \\
\int_{s+1}^\xi I_+(\varsigma)d\varsigma < K_\epsilon e^{-\beta(\xi-s)^2}; & \int_{s+1}^\xi \frac{PI_+}{|\varsigma-V|}d\varsigma < K_\epsilon e^{-\beta(\xi-s)^2}; & s+1 < \xi \\
\int_\xi^{s-1} I_-(\varsigma)d\varsigma < K_\epsilon e^{-\beta(\xi-s)^2}; & \int_\xi^{s-1} \frac{PI_-}{|\varsigma-V|}d\varsigma < K_\epsilon e^{-\beta(\xi-s)^2}; & s+1 > \xi
\end{array}$$

We can check the consistency in this results with the previously inequalities for $|u'(\xi)|$ and $|\rho'(\xi)|$.

[16]Keep in mind that all constants are absorbed by the term K_ϵ

Since the exponential are decaying their maximum value is one, and the terms elevated at $(\cdot)^{\frac{\alpha}{\epsilon}}$ in their corresponding domains are bounded between zero and one, these estimates imply the boundedness of ρ and u

$$|u(\xi)| < K_\epsilon; \qquad |\rho(\xi)| < K_\epsilon; \qquad -\infty < \xi < \infty$$

therefore we can establish the estimates for $\rho(\xi)$ and $u(\xi)$ from (3.168) and (3.169) or integrating from (3.166) and (3.166).

$$\begin{array}{rl}
|\rho(\xi) - \rho(s)| < K_\epsilon (s-\xi)^{\frac{\alpha}{\epsilon}}; & |\xi - s| < 1 \\
|\rho(\xi) - \rho(s)| < K_\epsilon e^{-\beta(\xi-s)^2}; & |\xi - s| > 1 \\
|u(\xi) - u(s)| < K_\epsilon (s-\xi)^{\frac{\alpha}{\epsilon}+1}; & |\xi - s| < 1 \\
|u(\xi) - u(s)| < K_\epsilon e^{-\beta(\xi-s)^2}; & |\xi - s| > 1
\end{array}$$

As checked with the obtained estimates that $|\rho(\xi)| < K_\epsilon$, $|u(\xi)| < K_\epsilon$ and $|u'(\xi)| < K_\epsilon$ this imply that $T(Y)$ of Y under the mapping T are bounded under the $C^0 \times C^1$ norm.

The next step is to prove that the operator T is compact, result that will be necessary later.

Theorem 12. *The operator $T : \bar{\Omega} \to X$ is a compact operator.*

Proof. The proof is separated in two parts, the first is to show that the operator is precompact and the second that is continuous, so the combination of this two facts gives compactness. First we will prove the precompactenss using the Ascoli- Arzela theorem, which needs the equicontinuity of sequences. Recall that a function is equicontinous if it is continuous and has bounded variation in an open set, so we will you use this facts to prove it. Let (ρ_n, u_n) be a sequence in $T(\bar{\Omega})$ since $|u'(\xi)| < K_\epsilon$ in $-\infty \le \xi \le \infty$ then u_n is equicontinous. Now for ρ_n, let $0 < \eta$ be given, the because

$$|\rho(\xi) - \rho(s)| < K_\epsilon |\xi - s|^{\frac{\alpha}{\epsilon}} \qquad |\xi - s| < 1$$

exists a $0 < \delta_1$ such that $|\rho(\xi_1) - \rho(\xi_2)| < \eta$ for all $\xi_1, \xi_2 \in I = (-\delta_1, \delta_1)$, this can be easily seen because of the right hand side of the inequality, this can be bounded by a number η and then you can only find δ_1 such that $K_\epsilon |\xi_1 - \xi_2|^{\frac{\alpha}{\epsilon}}$ is bounded by the same number. Now from the bounds

$$|\rho'(\xi)| < K_\epsilon |\xi - s|^{\frac{\alpha}{\epsilon}-1} \qquad |\xi - s| < 1$$

$$|\rho'(\xi)| < K_\epsilon e^{-\frac{\beta}{\epsilon}(\xi-s)^2} \qquad 1 < |\xi - s|$$

its obvious that ρ'_n is uniformly bounded in $I^c = (-\infty, -\delta_1) \cup (\delta_1, \infty)$ and from

$$|\rho(\xi) - \rho(s)| < K_\epsilon e^{-\frac{\beta}{\epsilon}(\xi - s)^2} \qquad 1 < |\xi - s|$$

we can find, using the same rezoning, a δ_2 such that $|\rho(\xi_1) - \rho(\xi_2)| < \eta$ for all $\xi_1, \xi_2 \in I = (-\delta_1, \delta_1)$ with $|\xi_1 - \xi_2| < \delta_2$. Take $\delta = min\{\delta_1, \delta_2\}$, choosing this, and letting $\eta \to 2\eta$ we can, for sure, bound ρ_n:

$$|\rho(\xi_1) - \rho(\xi_2)| < 2\eta$$

$$|\xi_1 - \xi_2| < \delta$$

So, for each δ it exists a 2η such that the two previous are true, so recalling a characterization of equicontinuity for sequences[17], then ρ_n fulfills this, so its equicontinous. The arguments for proving that u'_n is equicontinous are similar, and also we can use the bounded variation of the sequence, I mean the bounds of u''_n

$$|u''_n(\xi)| < K_\epsilon |\xi - s|^{\frac{\alpha}{\epsilon} - 1} \qquad |\xi - s| < 1$$

$$|u''_n(\xi)| < K_\epsilon e^{-\frac{\beta}{\epsilon}(\xi - s)^2} \qquad 1 < |\xi - s|$$

From

$$0 < \delta < \rho(\xi) < M \qquad \xi \in (-\infty, \infty)$$

$$|u(\xi)| < M \qquad \xi \in (-\infty, \infty)$$

$$A|\xi - s| < |u(\xi) - \xi| < B|\xi - s| \qquad \xi \in (-\infty, \infty), \xi \neq s$$

we see that ρ_n, u_n and u'_n are bounded, we need this because we want to use the Ascoli-Arzela theorem. This theorem states that a real valued sequence $\{f_n\}_{n \in \mathbb{N}}$ defined in a close and bounded set (that is equivalent to a compact set if the space is Euclidean, by Heine-Borel theorem), and if the sequence is bounded and equicontinous, then there exists a subsequence $\{f_{n_k}\}$ that converges uniformly, so all this requirements are fulfilled by ρ_n, u_n and u'_n, so there exists subsequences that converge. Taking a new subsequence of

[17] For a sequence $\{f_n\}$, countable, if for all $\epsilon > 0$ exists $\delta > 0$ such that $|f_n(x) - f(y)| < \epsilon$ if $|x - y| < \delta$, then the sequence is equicontinous.

u_{n_k} we can assume that $s_n \to s$ (the singular point).

Define $\rho_n \to \rho$, $u_n \to u$, $u'_n \to u_1$. Moreover $u_1 = u'$. Because of

$$|u'(\xi)| < K_\epsilon e^{-\frac{\beta}{\epsilon}(\xi-s)^2} \qquad 1 < |\xi - s|$$

$$|\rho(\xi) - \rho(s)| < K_\epsilon e^{-\frac{\beta}{\epsilon}(\xi-s)^2} \qquad 1 < |\xi - s|$$

and the definition of the set

$$\Omega = \{(P, V) \in Y : |V'(\xi)| < K\}$$

we can choose L such that

$$|u'_n(\xi)| < \eta \qquad L < |\xi$$

$$|u_n(\xi)| < \eta \qquad \xi < -L$$

$$|\rho'(\xi)| < \eta \qquad \xi < -L$$

$$|\rho_n(\xi) - (\rho_+ - \rho_-)| < \eta \qquad L < \xi$$

$$|u_n(\xi) - (\rho_+ - \rho_-)| < \eta \qquad L < \xi$$

Moreover, the limits satisfy this estimates and because ρ_n, u_n and u'_n converge uniformly in $[-L, L]$ it can by found an N such that

$$\|(\rho_n, u_n) - (\rho, u)\|_X = \sup_{-\infty<\xi<\infty} |\rho_n - \rho| + \sup_{-\infty<\xi<\infty} |u_n - u| + \sup_{-\infty<\xi<\infty} |u'_n - u'|$$

$$\leq \sup_{-\infty<\xi<\infty} |\rho_n| + \sup_{-\infty<\xi<\infty} |\rho| + \sup_{-\infty<\xi<\infty} |u_n| + \sup_{-\infty<\xi<\infty} |u| + \sup_{-\infty<\xi<\infty} |u'_n| + \sup_{-\infty<\xi<\infty} |u'|$$

$$< 6\eta$$

for $n > N$, so that $(\rho_n, u_n) \to (\rho, u) \in X$, which means that $T(\bar{\Omega})$ fulfills the requirements to be a precompact set.

