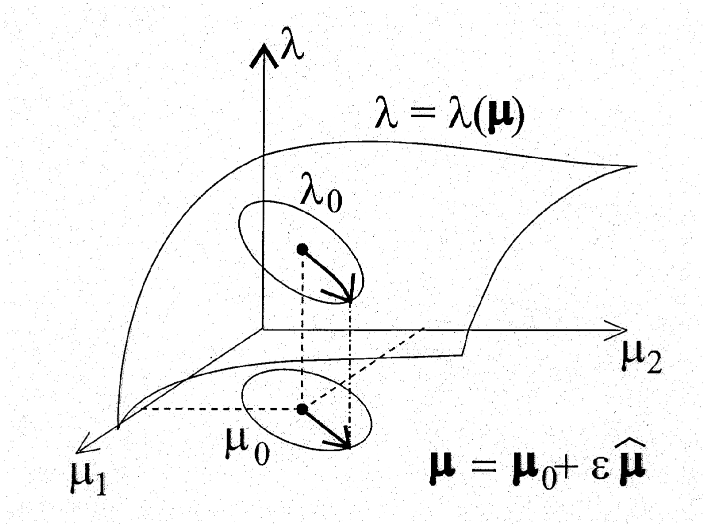
EIGENVALUE/EIGENVECTOR SENSITIVITY



- **Linear eigenvalue problem:**

$$(\mathbf{A}(\boldsymbol{\mu}) - \lambda(\boldsymbol{\mu})\,\mathbf{I})\,\mathbf{w} = \mathbf{0} \qquad\qquad \boldsymbol{\mu} \in \mathbb{R}^M$$

- **Unperturbed problem,** ε=0**:**

$$(\mathbf{A}_0 - \lambda_0\,\mathbf{I})\,\mathbf{w}_0 = \mathbf{0}, \quad \mathbf{A}_0 := \mathbf{A}(\boldsymbol{\mu}_0), \quad \lambda_0 := \lambda(\boldsymbol{\mu}_0)$$

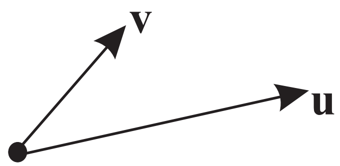
- **Perturbed problem,** ε≠0**:**

$$(\mathbf{A}(\varepsilon) - \lambda(\varepsilon)\,\mathbf{I})\,\mathbf{w}(\varepsilon) = \mathbf{0}$$

- **Properties of the eigenvectors:**

a) λ_0 **is simple (multiplicity *m* = 1)**

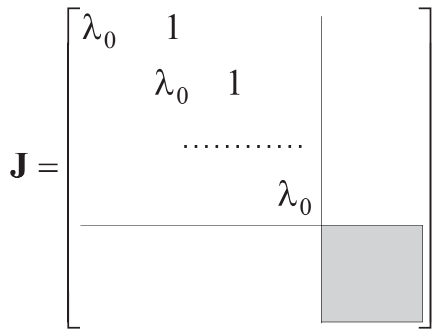
$$(\mathbf{A}_0 - \lambda_0 \mathbf{I})\mathbf{u} = 0$$
$$(\mathbf{A}_0 - \lambda_0 \mathbf{I})^H \mathbf{v} = 0$$



$$\mathbf{v}^H \mathbf{u} = 1$$

b) λ_0 **is multiple (*m* > 1) (A$_0$ non-derogatory)**

$$(\mathbf{A}_0 - \lambda_0 \mathbf{I})\mathbf{u}_1 = 0$$
$$(\mathbf{A}_0 - \lambda_0 \mathbf{I})^H \mathbf{v}_m = 0$$

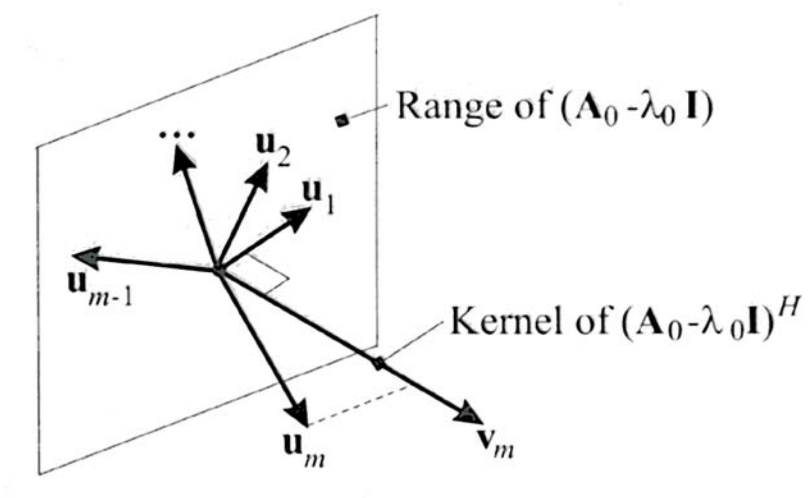


Semi-simple eigenvalues and derogatory matrices can also occur.

