Bifurcation Theory

Lecture 1

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ANALYTICAL TOOLS FOR PERTURBATION ANALYSIS

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OUTLINE

- **Introductory concepts**
- **Algebraic Eigenvalue Problems**
- **Initial Value Problems**
- **Advanced Topics**

INTRODUCTORY CONCEPTS

- **A preliminary example: a non linear algebraic equation containing a small parameter.**

- **How to introduce in the equation a perturbation parameter?**

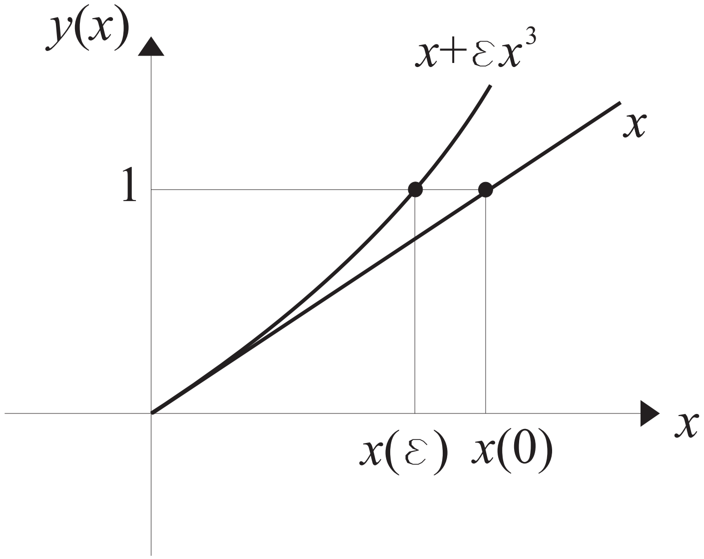
- **Regular and singular perturbation problems: *the compatibility condition*.**

A PRELIMINARY EXAMPLE

- **nonlinear algebraic equation**

$$x + \varepsilon x^3 = 1, \ \ \varepsilon \ll 1, \ \ x \in R$$

$$\lim_{\varepsilon \to 0} x(\varepsilon) = 1$$



- **Series expansion**

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots \qquad x_k = \mathrm{O}(1) \text{ independent of } \varepsilon$$

- **Collecting the same ε-order terms**

$$(x_0 - 1) + \varepsilon\left(x_1 + x_0^3\right) + \varepsilon^2\left(x_2 - 3x_0^2 x_1\right) + \ldots = 0 \qquad \forall \varepsilon$$

- **Perturbation equations**

$$\varepsilon^0: \ x_0 = 1$$

$$\varepsilon^1: \ x_1 = -x_0^3$$

$$\varepsilon^2: \ x_2 = -3x_0^2 x_1$$

A sequence of uncoupled linear equations in drawn. By solving them in chain, one gets:

$$x_0 = 1, \quad x_1 = -1, \quad x_2 = 3, \ \ldots$$

- **Solution**

$$x = 1 - \varepsilon + 3\varepsilon^2 + \ldots$$

Note: only the solution ‘close’ to x=1 is found.

INTRODUCING IN THE EQUATION A PERTURBATION PARAMETER

- **Example: the discrete non linear elastic problem**

$$\mathbf{Kx} + \mathbb{C}\mathbf{x}^3 = \mathbf{p} \qquad \mathbf{x} \in \mathbf{R}^N$$

with **K** the stiffness matrix, $\mathbb{C}$ the four-dimensional non linear coefficient matrix and **p** the load vector. (Here $\mathbb{C}\mathbf{x}^3 := \sum_{j,h,k} x_j x_h x_k \mathbf{e}_i$ and $\mathbf{e}_i$ are unit vectors).

- **Two cases occur:**

$$\|\mathbb{C}\| = \begin{cases} O(\varepsilon) \rightarrow \quad \mathbb{C} = \varepsilon\,\hat{\mathbb{C}}, \quad \|\hat{\mathbb{C}}\| = O(1) \\ O(1) \rightarrow \quad \mathbf{x} = \varepsilon^{1/2}\hat{\mathbf{x}}, \quad \|\hat{\mathbf{x}}\| = O(1); \;\; \mathbf{p} = \varepsilon^{1/2}\hat{\mathbf{p}}, \;\; \|\hat{\mathbf{p}}\| = O(1) \end{cases}$$

✓*Small* nonlinear coefficients case:

$$\mathbf{Kx} + \varepsilon \hat{\mathbb{C}} \mathbf{x}^3 = \mathbf{p}$$

where **x** and **p** are of order-1.