Lets prove now that T is continuous. Let $(\rho_n, u_n) \in \Omega$, and $(P_n, V_n) \to (P, V) \in X$. Define $(\rho_n, u_n) = T(P_n, V_n)$ and $(\rho, u) = T(P, V)$. Because of the first part of sequence $\{(\rho_n, u_n)\}$ has a convergent subsequence $\{(\rho_{n_k}, u_{n_k})\}$ such that $(\rho_{n_k}, u_{nk}) \to (\rho^o, u^o)$. Then

$$\rho_{n_k} = \begin{cases} (\rho_+ - \rho_-) - c_+^{n_k} \int_\xi^\infty \frac{P_{n_k}(\zeta) I_+^{n_k}}{\zeta - N_{n_k}(\zeta)} d\zeta & s_{n_k} < \xi \\ c_-^{n_k} \int_{-\infty}^\xi \frac{P_{n_k}(\zeta) I_-^{n_k}}{\zeta - N_{n_k}(\zeta)} d\zeta & \xi < s_{n_k} \end{cases}$$

$$u_{n_n} = \begin{cases} (u_+ - u_-) - c_+^{n_k} \int_\xi^\infty I_+^{n_k} d\zeta & s_{n_k} < \xi \\ c_-^{n_k} \int_{-\infty}^\xi I_-^{n_k} d\zeta & \xi < s_{n_k} \end{cases}$$

where $c_+^{n;k}$ and $c_-^{n;k}$ are solutions of the linear system describes in (3.170) and (3.171) with $I_\pm^{n_k}$ given by

$$I_\pm^{n_k}(\xi) = exp\left\{ -\frac{1}{\epsilon} \int_{\pm 1}^{\xi} \frac{\{(\zeta - V_{n_k})^2 - p'(P_{n_k})\} P_{n_k}}{\zeta - V_{n_k}} d\zeta \right\}$$

since $I_\pm$ are bounded from below and above its valid to put $\lim_{n_k \to \infty}$ inside the integrals, which allow us to write

$$\rho^o = \begin{cases} (\rho_+ - \rho_-) - c_+^o \int_\xi^\infty \frac{P^o(\zeta) I_+^o}{\zeta - V^o(\zeta)} d\zeta & s < \xi \\ c_-^o \int_{-\infty}^\xi \frac{P^o(\zeta) I_-^o}{\zeta - V^o(\zeta)} d\zeta & \xi < s \end{cases}$$

$$u^o = \begin{cases} (u_+ - u_-) - c_+^o \int_\xi^\infty I_+^o d\zeta & s < \xi \\ c_-^o \int_{-\infty}^\xi I_-^o d\zeta & \xi < s \end{cases}$$

and form the definition of T we have

$$(\rho_n, u_n) = T(P, V) = (\rho, u)$$

And by the sequential characterization of continuity[18] we see that the function T satisfies this, because $(\rho_{n_k}, u_{nk}) \to (\rho^o, u^o)$ then $T(\rho_{n_k}, u_{nk}) \to (\rho, u) = (\rho^o, u^o)$, so T is continuous. Now precompactness and continuity means compact.

□

We will prove now that the problem (P_ϵ) has a solution under the assumption $0 < \delta_\epsilon < \rho$, with the help of the Leray-Schauder theorem.

[18] if $f : X \to Y$ and $x_n \to x \in X$ and $f(x_n) \to f(x) \in Y$ then f is continuous.

Theorem 13. *Suppose that the pressure function $p(\rho)$ satisfies*

$$p'(\rho) > 0 \quad \rho > 0 \tag{3.172}$$

$$p(\rho) \to \infty \quad \rho \to \infty$$

$$p(\rho) \to 0 \quad \rho \to 0$$

If the solutions (ρ, u) of (P_ϵ^μ) satisfy $0 < \delta_\epsilon < \rho$, then the boundary value problem (P_ϵ) has a solution with $(\rho(\xi), u(\xi))$ for any $\epsilon > 0$.

Proof. We define $F : [0,1] \times \bar{\Omega} \to X$ by $F(\mu, P, V) = (\rho_-, u_-) + \mu T(P, V)$ in Ω, then (ρ, u) is a solution of (P_ϵ^μ) with $(\rho, u) \in \Omega$. To apply the Leray-Schauder theorem to a mapping of the for $I + G$ where I is the identity map we need G to be compact. If this is true then we have to compute the degree of the map, this will tell us if the whole mapping, as a fix point, has a zero inside the domain which it is define, in other words, if the fix point problem has a solution. Lets show it. We want to solve

$$(\rho, u) + \mu T(\rho, u) = (\rho_-, u_-) \qquad \mu \in [0.1]$$

As T is compact, μT is compact in X, so we can compute the degree of $(\rho, u) + \mu T(\rho, u)$. For any solution (ρ, u) of the previous, $\frac{1}{\mu}\{(\rho, u) - (\rho_-, u_-)\} \in T(\bar{\Omega})$. So u satisfies $|u'(\xi)| < K_\epsilon$. So by the theorem that says that we have four different classes of solutions, (3.153), (3.154) and (3.157) then a solution has to lie in the interior of Ω, so the degree has to be

$$d(I - \mu T, \Omega, (\rho_-, uu)) = d(I, \Omega, (\rho_-, uu)) = 1 \quad \mu \in [0.1]$$

so

$$(\rho, u) + \mu T(\rho, u) = (\rho_-, u_-)$$

admits at least one solution for each μ, this means has a fix point and our problem has a solution.

□

Structure of solutions of Riemann problem

In this section we will consider a sequence $(\rho_\epsilon, u_\epsilon)$ of solutions of (P_ϵ) obeying the estimates (3.153),(3.154) and (3.157). By taking subsequences we assume that the limits belong to one of the categories explained in previous sections, and also assume that $s_\epsilon \to s$ as $\epsilon \to 0$. The limits ρ and u inherit the monotonicity properties, but they are not longer strict.

Solution of the Riemann problem

In this section we construct solutions of (P) using $(\rho_\epsilon, u_\epsilon)$, solutions of (P_ϵ), and study what happens with the limit. Lets do this assuming first that $\rho > 0$, so in this case (3.153),(3.154) and (3.157) are independent of ϵ.

Theorem 14. *Let $(\rho_\epsilon, u_\epsilon)$, $\epsilon > 0$ be a solution of (P_ϵ). Then there exists a subsequence $(\rho_{\epsilon_n}, u_{\epsilon_n})$, $\epsilon \to 0$ such that the sequence of singular points $s_{\epsilon_n} \to s$ and $(\rho_{\epsilon_n}, u_{\epsilon_n})$ converges pointwise to a weak solution (ρ, u) of (P). Furthermore, if $\rho > 0$, then there exists constants $\beta_- < \alpha_- < s < \alpha_+ < \beta_+$ such that*

$$(\rho(\xi), u(\xi)) = \begin{cases} (\rho_-, u_-), & \xi < \beta_- \\ (\rho(s), u(s)), & \alpha_- < \xi < \alpha_+ \\ (\rho_+, u_+), & \beta_+ < \xi \end{cases} \tag{3.173}$$

Proof. We will omit the subindex ϵ_n and use ϵ for the subsequence. Now lets integrate $-\xi(\rho_n u_n)' + (\rho_n u_n^2 + p_n)' = \epsilon u_n''$ in the set (s_ϵ, ξ):

$$-\int_{s_\epsilon}^{\xi} \zeta(\rho_\epsilon u_\epsilon)' d\zeta + \int_{s_\epsilon}^{\xi} (\rho_\epsilon u_\epsilon^2 + p_\epsilon)' d\zeta = \int_{s_\epsilon}^{\xi} \epsilon u_\epsilon'' d\zeta$$

integrating the first term by parts give:

$$-\int_{s_\epsilon}^{\xi} \zeta(\rho_\epsilon u_\epsilon)' d\zeta = -\xi\rho_\epsilon(\xi)u_\epsilon(\xi) + s_\epsilon\rho_\epsilon(s_\epsilon)u_\epsilon(s_\epsilon) + \int_{s_\epsilon}^{\xi} \rho_\epsilon u_\epsilon d\zeta$$

the second term gives

$$\rho_\epsilon(\xi)u_\epsilon^2(\xi) + p_\epsilon(\rho_\epsilon(\xi)) - \rho_\epsilon(s_\epsilon)u_\epsilon^2(s_\epsilon) - p_\epsilon(\rho_\epsilon(s_\epsilon))$$

and the third gives

$$\epsilon u_\epsilon'(\xi) - \epsilon u_\epsilon'(s_\epsilon) = \epsilon u_\epsilon'(\xi) - 0$$

where we use the property $u'(s) = 0$ from above sections. Gathering all the results we get