- **Chain of right *Generalized Eigenvectors*:**

$$(\mathbf{A}_0 - \lambda_0 \mathbf{I})\mathbf{u}_k = \mathbf{u}_{k-1} \quad k = 2, 3, \ldots, m$$

Orthogonality properties:



$$\mathbf{v}_m^H \mathbf{u}_k = \delta_{km}$$

a) Sensitivity of a simple eigenvalue λ_0:

$$\mathbf{A} = \mathbf{A}_0 + \varepsilon\,\mathbf{A}_1 + \varepsilon^2\,\mathbf{A}_2 + \dots$$

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\,\lambda_2 + \dots$$

$$\mathbf{w} = \mathbf{w}_0 + \varepsilon\,\mathbf{w}_1 + \varepsilon^2\,\mathbf{w}_2 + \dots$$

- **Perturbation equations:**

$$\varepsilon^0 : \; (\mathbf{A}_0 - \lambda_0\,\mathbf{I})\,\mathbf{w}_0 = \mathbf{0}$$

$$\varepsilon^1 : \; (\mathbf{A}_0 - \lambda_0\,\mathbf{I})\,\mathbf{w}_1 = \lambda_1\,\mathbf{w}_0 - \mathbf{A}_1\,\mathbf{w}_0$$

$$\varepsilon^2 : \; (\mathbf{A}_0 - \lambda_0\,\mathbf{I})\,\mathbf{w}_2 = \lambda_2\,\mathbf{w}_0 + \lambda_1\,\mathbf{w}_1 - \mathbf{A}_2\,\mathbf{w}_0 - \mathbf{A}_1\,\mathbf{w}_1$$

- **Generating solution**:

$$\mathbf{w}_0 = \mathbf{u}$$

- ε**-order equation and solvability**:

$$(\mathbf{A}_0 - \lambda_0 \mathbf{I})\,\mathbf{w}_1 = \lambda_1 \mathbf{u} - \mathbf{A}_1 \mathbf{u} \qquad \Rightarrow \qquad \lambda_1 = \mathbf{v}^H \mathbf{A}_1 \mathbf{u}$$

Analogously, at higher orders, $\lambda_2, \lambda_3, \ldots$ ***are found***.

b) Sensitivity of a defective eigenvalue λ_0 of multiplicity *m*

- **The previous perturbation scheme fails:**

$$\underbrace{\mathbf{v}_m^H (\lambda_1 \mathbf{u}}_{=0} - \mathbf{A}_1 \mathbf{u}) = 0 \qquad \Rightarrow \qquad \lambda_1 = ?$$

- ***Fractional* power expansions:**

$$\mathbf{w} = \mathbf{w}_0 + \varepsilon^{1/m}\,\mathbf{w}_1 + \varepsilon^{2/m}\,\mathbf{w}_2 + \dots$$
$$\lambda = \lambda_0 + \varepsilon^{1/m}\lambda_1 + \varepsilon^{2/m}\,\lambda_2 + \dots$$

- **Perturbation equations:**

$$\varepsilon^0 : (\mathbf{A}_0 - \lambda_0 \mathbf{I})\,\mathbf{w}_0 = \mathbf{0}$$
$$\varepsilon^{1/m} : (\mathbf{A}_0 - \lambda_0 \mathbf{I})\,\mathbf{w}_1 = \lambda_1 \mathbf{w}_0$$
$$\varepsilon^{2/m} : (\mathbf{A}_0 - \lambda_0 \mathbf{I})\,\mathbf{w}_2 = \lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\varepsilon : (\mathbf{A}_0 - \lambda_0 \mathbf{I})\,\mathbf{w}_m = \lambda_1 \mathbf{w}_{m-1} + \lambda_2 \mathbf{w}_{m-2} + \ldots - \mathbf{A}_1 \mathbf{w}_0$$
$$\varepsilon^{1+1/m} : (\mathbf{A}_0 - \lambda_0 \mathbf{I})\,\mathbf{w}_{m+1} = \lambda_1 \mathbf{w}_m + \lambda_2 \mathbf{w}_{m-1} + \ldots$$

- **Solutions up-to ε$^{(m\text{-}1)/m}$-order:**

$$\varepsilon^0 : \mathbf{w}_0 = \mathbf{u}_1$$
$$\varepsilon^{1/m} : \mathbf{w}_1 = \lambda_1 \mathbf{u}_2$$
$$\varepsilon^{2/m} : \mathbf{w}_2 = \lambda_1^2 \mathbf{u}_3 + \lambda_2 \mathbf{u}_2$$
$$\cdots\cdots\cdots$$

since the known terms belong to the Range of the operator.

- **Solvability at order** ε:

$$\lambda_1^m = \mathbf{v}_m^H \, \mathbf{A}_1 \, \mathbf{u}_1 \quad \Rightarrow \quad m \text{ roots}$$

- **Solvability at higher orders:**

$$\begin{aligned} \lambda_2 \lambda_1^{m-1} &= f(\lambda_1) \\ \lambda_3 \lambda_1^{m-1} &= f(\lambda_1, \lambda_2) \\ \dots\dots\dots \end{aligned} \qquad \text{for each of the } m \text{ roots}$$

- ***Reconstitution* of the characteristic polynomial:**

$$\Delta\lambda^m + c_1(\boldsymbol{\mu})\,\Delta\lambda^{m-1} + \ldots + c_m(\boldsymbol{\mu}) = 0, \quad \text{where } \Delta\lambda := \lambda - \lambda_0 \equiv \lambda_1 + \lambda_2 + \ldots$$

Eigenvector Sensitivity