✓*Order*-1 nonlinear coefficients case:

$$\mathbf{K\hat{x}} + \varepsilon \mathbb{C} \mathbf{\hat{x}^3} = \mathbf{\hat{p}}$$

where **x** and **p** must be small, of order $\varepsilon^{1/2}$.

Note: the two equations are formally equal.

• **Perturbation solution** $\left(\|\mathbb{C}\| = \mathrm{O}(1) \text{ case}\right)$

✓Series expansion:

$$\hat{x} = \hat{x}_0 + \varepsilon \hat{x}_1 + \ldots$$

✓Perturbation equations:

$$\varepsilon^0 : \mathbf{K}\hat{\mathbf{x}}_0 = \hat{\mathbf{p}} \qquad \rightarrow \hat{\mathbf{x}}_0 = \mathbf{K}^{-1}\hat{\mathbf{p}}$$

$$\varepsilon^1 : \mathbf{K}\hat{\mathbf{x}}_1 = -\mathbb{C}\hat{\mathbf{x}}_0^3 \qquad \rightarrow \hat{\mathbf{x}}_1 = -\mathbf{K}^{-1}\mathbb{C}\left(\mathbf{K}^{-1}\hat{\mathbf{p}}\right)^3$$

✓Coming back to $\mathbf{x} = \varepsilon^{1/2}\,\hat{\mathbf{x}}$, and reabsorbing ε:

$$\mathbf{x} = \mathbf{K}^{-1}\left(\varepsilon^{1/2}\hat{\mathbf{p}}\right) - \mathbf{K}^{-1}\mathbb{C}\left(\varepsilon^{3/2}\left(\mathbf{K}^{-1}\hat{\mathbf{p}}\right)^3\right)$$

$$= \mathbf{K}^{-1}\mathbf{p} - \mathbf{K}^{-1}\mathbb{C}\left(\mathbf{K}^{-1}\mathbf{p}\right)^3$$

Formally equivalent method: ‘drop the hat and put $\varepsilon = 1$’

SINGULAR PERTURBATION PROBLEMS

- **General perturbation equation**

$$\mathbf{Ax} = \mathbf{b}$$

Does it always admit solution?

- **Discussion**

$$\det[\mathbf{A}] = \begin{cases} \neq 0 \rightarrow \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} & \text{(regular perturbation)} \\ = 0 \rightarrow \begin{cases} \mathbf{x} \text{ is undetermined} \\ \mathbf{x} \text{ does not exist} \end{cases} & \text{(singular perturbation)} \end{cases}$$

Regular perturbation problems are quite trivial. Attention is focused ahead on singular problems.

- **Compatibility (or *solvability*) condition**

✓Direct and *adjoint* problems:

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A}^T\mathbf{y} = \mathbf{c}$$

✓Bilinear identity:

$$\mathbf{y}^T\mathbf{Ax} = \mathbf{x}^T\mathbf{A}^T\mathbf{y} \quad \rightarrow \quad \mathbf{y}^T\mathbf{b} = \mathbf{x}^T\mathbf{c}$$

✓If **c**=**0**:

$$\mathbf{y}^T\mathbf{b} = \mathbf{0} \quad \forall \mathbf{y} \,\middle|\, \mathbf{A}^T\mathbf{y} = \mathbf{0}$$

i.e. **b** is orthogonal to *all* the solutions of the homogeneous adjoint problem.

✓Viceversa, if the previous property holds, then:

$$\mathbf{y}^T(\mathbf{b} - \mathbf{Ax}) = \mathbf{0} \quad \forall \mathbf{y} \quad \Rightarrow \quad \mathbf{Ax} = \mathbf{b}$$

<u>In order the direct problem to admit solution, the known term **b** must be orthogonal to *all* the solutions of the homogeneous adjoint problem.</u>

ALGEBRAIC EIGENVALUE PROBLEMS

- **Non linear eigenvalue problems**
  - ✓Example: buckling of a nonlinear structure.

- **Linear eigenvalue problem**
  - ✓Example: modification of linear structures.

The two classes of problems are *formally similar* from the Perturbation Method point of view.