$$\begin{aligned} \epsilon u_\epsilon'(\xi) &= -\xi\rho_\epsilon(\xi)u_\epsilon(\xi) + s_\epsilon\rho_\epsilon(s_\epsilon)u_\epsilon(s_\epsilon) + \int_{s_\epsilon}^{\xi} \rho_\epsilon u_\epsilon d\zeta + \rho_\epsilon(\xi)u_\epsilon^2(\xi) + p_\epsilon(\rho_\epsilon(\xi)) \\ &\quad -\rho_\epsilon(s_\epsilon)u_\epsilon^2(s_\epsilon) + p_\epsilon(\rho_\epsilon(s_\epsilon)) \end{aligned}$$

and using $u_\epsilon(s_\epsilon) = s_\epsilon$we can cancel two terms and get

$$\epsilon u'_\epsilon(\xi) = -\xi\rho_\epsilon(\xi)u_\epsilon(\xi) + \int_{s_\epsilon}^{\xi} \rho_\epsilon u_\epsilon d\zeta + \rho_\epsilon(\xi)u_\epsilon^2(\xi) + p_\epsilon(\rho_\epsilon(\xi)) - p_\epsilon(\rho_\epsilon(s_\epsilon))$$

because $\rho_\epsilon(\xi)$, $u_\epsilon(\xi)$ and $p_\epsilon(\rho_\epsilon(\xi))$ are bounded by M

$$\begin{aligned} \epsilon u'_\epsilon(\xi) &\leq M^3 + M - \xi M^2 + M^2 \int_{s_\epsilon}^{\xi} d\zeta - p_\epsilon(\rho_\epsilon(s_\epsilon)) \\ &\leq M^3 + M - \xi M^2 + M^2 \int_{s_\epsilon}^{\xi} d\zeta \\ &= M^3 + M - \xi M^2 + M^2(\xi - s_\epsilon) \end{aligned}$$

because s_ϵ is bounded by M so we can cancel the M^3, and taking absolute value we get

$$\epsilon|u'_\epsilon(\xi)| \leq M + 2|\xi|M^2$$

so

$$\epsilon|u'_\epsilon(\xi)| \leq M(|\xi| + 1) \qquad -\infty \leq \xi \leq \infty$$

Lets multiply u_ϵ the equation $-\xi(\rho_n u_n)' + (\rho_n u_n^2 + p_n)' = \epsilon u''_n$ and integrating in an interval (a_o, b_o) we have

$$\epsilon \int_{a_o}^{b_o} u_\epsilon u'_\epsilon d\zeta = -\int_{a_o}^{b_o} \zeta u_\epsilon (\rho_\epsilon u_\epsilon)' d\zeta + \int_{a_o}^{b_o} u_\epsilon (\rho_\epsilon u_\epsilon^2 + p_\epsilon)'' d\zeta$$

the first term gives

$$\epsilon \int_{a_o}^{b_o} u_\epsilon u'_\epsilon d\zeta = \epsilon u_\epsilon(b_o)u'_\epsilon(b_o) - \epsilon u_\epsilon(a_o)u'_\epsilon(a_o) - \int_{a_o}^{b_o} (u'_\epsilon)^2 d\zeta$$

so

$$
\begin{aligned}
\epsilon u_\epsilon(b_o)u'_\epsilon(b_o) - \epsilon u_\epsilon(a_o)u'_\epsilon(a_o) - \int_{a_o}^{b_o}(u'_\epsilon)^2 d\zeta &= -\int_{a_o}^{b_o}\zeta u_\epsilon(\rho_\epsilon u_\epsilon)' d\zeta + \int_{a_o}^{b_o} u_\epsilon(\rho_\epsilon u_\epsilon^2 + p_\epsilon)' d\zeta \\
\int_{a_o}^{b_o}(u'_n)^2 d\zeta &= \int_{a_o}^{b_o}\zeta u_\epsilon(\rho_\epsilon u_\epsilon)' d\zeta + \epsilon u_\epsilon(b_o)u'_\epsilon(b_o) - \epsilon u_\epsilon(a_o)u'_\epsilon(a_o) \\
&\quad - \int_{a_o}^{b_o} u_\epsilon(\rho_\epsilon u_\epsilon^2 + p_\epsilon)' d\zeta \\
&\le M\int_{a_o}^{b_o}\zeta(\rho_\epsilon u_\epsilon)' d\zeta + M\left(\epsilon u'_\epsilon(b_o) - \epsilon u'_\epsilon(a_o)\right) \\
&\quad - \int_{a_o}^{b_o} u_\epsilon(\rho_\epsilon u_\epsilon^2 + p_\epsilon)' d\zeta
\end{aligned}
$$

notice that $-M\int_{a_o}^{b_o}(\rho_\epsilon u_\epsilon^2 + p_\epsilon)' d\zeta < -\int_{a_o}^{b_o} u_\epsilon(\rho_\epsilon u_\epsilon^2 + p_\epsilon)' d\zeta$ both negative, so

$$
\begin{aligned}
\epsilon\int_{a_o}^{b_o}(u'_\epsilon)^2 d\zeta &\le \int_{a_o}^{b_o}\zeta u_\epsilon(\rho_\epsilon u_\epsilon)' d\zeta + M\epsilon\left(u'_\epsilon(b_o) - \epsilon u'_\epsilon(a_o)\right) \\
&\le M(|b_o| + |a_o| + 1) + M\epsilon\left(u'_\epsilon(b_o) - \epsilon u'_\epsilon(a_o)\right)
\end{aligned}
$$

and using

$$
\epsilon|u'_\epsilon(\xi)| \le M(|\xi| + 1) \qquad -\infty \le \xi \le \infty
$$

the inequality

$$
\epsilon(u'_\epsilon(b_o) - \epsilon u'_\epsilon(a_o)) \le M(|b_o| + |a_o| + 1)
$$

holds, so

$$
\epsilon\int_{a_o}^{b_o}(u'_\epsilon)^2 d\zeta \le M(|b_o| + |a_o| + 1)
$$

Set φ be a function with $supp\varphi \subset (a_o, b_o)$, and multiply φ the equation $-\xi(\rho_\epsilon u_\epsilon)' + (\rho_\epsilon u_\epsilon^2 + p_\epsilon)' = \epsilon u_\epsilon''$, integrate

$$-\int_{-\infty}^{\infty} \varphi\zeta(\rho_\epsilon u_\epsilon)' d\zeta + \int_{-\infty}^{\infty} \varphi(\rho_\epsilon u_\epsilon^2 + p_\epsilon)' d\zeta = \int_{-\infty}^{\infty} \epsilon\varphi u_\epsilon'' d\zeta$$

take absolute value

$$|\int_{-\infty}^{\infty} \varphi(\rho_\epsilon u_\epsilon^2 + p_\epsilon)' d\zeta - \int_{-\infty}^{\infty} \varphi\zeta(\rho_\epsilon u_\epsilon)' d\zeta| = |\int_{-\infty}^{\infty} \epsilon\varphi u_\epsilon'' d\zeta|$$

so integrating the last term

$$\begin{aligned} |\int_{-\infty}^{\infty} \epsilon\varphi u_\epsilon'' d\zeta| &= | \epsilon\varphi u_\epsilon'|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \epsilon\varphi' u_\epsilon' d\zeta| \\ &= |\int_{-\infty}^{\infty} \epsilon\varphi' u_\epsilon' d\zeta| \\ &\leq \epsilon \left[\int_{a_o}^{b_o} (\varphi')^2 d\zeta\right]^{\frac{1}{2}} \left[\int_{a_o}^{b_o} (u_\epsilon')^2 d\zeta\right]^{\frac{1}{2}} \end{aligned}$$

where the last inequality is because of Cauchy-Schwarz inequality and the compact support of φ. By $\epsilon \int_{a_o}^{b_o} (u_\epsilon')^2 d\zeta \leq M(|b_o| + |a_o| + 1)$ we have

$$|\int_{-\infty}^{\infty} \epsilon\varphi u_\epsilon'' d\zeta| \leq \epsilon^{\frac{1}{2}} M(|b_o| + |a_o| + 1) \left[\int_{a_o}^{b_o} (\varphi')^2 d\zeta\right]^{\frac{1}{2}}$$

so with the $lim_{\epsilon \to 0}$ we get

$$\int_{-\infty}^{\infty} \epsilon\varphi u_\epsilon'' d\zeta = 0$$

we will use this result in a moment. Now $-\int_{-\infty}^{\infty} \varphi\zeta(\rho u)' d\zeta + \int_{-\infty}^{\infty} \varphi(\rho u^2 + p)' d\zeta$, the first term is, integrating by parts (notice that it doesn't have ϵ)

$$-\int_{-\infty}^{\infty} \varphi\zeta(\rho u)' d\zeta = -(\varphi\zeta)'(\rho u)|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (\varphi\zeta)' \rho u d\zeta = \int_{-\infty}^{\infty} (\varphi\zeta)' \rho u d\zeta$$

the term $\int_{-\infty}^{\infty} \varphi(\rho u^2+p)' d\zeta$ gives

$$\int_{-\infty}^{\infty} \varphi(\rho u^2+p)'d\zeta = \varphi(\rho u^2+p)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \varphi'(\rho u^2+p)d\zeta = -\int_{-\infty}^{\infty} \varphi'(\rho u^2+p)d\zeta$$

in conclusion

$$\begin{aligned}\int_{-\infty}^{\infty} (\varphi\zeta)'\rho u d\zeta - \int_{-\infty}^{\infty} \varphi'(\rho u^2+p)d\zeta &= lim_{\epsilon\to 0}\int_{-\infty}^{\infty} (\varphi\zeta)'\rho_\epsilon u_\epsilon d\zeta \\ &\quad -lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \varphi'(\rho_\epsilon u_\epsilon^2+p_\epsilon)d\zeta \\ &= -lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \epsilon\varphi u_\epsilon'' d\zeta = 0\end{aligned}$$

so (ρ, u) is a weak solution of (P).

