Bifurcation Theory

Lecture 3

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CENTER MANIFOLD AND NORMAL FORM THEORIES

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THE CENTER MANIFOLD METHOD

Existence of an invariant manifold

- Linear systems

The state-space $\mathcal{X} \subseteq \mathbb{R}^N$ of the linear system $\dot{\mathbf{x}}(t) = \mathbf{J}\mathbf{x}(t)$ is direct sum of three invariant sub-spaces, i.e. $\mathcal{X} = \mathcal{X}_c \oplus \mathcal{X}_s \oplus \mathcal{X}_u$, where:

- $\mathcal{X}_c$ is the *center subspace*, of dimension N_c, spanned by the (generalized) eigenvectors associated with non-hyperbolic eigenvalues of **J**;
- $\mathcal{X}_s$ is the *stable subspace*, of dimension N_s, spanned by the (generalized) eigenvectors associated with hyperbolic eigenvalues of **J** having negative real part;
- $\mathcal{X}_u$ is the *unstable subspace*, of dimension N_u, spanned by the (generalized) eigenvectors associated with non-hyperbolic eigenvalues of **J** having positive real part.

- Nonlinear systems

We consider the nonlinear system (in local form):

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \boldsymbol{\mu})$$

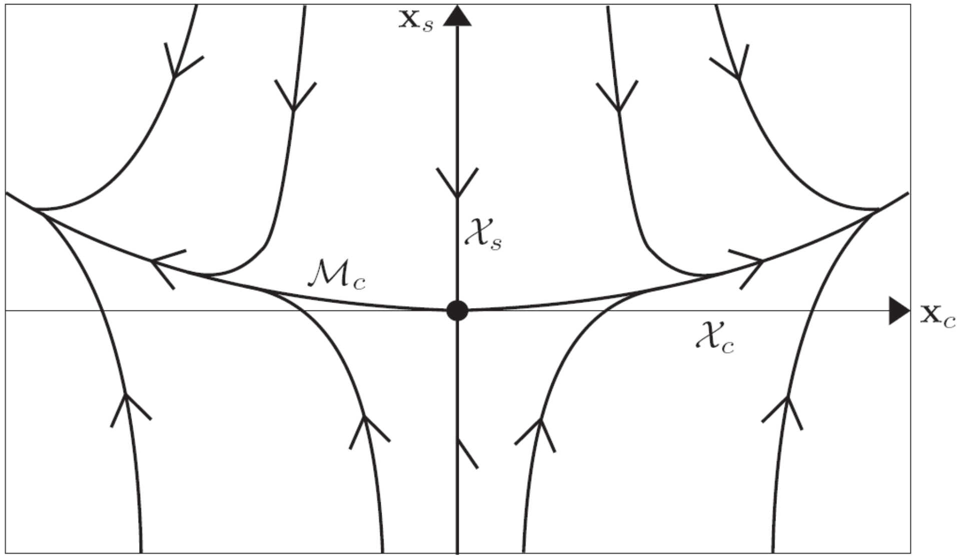
admitting the critical equilibrium $(\mathbf{x} = \mathbf{0}, \boldsymbol{\mu}_c = \mathbf{0})$. We assume that $\mathbf{J} := \mathbf{F}_{\mathbf{x}}(\mathbf{0}, \mathbf{0})$ posses N_c >0 critical eigenvalues, N_s >0 stable eigenvalues and N_u =0 unstable eigenvalues.

The *Center Manifold Theorem* states that the asymptotic dynamics of the system around the equilibrium point $\mathbf{x} = \mathbf{0}$, at the critical value of the parameters $\boldsymbol{\mu} = \boldsymbol{\mu}_c$, takes place on a (critical) manifold $\mathcal{M}_c \in \mathcal{X}$, which has the following properties:

- $\mathcal{M}_c$ has dimension N_c;
- $\mathcal{M}_c$ is *tangent* to the critical subspace $\mathcal{X}_c$ at $\mathbf{x} = \mathbf{0}$;
- $\mathcal{M}_c$ is *attractive*, i.e. all the orbits tend to it when $t \to \infty$.

The center manifold is therefore an N_c-dimensional surface in the $N=N_c+N_s$-dimensional state-space.

- Example:



- To analyze the asymptotic dynamics, it needs:

  (a) to find the center manifold $\mathcal{M}_c$;

  (b) to obtain the *reduced* N_c –dimensional equations governing the motion on $\mathcal{M}_c$ (*bifurcation equations*).