- **Right and left eigenvalues**

$$(\mathbf{A} - \lambda_k \mathbf{I})\mathbf{u}_k = \mathbf{0}$$

$$(\mathbf{A} - \lambda_k \mathbf{I})^H \mathbf{v}_k = \mathbf{0}$$

- **Bi-orthogonality conditions:**

$$\mathbf{v}_j^H \mathbf{u}_k = 0 \quad \text{if } j \neq k$$

- **Normalization:**

$$\mathbf{v}_k^H \mathbf{u}_k = 1$$

NON LINEAR ALGEBRAIC EIGENVALUE PROBLEMS

- **General problem**

$$\left(\mathbf{A} - \lambda\mathbf{I}\right)\mathbf{x} + \varepsilon\,\mathbb{C}\mathbf{x}^3 = \mathbf{0}$$

✓Series expansions:

$$\mathbf{x} = \mathbf{x}_0 + \varepsilon\mathbf{x}_1 + \ldots$$

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \ldots$$

✓Perturbation equations:

$$\varepsilon^0 : \left(\mathbf{A}\text{-}\lambda_0\mathbf{I}\right)\mathbf{x}_0 = 0 \qquad \rightarrow \left(\lambda_0, \mathbf{x}_0\right) = \left(\lambda_0^{(k)}, a\mathbf{u}_k\right), \left\|\mathbf{u}_k\right\| = 1$$

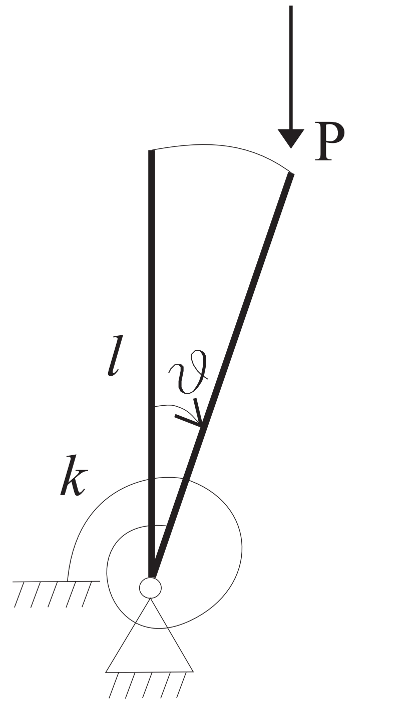
$$\varepsilon\ : \left(\mathbf{A}\text{-}\lambda_0\mathbf{I}\right)\mathbf{x}_1 = \lambda_1\,\mathbf{x}_0 - \mathbb{C}\mathbf{x}_0^3 \qquad \rightarrow \quad \lambda_1 = \frac{\mathbf{v}_k^H\,\mathbb{C}(a\mathbf{u}_k)^3}{a\,\underbrace{\mathbf{v}_k^H\mathbf{u}_k}_{=1}} = a^2\mathbf{v}_k^H\,\mathbb{C}\mathbf{u}_k^3$$

- **Example: a buckling problem**

✓Equilibrium equations, expanded in series:

$$\vartheta - \mu \sin\vartheta = 0, \qquad \mu := Pl/k$$

$$(1-\mu)\vartheta + \mu\vartheta^3/6 + \ldots = 0$$



✓Perturbation equations:

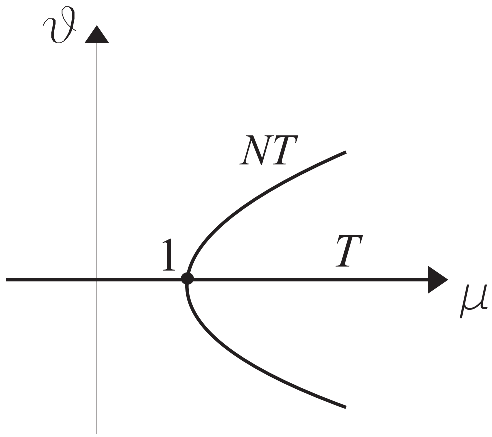
$$\vartheta \to \varepsilon^{1/2}\vartheta, \qquad \vartheta = \vartheta_0 + \varepsilon\vartheta_1 + \ldots, \qquad \mu = \mu_0 + \varepsilon\mu_1 + \ldots$$

$$\varepsilon^0 : (1-\mu_0)\vartheta_0 = 0 \qquad \to (\mu_0, \vartheta_0) = (1, a)$$

$$\varepsilon^1 : (1-\mu_0)\vartheta_1 = \mu_1\vartheta_0 - \mu_0\frac{\vartheta_0^3}{6} \qquad \to \mu_1 = \frac{1}{6}a^2$$