Now we consider the structure of the limit solution under the condition $\rho > 0$. In that case there exist constants $\delta > 0$ and $M > 0$ such that

$$\delta < \rho(\xi) < M \qquad \xi \in \mathbb{R}$$

for small $\epsilon > 0$. From (3.172) there exist constants $a_o > 0$ and $A_o > 0$ such that

$$a_o \leq p' \leq A_o$$

so the constants $\beta_- < \alpha_- < s < \alpha_+ < \beta_+$ are independent of ϵ so taking the limit $\epsilon \to 0$ of

$$\begin{aligned}|u'(\xi)| &\leq |u'(\alpha_+)| \left\|\frac{\xi-s}{\alpha_+-s}\right\|^{\frac{\alpha}{\epsilon}}, \qquad s < \xi < \alpha_+ \\ |u'(\xi)| &\leq |u'(\alpha_-)| \left\|\frac{\xi-s}{\alpha_--s}\right\|^{\frac{\alpha}{\epsilon}}, \qquad \alpha_- < \xi < s\end{aligned}$$

$$\begin{aligned}|u'(\xi)| &\leq |u'(\beta_+)| \exp\left\{-\frac{\beta}{\epsilon}\left(\left(\frac{\xi-s}{\beta_+-s}\right)^2-1\right)\right\}, \qquad \beta_+ < \xi \\ |u'(\xi)| &\leq |u'(\beta_-)| \exp\left\{-\frac{\beta}{\epsilon}\left(\left(\frac{\xi-s}{\beta_--s}\right)^2-1\right)\right\}, \qquad \beta_- < \xi\end{aligned} \tag{3.174}$$

the bound of $|u'| < K_\epsilon$ and the initial conditions we see that near s the value $|u'(\alpha_\pm)| \approx 0$, so $|u'(\xi)| \approx 0$, so we have

$$(\rho(\xi), u(\xi)) = \begin{cases} (\rho_-, u_-), & \xi < \beta_- \\ (\rho(s), u(s)), & \alpha_- < \xi < \alpha_+ \\ (\rho_+, u_+), & \beta_+ < \xi \end{cases}$$

□

The next theorem tells us about the Rankine-Hugoniot jump condition.

Theorem 15. *The solution (ρ, u) satisfies the Rankine-Hugoniot jump condition at all $\xi \in S$*

$$\xi[\rho(\xi_+) - \rho(\xi_-)] = \rho(\xi_+)u(\xi_+) - \rho(\xi_-)u(\xi_-)$$

$$\xi[\rho(\xi_+)u(\xi_+) - \rho(\xi_-)u(\xi_-)] = \rho(\xi_+)u^2(\xi_+) + p(\rho(\xi_+)) - \rho(\xi_-)u^2(\xi_-) - p(\rho(\xi_-))$$

Proof. Integrate

$$[(\xi - u)\rho]_a^b + \int_a^b \rho d\xi = 0$$

over the set $(\xi - \delta, \xi + \delta)$, and take the limit $\epsilon \to 0$, to get

$$(\xi + \delta - u(\xi + \delta))\rho(\xi + \delta) - (\xi - \delta - u(\xi - \delta))\rho(\xi - \delta) + \int_{\xi-\delta}^{\xi+\delta} \rho d\xi = 0$$

take the limit $\delta \to 0$ and put $\xi + \delta \to \xi_+$ and $\xi - \delta \to \xi_-$ to get

$$\xi[\rho(\xi_+) - \rho(\xi_-)] = \rho(\xi_+)u(\xi_+) - \rho(\xi_-)u(\xi_-)$$

for the second one is done the same but using

$$[(\xi - u)\rho u - p + \epsilon u']_{\xi-\delta}^{\xi+\delta} + \int_{\xi-\delta}^{\xi+\delta} \rho u d\xi = 0$$

□

In this section we study the structure of the limit (ρ, u) without the strict positiveness of the density limit, i.e., $\rho \geq 0$, but with the convexity hypothesis

$$p''(\rho) > 0 \qquad \rho > 0$$

The next theorem takes two important properties of solutions to (P_ϵ^μ) under the previous hypothesis of the pressure.

Theorem 16. *Let $(\rho_\epsilon, u_\epsilon)$ be a solution of (P_ϵ^μ) and s_ϵ be a singular point. Then (i) if u_ϵ in increasing on (s_ϵ, ∞), then $u_\epsilon' \leq 1$ on this interval. (ii) If u_ϵ is increasing in $(-\infty, s_\epsilon)$, then $u_\epsilon' \leq 1$ on this interval. (iii) If u_ϵ is decreasing on (s_ϵ, ∞), then there exists exactly one $\xi \in (s_\epsilon, \infty)$ such that $u_\epsilon''(\xi) = 0$. (iv) if u_ϵ is increasing on $(-\infty, s_\epsilon)$, then there exists exactly one $\xi \in (-\infty, s_\epsilon)$, such that $u_\epsilon''(\xi) = 0$.*

Proof. Let u_ϵ be increasing on (s_ϵ, ∞) and suppose there exist $\xi \in (s_\epsilon, \infty)$ such that $u_\epsilon'(\xi) > 1$. So the proof will be done by contradiction. Since we know that $u_\epsilon'(s_\epsilon) = 0$ be previous results, and $u_\epsilon'(\xi) \to 0$ as $\xi \to \infty$, it should exist $\xi_1, \xi_2 \in (s_\epsilon, \infty)$ such that $u_\epsilon'(\xi_1) = u_\epsilon'(\xi_2) = 1$, this because the function is increasing and the derivative has to go from zero, increase, and zero again, moreover $u_\epsilon''(\xi_1) > 0$ and $u_\epsilon''(\xi_2) < 0$ i.e. should be two points where the function changes concavity. In this case, because u_ϵ in increasing, then $(\xi - u_\epsilon)^2$ is decreasing on (ξ_1, ξ_2) and using the equations of (P) we find, from the second equation

$$\frac{\epsilon u''}{\rho'} = (u - \xi)\frac{\rho u'}{\rho'} + \frac{p'\rho'}{\rho'}$$

and from the first

$$(u - \xi) = -\frac{\rho u'}{\rho'}$$

combining this we have

$$p'(\rho) - (u - \xi)^2 = \epsilon \frac{u''}{\rho'}$$

and form the first equation again we know

$$\rho' = -\frac{\rho u'}{(u - \xi)}$$

so as before we can use this to find the signs. Using this we get

$$p'(\rho_\epsilon(\xi_1)) - (u_\epsilon(\xi_1) - \xi_1)^2 = \epsilon \frac{u''_\epsilon(\xi_1)}{\rho'(\xi_1)} > 0$$

$$p'(\rho_\epsilon(\xi_2)) - (u_\epsilon(\xi_2) - \xi_2)^2 = \epsilon \frac{u''_\epsilon(\xi_2)}{\rho'(\xi_2)} < 0$$

so

$$(u_\epsilon(\xi_1) - \xi_1)^2 < p'(\rho_\epsilon(\xi_1))$$

$$p'(\rho_\epsilon(\xi_2)) < (u_\epsilon(\xi_2) - \xi_2)^2$$

combining

$$p'(\rho_\epsilon(\xi_2)) < (u_\epsilon(\xi_2) - \xi_2)^2 < (u_\epsilon(\xi_1) - \xi_1)^2 < p'(\rho_\epsilon(\xi_1))$$

because p' is increasing then $\rho_\epsilon(\xi_2) < \rho_\epsilon(\xi_1)$ but we know that if u_ϵ increase ρ_ϵ increase in (s_ϵ, ∞) so this is a contradiction. Doing the same we get the other result for case (ii).

Now the other two results. Let u_ϵ decrease in (s_ϵ, ∞).Since u_ϵ is decreasing then ρ_ϵ decreases too in this set. Since $u'_\epsilon(\xi) \to 0$ as $\xi \to \infty$, and $u'_\epsilon(s_\epsilon) = 0$, there exists $\xi_1 \in (s_\epsilon, \infty)$ such that $u''_\epsilon(\xi_1) = 0$. Now suppose that there exists $\xi_2 > \xi_1$ such that $u''_\epsilon(\xi_2) = 0$ too, so the proof will be done by contradiction. As in the previous part of the proof, using

$$p'_\epsilon(\rho_\epsilon) - (u_\epsilon - \xi)^2 = \epsilon \frac{u''_\epsilon}{\rho'_\epsilon}$$

we get

$$p'(\rho_\epsilon(\xi_1)) - (u_\epsilon(\xi_1) - \xi_1)^2 = 0$$

$$p'(\rho_\epsilon(\xi_2)) - (u_\epsilon(\xi_2) - \xi_2)^2 = 0$$

combining

$$p'(\rho_\epsilon(\xi_1)) = (u_\epsilon(\xi_1) - \xi_1)^2 < (u_\epsilon(\xi_2) - \xi_2)^2 = p'(\rho_\epsilon(\xi_2))$$

now, p' is increasing then $\rho_\epsilon(\xi_1) < \rho_\epsilon(\xi_2)$, but ρ_ϵ decreases ($\rightarrow\leftarrow$), so there $\exists!\ \xi \in (s_\epsilon, \infty)$ such that $u''_\epsilon(\xi) = 0$. For the other part of the interval the proof is the same.

□

Properties (i) and (ii) of previous theorem provide the structure of rarefaction waves and (iii) and (iv) provide the structure of shock waves. Since (3.174) implies the convergence of the double index sequence $u_{\epsilon 1}(s_{\epsilon 2})$ for viscous solutions u_ϵ which belong to the category of $C1, C2, C3$, we have $u(s) = s$. For the case of $C4$, the convergence is from(i) and (ii) of previous theorem. So we get $u(s) = s$. In the following two theorems we study the continuity of the limit solution.

Theorem 17. *Let a solution (ρ, u) of (P) be a limit of viscous solutions $(\rho_\epsilon, u_\epsilon)$ of (P_ϵ) and s be the limit of singular point s_ϵ. Then $u(s) = s$ and ρ and u are continuous at $\xi = s$.*

Proof. First, we suppose $\rho(s) > 0$. Then, from (3.173), $\rho(\xi)$ and $u(\xi)$ are constant on a neighborhood of $\xi = s$. So ρ and u are continuous at $\xi = s$.

Now suppose $\rho(s) = 0$. Since $\rho_\pm$ are positive, this is possible only when $(\rho_\epsilon, u_\epsilon)$ belongs to Category $C4$. So , u_ϵ are increasing on $(-\infty, \infty)$ and $|u'_\epsilon(»)| \leq 1$. We already know that u_ϵ is uniformly bounded. The Ascoli-Arzela theorem implies the limit u is continuous on $(-\infty, \infty)$. From the Rankine-Hugoniot jump condition at $\xi = s$ we have $p(\rho(s_+)) = p(\rho(s_-))$, it means that the pressure doesn't change in the singular point. Now remember that $p' \geq 0$ if $\rho \geq 0$, i.e. increasing or constant, because of the continuity of the pressure ρ cannot jump in the discontinuity, if not the pressure will jump too, so $\rho_+ = \rho_-$, i.e. its continuous in s.

□

We consider now the continuity of rarefaction waves.

Theorem 18. *Let a solution (ρ, u) of (P) be a limit of viscous solutions $(\rho_\epsilon, u_\epsilon)$ of (P_ϵ) and s be a singular point. If u_ϵ in increasing on (s, ∞), then ρ and u are continuous on (s, ∞). If u_ϵ in increasing on $(-\infty, s)$, then ρ and u are continuous on $(-\infty, s)$.*

Proof. We know that $|u_\epsilon(\xi)|$ is bounded by a constant M that is independent of ϵ. We shown two theorems above that $|u_\epsilon(\xi)| \leq 1$. So the sequence $\{|u\}$ is uniformly bounded and equicontinous, so it fulfills the requirements for the Ascoli-Arzela theorem, and u is continuous.

Suppose that $\rho(s) > 0$. From

$$(\rho(\xi),u(\xi)) = \begin{cases} (\rho_-,u_-), & \xi<\beta_- \\ (\rho(s),u(s)), & \alpha_-<\xi<\alpha_+ \\ (\rho_+,u_+), & \beta_+<\xi \end{cases}$$

we now that ρ is constant in $(s, s+\delta)$ for some δ, and that this last is not dependent of ϵ, moreover remember that a in $A|\xi-s| < |u(\xi)-\xi| < B|\xi-s|$ is independent of ϵ $(A = 1-a)$ so because of the equation

$$|\rho_\epsilon'| = \frac{|\rho_\epsilon u_\epsilon'|}{|u_\epsilon-\xi|} < \frac{\rho_\epsilon}{|u_\epsilon-\xi|}$$

its bounded from above, on the interval $(s+c,\infty)$, for any $c>0$. So by the Ascoli-Arzela $|\rho|$ is continuous, so ρ is continuous on (s,∞).

Now the case $\rho(s)=0$. We divide the proof in two.

First: Suppose $\rho(\xi)>0$ on (s,∞). It is enough to prove that ρ is continuous on $(s+\delta,\infty)$ for any $\delta>0$.

We start assuming $u(s+\delta)=s+\delta$. Then, since $u_\epsilon'(\xi)\le 1$ for all ϵ, we can say that $\xi=s+\delta$ because we satisfy $u_\epsilon'(\xi)\le 1$ with our previously defined u having $u'(\xi)=1$ for all ϵ, so $u(\xi)=\xi$ and as s is a fixed point, i.e. $u(s)=s$ then ξ can take values on the interval $[s,s+\delta]$.

From $(u-\xi)\rho'+\rho u'=0$ we have

$$\rho = \frac{\overbrace{(u-\xi)}^{=0}\rho'}{u'} = 0$$

with $\xi\in[s,s+\delta]$. This contradicts our supposition $\rho(\xi)>0$ on (s,∞) therefore to satisfy $u_\epsilon'(\xi)\le 1$ for all ϵ, we remove the equality from the inequality getting $u(s+\delta)<s+\delta$.

As we can see from (3.155) and (3.156) we have

$$u(\xi)-\xi<0 \quad \text{on} \quad \xi\in(s,\infty) \qquad\qquad u(\xi)-\xi>0 \quad \text{on} \quad \xi\in(-\infty,s)$$

So taking the latter inequality we have

$$u(\xi)-\xi<0<s+\delta-u(s+\delta) \to u(\xi)-\xi<s+\delta-u(s+\delta) \qquad \xi \in (\text{s},\infty)$$

$$u(s+\delta)-s-\delta<0<u(\xi)-\xi \rightarrow u(s+\delta)-s-\delta<u(\xi)-\xi \qquad \xi \in(\text{-}\infty,\text{s})$$

Then taking into account the interval on which we are interested where$\xi \in (s,\infty)$

$$u(\xi)-\xi<s+\delta-u(s+\delta)<\xi-u(\xi) \qquad \xi \in(\text{s}+\delta,\infty)$$

Thus $|u(\xi)-\xi|>s+\delta-u(s+\delta)$ for $\xi\in(s+\delta,\infty)$. So because of

$$|\rho_\epsilon'|=\frac{|\rho_\epsilon u_\epsilon'|}{|u_\epsilon-\xi|}<\frac{\rho_\epsilon}{|u_\epsilon-\xi|}$$

is bounded in the set. Now yes, with this conditions, for the first case in the set (s,∞) , we can apply Ascoli-Arzela theorem and ρ is continuous.

Second: suppose that $\rho(\xi)=0$ on $(s,s+\tau]$ and $\rho(\xi)>0$ on $(s+\tau,\infty)$ for some $\tau>0$. The continuity, except in $\xi=s+\tau$, follows from the first case, we only need to prove the continuity at this point. The Rankine-Hugoniot is valid everywhere, so at $s+\tau$. Since $\rho(\xi)=0$ on $(s,s+\tau]$, then $\rho\left((s+\tau)_-\right)=0$, and the jump condition gives

$$(s+\tau)\rho\left((s+\tau)_+\right)=\rho\left((s+\tau)_+\right)u\left((s+\tau)_+\right)$$

$$(s+\tau)\rho\left((s+\tau)_+\right)u\left((s+\tau)_+\right)=\rho\left((s+\tau)_+\right)u^2\left((s+\tau)_+\right)+p\left(\rho\left((s+\tau)_+\right)\right)$$

form the first, cancelling $\rho\left((s+\tau)_+\right)$ we have

$$(s+\tau)=u\left((s+\tau)_+\right)$$

plug in this in the second one cancels the terms $(s+\tau)\rho\left((s+\tau)_+\right)u\left((s+\tau)_+\right)$ and $\rho\left((s+\tau)_+\right)u^2\left((s+\tau)_+\right)$ so

$$p\left(\rho\left((s+\tau)_+\right)\right)=0$$

so because of $p(\rho)=0$ implies $\rho=0$

$$\rho\left((s+\tau)_+\right)=0$$

so $\rho\left((s+\tau)_+\right)=\rho\left((s+\tau)_-\right)=0$ and ρ is continuous.