✓First order solution:

$$\begin{cases} \vartheta = (\varepsilon^{1/2}a) \\ \mu = 1 + \dfrac{\varepsilon a^2}{6} \end{cases} \to \mu = 1 + \frac{\vartheta^2}{6}$$



LINEAR ALGEBRAIC EIGENVALUE PROBLEMS

- **General problem**

$$\left(\mathbf{A}_0 + \varepsilon\mathbf{A}_1 - \lambda\right)\mathbf{x} = \mathbf{0} \qquad \varepsilon\mathbf{A}_1 : \text{imperfection/modification}$$

✓Series expansions:

$$\mathbf{x} = \mathbf{x}_0 + \varepsilon\mathbf{x}_1 + \ldots$$

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \ldots$$

✓Perturbation equations:

$$\varepsilon^0 : \left(\mathbf{A}\text{-}\lambda_0\mathbf{I}\right)\mathbf{x}_0 = \mathbf{0} \qquad \rightarrow \left(\lambda_0, \mathbf{x}_0\right) = \left(\lambda_0^{(k)}, a\mathbf{u}_k\right)$$

$$\varepsilon^1 : \left(\mathbf{A}\text{-}\lambda_0\mathbf{I}\right)\mathbf{x}_1 = \lambda_1\,\mathbf{x}_0 - \mathbf{A}_1\mathbf{x}_0 \qquad \rightarrow \; \lambda_1 = -\mathbf{v}_k^H\,\mathbf{A}_1\mathbf{u}_k$$

Imperfection play the same role than nonlinearities.

- **Mechanical example: a lightly damped system**

  ✓Eigenvalue problem:

  $$\ddot{q} + 2\xi\omega_0\dot{q} + \omega_0^2 q = 0 \qquad q \in \mathrm{R}, \quad \xi \ll 1: \text{perturbation parameter}$$

  $$q = x\exp(\lambda t) \;\Rightarrow\; \left(\lambda^2 + 2\xi\omega_0\lambda + \omega_0^2\right)\mathrm{x} = 0$$

  ✓Series expansions:

  $$\mathrm{x} = \mathrm{x}_0 + \xi\mathrm{x}_1 + \ldots, \qquad \lambda = \lambda_0 + \xi\lambda_1 + \ldots$$

  ✓Perturbation equations:

  $$\xi^0 : \left(\lambda_0^2 + \omega_0^2\right)\mathrm{x}_0 = 0 \qquad \rightarrow \left(\lambda_0, \mathrm{x}_0\right) = \left(\pm i\omega_0, a\right)$$

  $$\xi \;: \left(\lambda_0^2 + \omega_0^2\right)\mathrm{x}_1 = -2\lambda_0\left(\lambda_1 + \omega_0\right)\mathrm{x}_0 \qquad \rightarrow \; \lambda_1 = -\omega_0$$

  ✓Solution:

  $$\lambda = \pm i\omega_0 - \xi\omega_0$$

INITIAL VALUE PROBLEMS

- **Straightforward expansions**

  ✓The appearance of secular terms

  ✓The breakdown of the series

- **The Multiple Scale Method (MSM)**

  ✓Removing secular terms

  ✓Describing the slow-flow

STRAIGHTFORWARD EXPANSIONS AND SECULAR TERMS

- **A sample system**

The self-excited one-d.o.f. is considered:

$$\begin{cases} \ddot{x} - \omega_0^2 x + \varepsilon\left(-\mu \dot{x} + b\dot{x}^3 + cx^3\right) = 0 \\ x(0) = a, \quad \dot{x}(0) = 0 \end{cases}$$

- **Straightforward expansion**

✓Series expansion:

$$x = x_0 + \varepsilon x_1 + \ldots,$$

✓Perturbation equations:

$$\varepsilon^0:\begin{cases}\ddot{x}_0+\omega_0^2x_0=0\\ x(0)=a,\ \dot{x}(0)=0\end{cases}\qquad\qquad \varepsilon^1:\begin{cases}\ddot{x}_1+\omega_0^2x_1=\mu\dot{x}_0-b\dot{x}_0^3-cx_0^3\\ x_1(0)=0,\ \dot{x}(0)=0\end{cases}$$