The proof for the interval $\xi\in(-\infty,s)$ is similar.

□

The previous theorem provide regularity properties for the limit of the solution (ρ, u). Let S be the set of points of discontinuity of (ρ, u) and C be the set of points of continuity. If u is increasing, (ρ, u) is continuous due to the last theorem. If u is decreasing, we can easily verify that there exists at most one point of discontinuity in $(-\infty, s)$ and (s, ∞) from the theorem were we characterized the shock waves with in (iii),(iv).

Now we consider the relationship between the characteristic speeds of the problem (P) and the weak derivative of the limit solution (ρ, u) According to our original problem

$$\begin{cases} \rho_t + (\rho u)_x = 0 & x \in \mathbb{R} \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0 & t > 0 \end{cases}$$

Let

$$m = \rho u, \qquad U = \begin{pmatrix} \rho \\ m \end{pmatrix}, \qquad F(U) = \begin{pmatrix} m \\ \frac{m^2}{\rho} + p(\rho) \end{pmatrix}$$

We know that the eigenvalues $\lambda_{\pm}$of ∇F are given by

$$\lambda_{\pm}(\rho, u) = u \pm \sqrt{p'(\rho)}$$

We use the notation $\lambda_{\pm}(\xi) = \lambda_{\pm}(\rho(\xi), u(\xi))$ and $\lambda_{\pm}^{\epsilon} = \lambda_{\pm}(\rho_\epsilon(\xi), u_\epsilon(\xi))$.

Let $d\mu = (d\mu_1, d\mu_2) = \frac{dU}{d\xi}$ be the vector valued measure which corresponds to the weak derivative of U, i..e. corresponding to the linear functional:

$$\phi \to -\int \phi'(\xi) U(\xi) d\xi, \qquad \phi \in C_c^1(\mathbb{R})$$

We apply the Volpert product of $\nabla F(U)$ and $d\mu$ to the equation (P) to get

$$(\hat{F}(U) - \xi I) d\mu = 0$$

in the sens of measures, where the averaged superposition $\hat{\nabla F}(U)$ of U by ∇F is given by

$$(\hat{\nabla F}(U)(\xi) = \int_0^1 \nabla F(U(\xi-) + s(U(\xi+ = -U(\xi-)))ds$$

Let $\xi \in C \cap supp\mu$. Since there is at most one point of discontinuity, $\hat{\nabla F}(U) = \nabla F(U)$ in a neighborhood of ξ . Suppose the determinant of$(\nabla F(U)-\xi I)$ is not zero, for example $det(\nabla F(U)-\xi I)>0$. Then there exists a neighborhood N of ξ such that $det(\nabla F(U)-\varsigma I)>\delta>0$ for all $\varsigma\in N$. But from $(\hat{F}(U)-\xi I)d\mu=0$ the measures $det(\nabla F(U)-\varsigma I)d\mu_{1,2}=0$ on N, which contradicts the fact that $\xi\in$ suppμ. So ξ is an eigenvalue of $\nabla F(U(\xi))$. We summarize these facts in a theorem :

Theorem 19. *Let a solution (ρ,u) of (P) be a limit of viscosity solutions $(\rho_\epsilon,u_\epsilon)$ of $(P_\epsilon)$with a singular point s. Let $d\mu$ be the measure of $\phi\to-\int\phi'(\xi)U(\xi)d\xi$. Then we have: (i) If u is increasing, then$(\rho,u)$is continuous. If u is decreasing, then there exists at most one point of discontinuity in $(-\infty,s)$and (s,∞). (ii) If $(\frac{1}{\rho},u)$ is continuous at $\xi\in supp\mu$, $\xi=\lambda_+(\xi)$ on (s,∞) and $\xi=\lambda_-(\xi)$on $(-\infty,s)$.*

We conclude the section with a theorem which provides the structure of the limit solution (ρ,u) of the viscosity solutions $(\rho_\epsilon,u_\epsilon)$ which obey the a priori estimates.

Theorem 20. *Let (ρ,u) be a solution of the Riemann problem (P) through the method of self-similar zero-viscosity limits and s be the limit of singular points. (i) If u is increasing on (s,∞), then ¸$\lambda_+(\xi)$ is continuous on (s,∞) and*

$$\lambda_+(\xi)=\begin{cases}\lambda_+(s) & , \quad s<\xi<\lambda_+(s)\\ \xi & , \quad \lambda_+(s)<\xi<\lambda_+(\rho_+,u_+)\\ \lambda_+(\rho_+,u_+) & , \quad \lambda_+(\rho_+,u_+)<\xi\end{cases}$$

(ii) If u is increasing on $(-\infty,s)$, then ¸$\lambda_-(\xi)$ is continuous on $(-\infty,s)$ and

$$\lambda_-(\xi)=\begin{cases}\lambda_-(s) & , \quad s<\xi<\lambda_-(s)\\ \xi & , \quad \lambda_-(\rho_-,u_-)<\xi<\lambda_-(s)\\ \lambda_-(\rho_-,u_-) & , \quad \xi<\lambda_-(\rho_-,u_-)\end{cases}$$

(iii) If u is decreasing on (s,∞), then $\lambda_+(\xi)$ has an unique discontinuity on (s,∞) and

$$\lambda_+(\xi)=\begin{cases}\lambda_+(s) & , \quad s<\xi<\frac{\rho_+u_+}{(\rho_+-\rho(s))}\\ \lambda_+(\rho_+,u_+) & , \quad \frac{\rho_+u_+}{(\rho_+-\rho(s))}<\xi\end{cases}$$

(iv) If u is decreasing on $(-\infty,s)$, thenu has an unique discontinuity on $(-\infty,s)$ and

$$\lambda_{-}(\xi) = \begin{cases} \lambda_{-}(s) & , \quad \frac{\rho_{-}u_{-}}{(\rho_{-}-\rho(s))} < \xi < s \\ \lambda_{-}(\rho_{-}, u_{-}) & , \quad \xi < \frac{\rho_{-}u_{-}}{(\rho_{-}-\rho(s))} \end{cases}$$

Proof. We always consider the eigenvalue λ_- on the interval $(-\infty, s)$and λ_+ on the other side (s,∞). If u is increasing, then u and ρ are continuous. So they should be constant out of suppμ and $\lambda_\pm$ are continuous and increasing, too. With these facts we can easily check that suppμ is a connected subintervals of $(-\infty, s]$ or $[s,\infty)$. If not, $\lambda_\pm$ is discontinuous. So the structure of $\lambda_\pm$ should follow (i) and (ii).

If u is decreasing, then $\lambda_\pm$ are also decreasing and there exists at most one point of discontinuity on$(-\infty, s)$ and(s,∞). Since $\lambda_\pm = \xi$ on suppμ and $\lambda_\pm$ are decreasing, suppμ should be the point of discontinuity and $\lambda_\pm$ be constant before and after the discontinuity. We can find the point of the discontinuity from the Rankine-Hugoniot jump condition, and $\lambda_\pm$should follow (iii) and (iv).

□

In summary, the limit of viscosity solutions has an intermediate state which is connected to the boundary states by rarefaction waves ((i) and (ii)) or shocks ((iii) and (iv)).

Theorem 21. *If the system* (P) *(The P-system) is strictly hyperbolic, i.e. there exists*$c > 0$ *such that*

$$p\,'(\rho) \geq c^2 > 0, \qquad \rho > 0$$

then the emerging limit does not have a vacuum state.

Proof. The last theorem implies that $\lambda_\pm$ is constant on the interval $(s, \lambda_+(s)) \neq \phi$ and$(\lambda_+(\rho_+, u_+), \infty)$. Since $\sqrt{p\,'(\rho)}$ is increasing on (s,∞), $u(\xi)$ is also constant on those intervals. From $(u-\xi)\rho' + \rho u' = 0$, ρ is also constant. Now we consider the interval $(\lambda_+(s), \lambda_+(\rho_+, u_+))$. From $(u-\xi)\rho u' + p(\rho)' = 0$ we get $c^2\rho\,' \leq (\xi - u)\rho$, and hence there exists a constant C such that

$$\rho\,' \leq C\rho.$$

So we have

$$\rho(\xi) \leq \rho(\lambda_+(s))e^{C(\xi-\lambda_+(s))}.$$

Suppose the solution has a vacuum state, i.e. $\rho(s) = 0$. Then, since ρ is constant on $(s, \lambda_+(s))$, $\rho(\lambda_+(s)) = 0$, and hence ρ is zero on $(\lambda_+(s), \lambda_+(\rho_+, u_+))$. So $\rho(\infty) = 0$ which contradicts the boundary condition $\rho(\infty) = \rho_+ > 0$.

□

Convex pressure laws

In Lemma 6 the a priori estimates $0 < \delta < \rho(\xi) < M$, $|u(\xi)| < M$ for any $\xi \in (-\infty, \infty)$ are established except for the lower bound for ρ of the case C_4. In this section we complete the a priori estimates in two cases under the convex pressure laws

$$p''(\rho) \geq 0 \qquad \text{for} \qquad \rho > 0$$

. First, we consider the case of strictly hyperbolic systems.

The equation

$$(u - \xi)\rho' + \rho u' = 0 \qquad -\infty < \xi < \infty$$

can be written as a first order linear equation for ρ :

$$\rho' + \frac{u'}{u - \xi}\rho = 0, \qquad \xi \neq s$$

where s is the singular point. Then the solution is given by

$$\rho(\xi) = \begin{cases} \rho_+^\mu e^{-\int_\xi^\infty \frac{u'}{\varsigma - u} d\varsigma}, & s < \xi \\ \rho_-^\mu e^{-\int_{-\infty}^\xi \frac{u'}{\varsigma - u} d\varsigma}, & \xi < s \end{cases}$$

where $\rho_-^\mu = \rho_+$ and $\rho_+^\mu = \rho_- + \mu(\rho_+ - \rho_-)$ are the boundary conditions $\rho(\pm\infty) = \rho_\pm^\mu := \rho_- + \mu(\rho_\pm - \rho_-)$ for $0 \leq \mu \leq 1$

Theorem 22. *Let a solution (ρ, u) of (P_ϵ^μ) belong to the class C_4. If the system (P) is strictly hyperbolic, i.e.*

$$p'(\rho) \geq c^2 > 0, \qquad \rho > 0$$

then there exists a constant $\delta > 0$ which satisfies (3.153) and is independent of ϵ and μ.

Proof. Since u is increasing on $\mathbb{R}$, $u' \leq 1$ by previous theorems, and $\xi - u(\xi)$ is also increasing. Since the function is always increasing, and the derivative goes from zero at the singular point, to zero at infinity, then it should exist a $\xi_1 > s$ such that $0 < \xi - u(\xi) \leq \frac{c}{2}$ on $(s, \xi 1)$ and $\frac{c}{2} \leq \xi - u(\xi)$ on (ξ_1, ∞).

Then

$$\rho(\xi_1) = \rho_+^{\mu} e^{-\int_{\xi_1}^{\infty} \frac{u'}{\varsigma - u} d\varsigma} \geq \rho_+^{\mu} e^{-\frac{2}{c}(u_+ - u_-)} \tag{3.175}$$

From the equations in (P), we can write

$$\left[p'\left(\rho(\xi)\right) - (u(\xi) - \xi)^2\right] \rho'(\xi) = \epsilon u''(\xi)$$

Integrating this last on(s, ξ_1) we get, for the right side

$$\epsilon \int_s^{\xi_1} u''(\varsigma) d\varsigma = \epsilon[u'(\xi_1) - u'(s)] = \epsilon u'(\xi_1) \leq \epsilon \tag{3.176}$$

where we use $u'(s) = 0$ and the fact that $u\acute{} \leq 1$. Now for the left part.

$$\int_s^{\xi_1} (p'(\rho) - (\varsigma - u)^2)\rho' d\varsigma$$

In the set $(s, \xi 1)$, we have $0 < \xi - u(\xi) \leq \frac{c}{2}$ so

$$\begin{aligned} (\xi - u(\xi))^2 &\leq \frac{c^2}{4} \\ -\frac{c^2}{4} &\leq - (\xi - u(\xi))^2 \end{aligned}$$

so substituting this and using $p\acute{}(\rho) \geq c^2$, and integrating we have

$$\frac{3c^2}{4} \int_s^{\xi_1} \rho' d\varsigma = \int_s^{\xi_1} (c^2 - \frac{c^2}{4})\rho' d\varsigma \leq \int_s^{\xi_1} (p'(\rho) - (\varsigma - u)^2)\rho' d\varsigma$$

so

$$\frac{3c^2}{4}(\rho(\xi_1) - \rho(s)) \leq \int_s^{\xi_1} (p'(\rho) - (\varsigma - u)^2)\rho' d\varsigma$$

Combining this last result with (3.176)

$$\epsilon \leq \frac{3c^2}{4}(\rho(\xi_1) - \rho(s))$$

using (3.175) we can write the bound of ρ from below by

$$\min\{\rho_-,\rho_+\}e^{-\frac{2}{c}(u_+-u_-)}-\frac{4\epsilon}{3c^2}\leq \rho_+^{\mu}e^{-\frac{2}{c}(u_+-u_-)}-\frac{4\epsilon}{3c^2}\leq \rho(s)$$

Notice that if $\epsilon \to 0$ then ρ is bounded by a constant independent of ϵ.

□

We return to general convex pressure laws $p'(\rho) \geq c^2 > 0$, with $\rho > 0$ and consider the function

$$g(\rho)=\frac{p(\rho)}{\rho}, \qquad \rho > 0$$

. Either the function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is invertible or the system is strictly hyperbolic, as we will see in the next theorem. Consider the case when g has an inverse g^{-1}

Theorem 23. *Let a solution (ρ, u) of (P^{μ}_{ϵ})belong to the class C_4. If the boundary conditions $(\rho_\pm, u_\pm)$ satisfy*

$$u_+ - u_- < \max_{m>0}(m\ln(\frac{\rho_-^{\mu}}{g^{-1}(m^2)})) + \max_{m>0}(m\ln(\frac{\rho_+^{\mu}}{g^{-1}(m^2)})) \tag{3.177}$$

then there exists a constant $\delta > 0$ which satisfies (3.159) and is independent of ϵ and μ.

.

Proof. Let s be the singular point of the solution (ρ, u). Since u is increasing in $\mathbb{R}$, we have $u_- < u(s) < u_+$. If (3.177) holds,

$$u_+ - u(s) < \max_{m>0}(m\ln(\frac{\rho_+^{\mu}}{g^{-1}(m^2)}))$$

or

$$u(s) - u_- < \max_{m>0}(m\ln(\frac{\rho_-^{\mu}}{g^{-1}(m^2)}))$$

We assume that the first one holds. Then there exists $m > 0$ such that $u_+ - u(s) < m\ln(\frac{\rho_+^{\mu}}{g^{-1}(m^2)})$ or equivalently, taking the exponent,

$$g\left(\rho_+^{\mu}e^{-\frac{(u_+-u(s))}{m}}\right) - m^2 > 0.$$

Since $\xi - u(\xi)$ is increasing, as stated above in the previous theorem, we can find a $\xi_1 > s$ such that $0 < \xi - u(\xi) \leq m$ on (s, ξ_1) and $m \leq \xi - u(\xi)$ on (ξ_1, ∞). Then

$$\begin{aligned} m &\leq \xi - u(\xi) \\ \frac{1}{\xi - u(\xi)} &\leq \frac{1}{m} \\ -\frac{1}{m} &\leq -\frac{1}{\xi - u(\xi)} \end{aligned}$$

so $\rho_+^\mu e^{-\int_{\xi_1}^\infty \frac{u'}{m} d\varsigma} \leq \rho_+^\mu e^{-\int_{\xi_1}^\infty \frac{u'}{\varsigma - u} d\varsigma} = \rho(\xi_1)$ integrating the left hand side

$$\rho_+^\mu e^{-\int_{\xi_1}^\infty \frac{u'}{m} d\varsigma} = \rho_+^\mu e^{-\frac{(u_+ - u(\xi_1))}{m} d\varsigma}$$

and because $\xi_1 > s$

$$\rho_+^\mu e^{-\frac{(u_+ - u(s))}{m}} \leq \rho_+^\mu e^{-\int_{\xi_1}^\infty \frac{u'}{\varsigma - u} d\varsigma} = \rho(\xi_1) \tag{3.178}$$

We can easily check that $g(\rho)$ is increasing for $\rho > 0$, derivating respect of ξ

$$g'(\rho) = \frac{p'\rho'}{\rho} - \frac{p\rho'}{\rho^2}$$

the first term is positive because $p'(\rho) = p'\rho' \geq c^2$, and p is positive then g' is positive because $\rho' \leq 0$ **so** the second term is positive and g is increasing. Using (3.178) and the positiveness of the derivative of g its obvious that

$$g(\rho(\xi_1)) - m^2 > 0$$

.