✓ε^0-order solution:

$$x_0=\frac{a}{2}e^{i\omega_0 t}+c.c.$$

✓ε-order equation:

$$\ddot{x}_1+\omega_0^2x_1=\frac{a}{2}\left(i\mu\omega_0-\frac{3}{4}i\omega_0^3ba^2-\frac{3}{4}ca^2\right)e^{i\omega_0 t}+\frac{a^2}{8}\left(i\omega_0^3b-c\right)e^{3i\omega_0 t}+c.c$$

$$=:\underbrace{f_1e^{i\omega_0 t}}_{\text{resonant}}+f_3e^{3i\omega_0 t}+c.c.$$

✓ ε-order solution

$$x_1 = \underbrace{\frac{1}{2} a_1 e^{i(\omega_0 t + \vartheta_1)}}_{\text{complementary solution}} - i \underbrace{\frac{1}{2\omega_0} f_1 \, t e^{i\omega t}}_{\text{secular terms}} - \underbrace{\frac{1}{8\omega_0^2} f_3 \, e^{3i\omega t}}_{\text{non-secular terms}} + c.c.$$

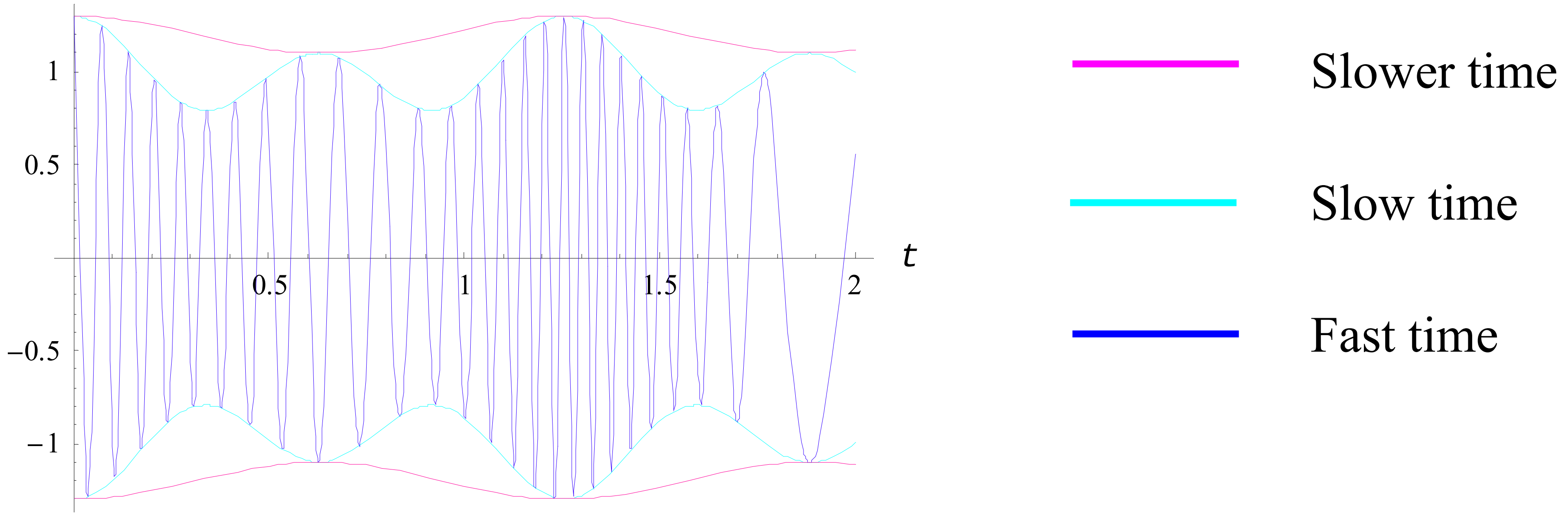
✓ Discussion:

- When $t \geq O(\varepsilon^{-1})$, then $\varepsilon x_1 \geq O(x_0)$, i.e. the latter it is *not* a small correction of x_0. The series is *not uniformly valid* in the interval $[0, \infty]$.
- The drawback is due to the unlimited domain. In *limited* spatial problems, secular terms as $s \exp(i\alpha s)$ do not entail any inconvenience.

THE MULTIPLE SCALE METHOD

- **The basic ideas**

✓Small nonlinearities slowly modulate amplitude and phases.
✓The solution depends on several independent time scales.
✓Removing secular terms provides modulation laws.



- **The self-excited oscillator**

✓Equation of motion:

$$\ddot{x} + \omega_0^2 x + \varepsilon\left(-\mu\dot{x} + b\dot{x}^3 + cx^3\right) = 0$$

✓Independent time scales:

$$x = x\left(\varepsilon; t_0, t_1, t_2, \ldots\right)$$

$$t_0 := t,\ t_1 := \varepsilon t,\ t_2 := \varepsilon^2 t,\ \ldots$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2} + \ldots = \sum_{k=0}^{\infty}\varepsilon^k d_k \quad , \qquad d_k := \frac{\partial}{\partial t_k}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{\partial^2}{\partial t_0^2} + 2\varepsilon\frac{\partial}{\partial t_0}\frac{\partial}{\partial t_1} + \varepsilon^2\left(\frac{\partial^2}{\partial t_1^2} + 2\frac{\partial}{\partial t_0}\frac{\partial}{\partial t_2}\right) + \ldots = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varepsilon^{k+j} d_k d_j$$

✓Series expansion:

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 \dots$$

✓Perturbation equations:

$$\varepsilon^0 : d_0^2 x_0 + \omega_0^2 x_0 = 0$$

$$\varepsilon^1 : d_0^2 x_1 + \omega_0^2 x_1 = -2 d_0 d_1 x_0 + \mu d_0 x_0 - b\left(d_0 x_0\right)^3 - c x_0^3$$

....

- **Solution**

✓ε^0-order solution:

$$x_0 = A\left(t_1, t_2, \dots\right) e^{i\omega_0 t_0} + c.c. \qquad A = \frac{1}{2} a e^{i\theta} \in C$$

✓ ε-order equation:

$$d_0^2 x_1 + \omega_0^2 x_1 = \left[-2i\omega_0 d_1 A + i\omega_0 \mu A - 3\left(i\omega_0^3 b + c\right) A^2 \overline{A} \right] e^{i\omega_0 t_0}$$

$$+\left(i\omega^3 b - c\right) A^3 e^{3i\omega_0 t_0} + c.c.$$

✓Removing secular terms:

Zeroing the coefficient of $e^{i\omega_0 t_0}$ one obtains:

$$d_1 A = \frac{1}{2}\mu A + \frac{3}{2}\left(-\omega_0^2 b + ic\right) A^2 \overline{A}$$

which governs the A-modulation of the t_1-scale (*Amplitude Modulation Equation*).

✓Real AME's:

Since:

$$d_1 A = d_1 \left(\frac{1}{2} a e^{i\theta}\right) = \frac{1}{2}\left(a' + i a \vartheta'\right) e^{i\theta} \qquad (...)' = \frac{\partial}{\partial t_1}$$

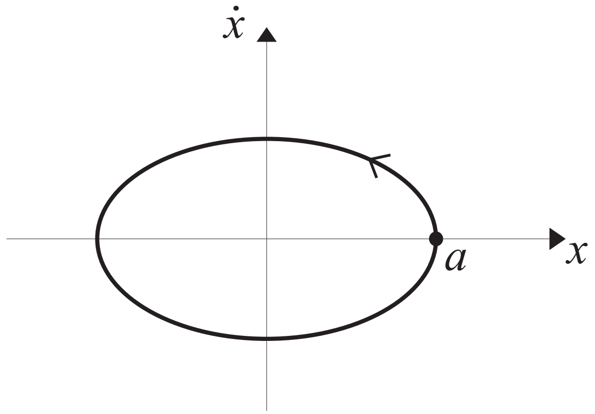
by separating the real and imaginary parts of AME, it follows:

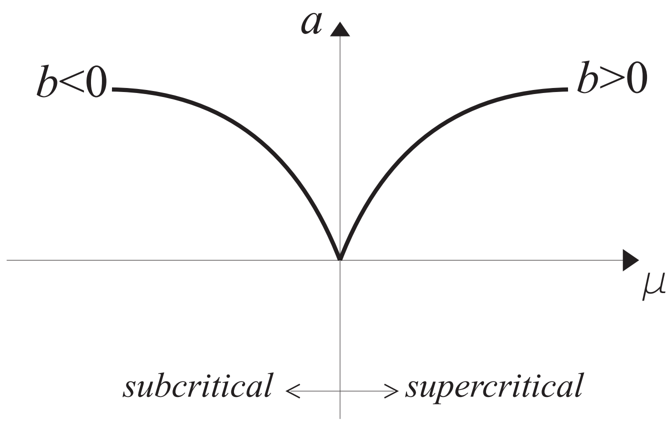
$$\begin{cases} a' = \frac{1}{2} a \left(\mu - \frac{3}{4}\omega_0^2 b a^2\right) \\ a\vartheta' = \frac{3}{8} c a^3 \end{cases}$$