Integrating

$$\left[p'\left(\rho(\xi)\right) - \left(u(\xi) - \xi\right)^2\right] \rho'(\xi) = \epsilon u''(\xi)$$

on (s, ξ_1) we get, as in previous theorem

$$\epsilon \int_s^{\xi_1} u''(\varsigma) d\varsigma = \epsilon u'(\xi_1) \leq \epsilon \tag{3.179}$$

Before doing the second part, see that in the set (s, ξ_1) its true $0 < \xi - u(\xi) \leq m$ so its also true

$$-m^2 \leq -(\xi - u(\xi))^2$$

so the second integral can be bounded by

$$\int_s^{\xi_1} (p'(\rho) - m^2)\rho' d\xi \leq \int_s^{\xi_1} (p'(\rho) - (\varsigma - u)^2)\rho' d\xi$$

and the right hand side is

$$\begin{aligned}
\int_s^{\xi_1} (p'(\rho) - m^2)\rho' d\xi &= \int_s^{\xi_1} (p\rho)' - m^2\rho' d\xi \\
&= (p(\xi_1) - p(s))(\rho(\xi_1) - \rho(s)) - m^2(\rho(\xi_1) - \rho(s)) \\
&= \left[\frac{p(\xi_1) - p(s)}{\rho(\xi_1) - \rho(s)} - m^2\right](\rho(\xi_1) - \rho(s))
\end{aligned}$$

so

$$\left[\frac{p(\xi_1) - p(s)}{\rho(\xi_1) - \rho(s)} - m^2\right](\rho(\xi_1) - \rho(s)) \leq \int_s^{\xi_1} (p'(\rho) - (\varsigma - u)^2)\rho' d\xi$$

The convexity hypothesis implies that p is increasing, and because $\rho(s) > 0$ then

$$\frac{p(\rho(\xi_1))}{\rho(\xi_1)} - m^2 = \frac{p(\rho(\xi_1))}{\rho(\xi_1) - \rho(s)} - m^2 < \frac{p(\rho(\xi_1)) - p(\rho(s))}{\rho(\xi_1) - \rho(s)} - m^2$$

so

$$0 < g(\rho(\xi_1)) - m^2 = \frac{p(\rho(\xi_1))}{\rho(\xi_1) - \rho(s)} - m^2 < \frac{p(\rho(\xi_1)) - p(\rho(s))}{\rho(\xi_1) - \rho(s)} - m^2$$

Remember that this should be compared with the left side integrated before, i.e. (3.179) so

$$\begin{aligned}
\left(g(\rho(\xi_1)) - m^2\right)(\rho(\xi_1) - \rho(s)) &\leq \epsilon \\
\rho(\xi_1) - \rho(s) &\leq \frac{\epsilon}{g(\rho(\xi_1)) - m^2}
\end{aligned}$$

and

$$\begin{aligned} \rho(\xi_1) - \frac{\epsilon}{g(\rho(\xi_1)) - m^2} &\leq \rho(s) \\ \rho(\xi_1) - \frac{\epsilon}{g(\rho(\xi_1)) - m^2} &\leq \rho(\xi_1) - \frac{\epsilon}{g(\rho(\xi_1)) - m^2} \\ \rho_+^{\mu} e^{-\frac{(u_+ - u(s))}{m}} - \frac{\epsilon}{g(\rho(\xi_1)) - m^2} &\leq \rho(s) \end{aligned}$$

where the last inequality comes from (3.178). So the density ρ is bounded below by

$$\min\{\rho_+, \rho_-\} e^{-\frac{1}{m(u_+ - u(s))}} - \frac{\epsilon}{g(\rho(\xi_1)) - m^2} \leq \rho_+^{\mu} e^{-\frac{(u_+ - u(s))}{m}} - \frac{\epsilon}{g(\rho(\xi_1)) - m^2} \leq \rho(s)$$

for a sufficiently small $\epsilon > 0$. The situation is similar if the second condition holds.

□

One can check that, if

$$u_+ - u_- < \max_{m>0}(m \ln(\frac{\rho_-^{\mu}}{g^{-1}(m^2)})) + \max_{m>0}(m \ln(\frac{\rho_+^{\mu}}{g^{-1}(m^2)})) \tag{3.180}$$

holds and $\rho_- \leq \rho+$, then (3.177) holds. If $\rho_- > \rho_+$ and instead of using the continuity of the boundary data

$$\begin{aligned} \rho(\pm\infty) &= \rho_- + \mu(\rho_\pm - \rho_-) \qquad 0 \leq \mu \leq 1 \\ u(\pm\infty) &= u_- + \mu(u_\pm - u_-) \end{aligned}$$

one uses

$$\begin{aligned} \rho(\pm\infty) &= \rho_+ + \mu(\rho_\pm - \rho_+) \qquad 0 \leq \mu \leq 1 \\ u(\pm\infty) &= u_+ + \mu(u_\pm - u_+) \end{aligned}$$

and the (3.177) holds. So (3.180) is sufficient condition (but not necessary) such that the vacuum won't appear. As seen in [Smoller], admissible solutions of the gas dynamics don't have vacuum state iff

$$u_+ - u_- < \int_0^{\rho_-} \frac{\sqrt{p'(\rho)}}{\rho} d\rho + \int_0^{\rho_+} \frac{\sqrt{p'(\rho)}}{\rho} d\rho$$

We state the last theorem.

Theorem 24. *Suppose that $p(\rho)$ satisfies*

$$p'(\rho) > 0 \qquad \rho > 0$$

$$p(\rho) \to \infty \qquad \rho \to \infty$$

$$p(\rho) \to 0 \qquad \rho \to 0$$

If the system

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2 + p(\rho))_x &= 0 \end{aligned}$$

is strictly hyperbolic or the initial data $(\rho_\pm, u_\pm)$ satisfy (3.180), then the boundary value problem (P) has a solution (ρ, u) which is a $\epsilon \to 0$ of solutions of (P_ϵ). The function (ρ, u) has structure stated in the previous theorems and does not contain vacuum, moreover, $\left(\rho(\frac{x}{t}), u(\frac{x}{t})\right)$ is a solution of the Riemann problem.

Bibliography

[1] Smoller Joel, *Shock Waves and Reaction-Diffusion Equation*, second ed., Springer Verlag, 1994.

[2] Nelson David & Cox Michael, *Lehninger Principles of Biochemistry*, fifth ed., W.H. Freeman and Company, 2008.

[3] Chapman S., Fowler A.C. & Hinch R., *Introduction to Mathematical Physiology*, first ed., Oxford University, 2006.

[4] Duan P. Chen,* Robert S. Eisenberg,* Joseph W. Jerome,t and Chi-Wang Shu., *Hydrodynamic Model of Temperature Change in Open Ionic Channels*, Biophysical Journal Volume 69 December 1995.

[5] Yong Jung Kim., *A self-similar viscosity approach for the Riemann problem in isentropic gas dynamics and the structure of the solutions*, Preprint submitted to Quart. Appl. Math. September 1999.

[6] Bruce Alberts, Alexander Johnson, Julian Lewis, Martin Raff, Keith Roberts, Peter Walter., *Molecular Biology of the Cell*, fourth edition, edited by Bruce Alberts (2002) published by Garland Science.

[7] Barnett MW, Larkman PM., *The action potential*, Pract Neurol 2007;7:192-197.

[8] Fromhertz Peter, *Joining microelectronics and microionics: Nerve cells and brain tissue on semiconductor chips*, Max Plank Institute of Biochemistry, Departament of Membrane and Neurophysics, Marinsried, Germany, 2008

[9] Fromhertz Peter, *Neuroelectronic Interfacing: Seminconductor Chips with Ion Channels, Nerve cells, and Brain*, Max Plank Institute of Biochemistry, Departament of Membrane and Neurophysics, Marinsried, Germany, 2003

[10] F. Di Michele et al, *An easy model of ions flux in cellular membrane including ionic pumps*, paper in preparation, University of L'Aquila, Department of Applied Mathematics, L'Aquila, Italy, 2011

[11] Tzavaras, Anthanasios E, *Elastic as Limit of Viscoelastic Response, in a Context of Self-Similar Viscous Limits*, Wisconsin Univ-Madison Center for Mathematical Sciences, March, 1994

[12] Mason Cara, *NanoTechnology and it's Prospects for Alzheimer's disease*, Merrit research paper based on pathology a Medlink, 2007

[13] Knowles JR, *Enzyme-catalyzed phosphoryl transfer reactions*, Annu. Rev. Biochem. 49: 877–919, 1980

[14] *Nature's Batteries' May Have Helped Power Early Lifeforms*, Science Daily, May 25, 2010

[15] C. F. Dafermos, *Solution of the Riemann problem for a class of hyperbolic conservation laws by the viscosity method,* Arch. Rational Mech. Analysis 52 (1973) 1-9

[16] M. Slemrod, *Resolution of the spherical piton problem for compressible isentropic gas dynamics via a self-similar viscous limit*, Proc. Roy Soc. Edinburgh Sec A 126 (196) 1309-1340

[17] M. Slemrod, A. Tzavaras, *A limiting viscosity approach for the Riemann Problem in isentropic gas dynamics*, Indiana University Math. J. 38 (1989) 1047-1074