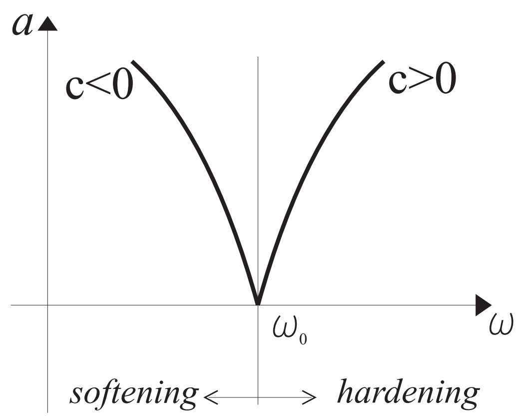
- **Steady first-order solution**

$$a = a_s = \text{const} \rightarrow \mu = \frac{3}{4}\omega_0^2 b a^2, \quad \vartheta = \frac{3}{8} c a^2 t =: \nu t$$

$$x_0 = \frac{1}{2} a_s e^{i\left(\omega_0 t + \vartheta\right)} + c.c. = a_s \cos\left[\left(\omega_0 + \nu\right) t\right]$$







A one-parameter family of limit cycles is found**.**

Note that, due to the fact the fast dynamics has been filtered:

- *periodic orbits* for the original equations become *equilibrium point* for the AME
- similarly: *quasi-periodic orbits* become *periodic orbits*

- **Stability of steady solutions**

The MSM also permits to study stability. By letting:

$$a = a_s + \delta a \qquad\qquad \delta a \ll a_s$$

and linearizing in δa, one gets the *variational equation:*

$$\delta a' = \frac{1}{2}\left(\mu - \frac{9}{4}\omega_0^2 b a_s^2 \right)\delta a$$

By expressing a_s as a function of μ:

$$\delta a' = -\mu\delta a \quad \rightarrow \quad \delta a = e^{-\mu t} \begin{cases} a_s \text{ stable, if } \mu > 0 \text{ (supercritical)} \\ a_s \text{ unstable, if } \mu < 0 \text{ (subcritical)} \end{cases}$$

- **Higher-order solutions**

By going on to higher-orders, solvability conditions of the following type are found:

$$\varepsilon^1: \; d_1 A = f_1 \left(A\right)$$

$$\varepsilon^2: \; d_2 A = f_2 \left(A\right)$$

....

They can be recombined in a unique equation (*reconstitution method*):

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \varepsilon\, d_1 A + \varepsilon^2 d_2 A + \ldots = \varepsilon\, \mathrm{f}_1\left(A\right) + \varepsilon^2 \mathrm{f}_2\left(A\right) + \ldots =: \mathrm{F}\left(A\right)$$

governing the modulation on the true time-scale t.

- **Internal resonances (I)**

When dealing with multi-d.o.f. systems, internal resonances can occur.

✓For example, let us consider the system:

$$\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} + \varepsilon\,\mathbb{C}\mathbf{x}^3 = 0 \qquad\qquad \mathbf{x} \in R^{\mathbf{N}}$$

where **A** admits the eigenvalues:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{0} \qquad \rightarrow \lambda = i\omega_k,\ i\omega_j,\ \alpha_l + i\omega_l\ , \quad \text{with} \quad \omega_j \simeq 3\omega_k$$

✓One *cannot take*, (e.g. for special initial conditions) a *monomodal generating solution*:

$$\mathbf{x}_0 = A_k(t_1, \ldots)\mathbf{u}_k \mathrm{e}^{i\omega_k t_0} + c.c.$$

since the forcing frequency $3i\omega_k t_0$ generated by x_0^3 would be in resonance with the proper frequency $\omega_j \simeq 3\omega_k$.

- **Internal resonances (II)**

In these cases one has to take a *multimodal generating solution*:

$$\mathbf{x}_0 = A_k\left(t_1,\ldots\right)\mathbf{u}_k \mathrm{e}^{i\omega_k t_0} + A_j\left(t_1,\ldots\right)\mathbf{u}_j \mathrm{e}^{i\omega_j t_0} + c.c.$$

leading to:

$$\begin{aligned} d_1 A_k &= \mathrm{f}_k(A_k, A_j) \\ d_1 A_j &= \mathrm{f}_j(A_k, A_j) \end{aligned}$$

ADVANCED TOPICS

- **High sensitive-systems**

Q: Do order-ε perturbations always produce variations of the same order-ε? There exist systems high-sensitive to perturbations?

A: *Defective systems* (i.e. possessing an incomplete set of eigenvectors) are high-sensitive to perturbations.

DEFECTIVE SYSTEMS

- **Example: matrix eigenvalue sensitivity**

<table>
<tr><th></th><th colspan="2">Multiple Eigenvalues</th></tr>
<tr><th>Distinct Eigenvalues</th><th>Hermitian Matrix</th><th>Jordan-Block</th></tr>
<tr>
<td>$\mathbf{A}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
$\hat{\mathbf{A}} = \begin{bmatrix} 1+\varepsilon & 0 \\ 0 & 2 \end{bmatrix}$</td>
<td>$\mathbf{A}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
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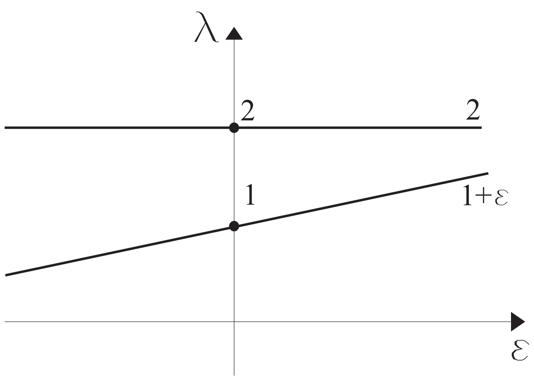
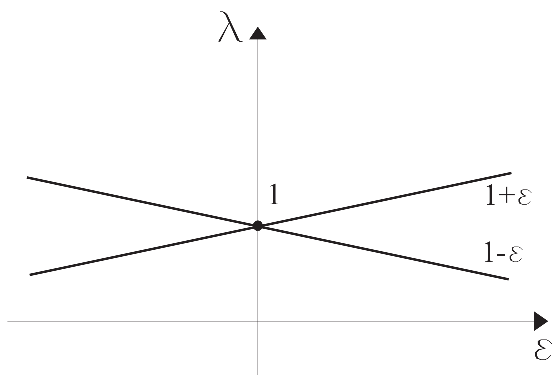
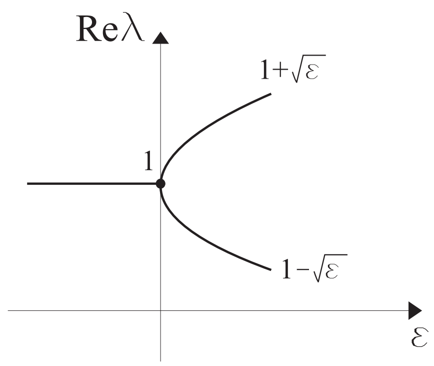
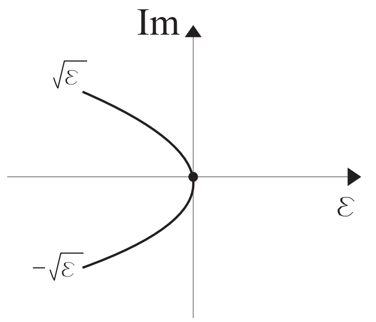
<td></td>
<td></td>
<td></td>
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✓Perturbations of defective matrices (containing Jordan blocks) require using series of *fractional powers* of the perturbation parameter:

$$\left(\mathbf{A}_0 + \varepsilon \mathbf{A}_1 - \lambda\right)\mathbf{u} = \mathbf{0}$$

$$\begin{cases} \lambda = \lambda_0 + \varepsilon^{1/m}\lambda_1 + \varepsilon^{2/m}\lambda_2 + \ldots \\ \mathbf{u} = \mathbf{u}_0 + \varepsilon^{1/m}\lambda_1 + \varepsilon^{2/m}\lambda_2 + \ldots \end{cases}$$

✓The Multiple Scale Method, when applied to defective dynamical systems, in addition requires using fractional ε - *power* time-scales:

$$\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} + \varepsilon\,\mathbb{C}\mathbf{x}^3 = \mathbf{0}$$

$$\mathbf{x} = \mathbf{x}_0 + \varepsilon^{1/m}\mathbf{x}_1 + \varepsilon^{2/m}\mathbf{x}_2 + \ldots$$

$$t_0 = t, \quad t_1 = \varepsilon^{1/m} t, \quad t_2 = \varepsilon^{2/m} t, \ldots . \;\rightarrow \frac{\mathrm{d}}{\mathrm{d}t} = d_0 + \varepsilon^{1/m} d_1 + \varepsilon^{2/m} d_2 + \ldots$$