- **Dependence of the CM on parameters**

Since we are interested not only in the dynamics at $\boldsymbol{\mu} = \boldsymbol{\mu}_c$, but also at the dynamics at $\boldsymbol{\mu}$ close to $\boldsymbol{\mu}_c$, we can use the 'trick' to consider $\boldsymbol{\mu}$ as additional 'critical' variables, by considering the *extended dynamical system*:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \boldsymbol{\mu}(t)) \\ \dot{\boldsymbol{\mu}}(t) = \mathbf{0} \end{cases}$$

Therefore, the critical subspace becomes $\mathcal{X}_c^+ := \mathcal{X}_c \oplus \mathcal{P}$ with $\mathcal{P} := \{\boldsymbol{\mu}\}$ the parameter space. Hence:

- $\mathcal{M}_c^+$ has dimension N_c+M;
- $\mathcal{M}_c^+$ is *tangent* to the critical subspace $\mathcal{X}_c^+$;
- $\mathcal{M}_c^+$ is *attractive*, i.e. all the orbits tend to it when $t \to \infty$.

- **Reduction process**
  - Equations of motion, expanded:

    By expanding $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \boldsymbol{\mu})$ (and ignoring the dummy equations $\dot{\boldsymbol{\mu}} = \mathbf{0}$) for small $\mathbf{x}(t)$ and $\boldsymbol{\mu}$ close to $\boldsymbol{\mu}_c$, we have:

$$\dot{\mathbf{x}}(t) = \mathbf{J}\mathbf{x}(t) + \mathbf{f}(\mathbf{x}(t), \boldsymbol{\mu}), \quad |\mathbf{f}| = \mathrm{O}(|\mathbf{x}(t)|^2, |\boldsymbol{\mu}|^2)$$

  - Linear transformation of the variables:

    After letting $\mathbf{x}(t) := (\mathbf{x}_c(t), \mathbf{x}_s(t))$, with $\mathbf{x}_c(t) \in \mathcal{X}_c$ *critical variables* and $\mathbf{x}_s(t) \in \mathcal{X}_s$ *stable variables*, a proper linear transformation uncouples the linear part of the equations (t omitted):

$$\begin{pmatrix} \dot{\mathbf{x}}_c \\ \dot{\mathbf{x}}_s \end{pmatrix} = \begin{pmatrix} \mathbf{J}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_s \end{pmatrix} \begin{pmatrix} \mathbf{x}_c \\ \mathbf{x}_s \end{pmatrix} + \begin{pmatrix} \mathbf{f}_c(\mathbf{x}_c, \mathbf{x}_s, \boldsymbol{\mu}) \\ \mathbf{f}_s(\mathbf{x}_c, \mathbf{x}_s, \boldsymbol{\mu}) \end{pmatrix}$$

- Center manifold: *active and passive* coordinates

  Cartesian equations for the CM:

$$\mathbf{x}_s = \mathbf{h}(\mathbf{x}_c, \boldsymbol{\mu})$$

  Tangency requirements:

$$\mathbf{h}(\mathbf{0},\mathbf{0}) = \mathbf{0}, \quad \mathbf{h}_{\mathbf{x}}(\mathbf{0},\mathbf{0}) = \mathbf{0}, \quad \mathbf{h}_{\boldsymbol{\mu}}(\mathbf{0},\mathbf{0}) = \mathbf{0}\,.$$

  The equation expresses the stable variables $\mathbf{x}_s$ as (unknown) functions of the critical variables $\mathbf{x}_c$; therefore $\mathbf{x}_c$ are also said *active coordinates*, and $\mathbf{x}_s$ *passive coordinates*.

- Time derivative of $\mathbf{x}_s$:

  The passive character of $\mathbf{x}_s$ also holds for time-derivatives. By using the chain rule and the upper partition of the equations of motion:

$$\begin{aligned}\dot{\mathbf{x}}_s &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{h}(\mathbf{x}_c, \boldsymbol{\mu}) = \mathbf{h}_{\mathbf{x}}(\mathbf{x}_c, \boldsymbol{\mu})\dot{\mathbf{x}}_c \\ &= \mathbf{h}_{\mathbf{x}}(\mathbf{x}_c, \boldsymbol{\mu})(\mathbf{J}_c\mathbf{x}_c + \mathbf{f}_c(\mathbf{x}_c, \mathbf{h}(\mathbf{x}_c), \boldsymbol{\mu}))\end{aligned}$$

- Equation for the center manifold:

The lower-partition of the equation of motion supplies the equation for determining $\mathcal{M}_c$:

$$\mathbf{h}_{\mathbf{x}}(\mathbf{x}_c,\boldsymbol{\mu})(\mathbf{J}_c\mathbf{x}_c+\mathbf{f}_c(\mathbf{x}_c,\mathbf{h}(\mathbf{x}_c),\boldsymbol{\mu}))=\mathbf{J}_s\mathbf{h}(\mathbf{x}_c)+\mathbf{f}_s(\mathbf{x}_c,\mathbf{h}(\mathbf{x}_c),\boldsymbol{\mu})$$

This equation can be solved, e.g., by using power series expansions, (starting from degree-2 polynomial, due to the required tangency):

$$\mathbf{h}(\mathbf{x}_c,\boldsymbol{\mu})=\boldsymbol{\alpha}_2\mathbf{z}^2+\boldsymbol{\alpha}_3\mathbf{z}^3+\cdots, \qquad \mathbf{z}:=\{\mathbf{x}_c,\boldsymbol{\mu}\}$$

- Bifurcation equations:

The upper-partition governs the dynamics on $\mathcal{M}_c$:

$$\dot{\mathbf{x}}_c=\mathbf{J}_c\mathbf{x}_c+\mathbf{f}_c(\mathbf{x}_c,\mathbf{h}(\mathbf{x}_c),\boldsymbol{\mu})$$

- **Example 1: a static bifurcation**

  - Nonlinear, 2-dimensional system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} xy + cx^3 \\ bx^2 \end{pmatrix}$$

  - Critical point, critical and stable coordinates:

$$\mu_c = 0, \quad \mathbf{x}_c = \{x\}, \quad \mathbf{x}_s = \{y\}$$

  - Equation for the Center Manifold:

$$y = h(x,\mu) \quad \Rightarrow \quad h_x(x,\mu)[\mu x + xh(x,\mu) + cx^3] = -h(x,\mu) + bx^2$$

  - Power series expansion:

$$y = h(x,\mu) = \{\alpha_{20}x^2 + \alpha_{11}x\mu + \alpha_{02}\mu^2\} + \{\alpha_{30}x^3 + \cdots\} + \cdots$$

- Zeroing separately the different powers:

$$\begin{aligned} x^2 &: -\alpha_{20} + b = 0 \\ x\mu &: -\alpha_{11} = 0 \\ \mu^2 &: -\alpha_{02} = 0 \\ &\ldots\ldots\ldots \end{aligned}$$

- By solving for α's coefficients, at the lowest order, we have:

$$y = bx^2 + \mathrm{O}(|(x, \mu)|^3)$$

- Bifurcation equation:

$$\dot{x} = \mu x + (b + c)x^3$$

If $b+c<0$, a supercritical (stable) pitchfork occurs; if $b+c>0$, a sub-critical (unstable) pitchfork occurs; if $b+c=0$, higher-order terms must be evaluated.

- **Example 2: a dynamical bifurcation**

  - 3-dimensional nonlinear system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \mu & -1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} xz \\ yz \\ x^2 + kxy + y^2 \end{pmatrix}$$

  - Equation for CM:

$$z = \alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 xy + \alpha_4 \mu x + \alpha_5 \mu y + \mathrm{O}(3)$$

  - z-equation (passive coordinate), transformed:

$$\begin{aligned} &2(\alpha_2 - \alpha_1)xy + \alpha_3(x^2 - y^2) + \alpha_5 \mu x - \alpha_4 \mu y \\ &= -(\alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 xy + \alpha_4 \mu x + \alpha_5 \mu y) + x^2 + kxy + y^2 + \mathrm{O}(3) \end{aligned}$$

- Zeroing separately the different powers in the z-equation:

$$\begin{cases} x^2: & \alpha_3+\alpha_1-1=0 \\ y^2: & -\alpha_3+\alpha_2-1=0 \\ xy: & 2(\alpha_2-\alpha_1)+\alpha_3-k=0 \\ \mu x: & \alpha_4+\alpha_5=0 \\ \mu y: & \alpha_5-\alpha_4=0 \end{cases} \Rightarrow \begin{cases} \alpha_1=1-k/5 \\ \alpha_2=1+k/5 \\ \alpha_3=k/5 \\ \alpha_4=0 \\ \alpha_5=0 \end{cases}$$

- Bifurcation equations:

$$\begin{cases} \dot{x}=\mu x-y+x[(1-k/5)x^2+k/5xy+(1+k/5)y^2] \\ \dot{y}=\mu y-x+y[(1-k/5)x^2+k/5xy+(1+k/5)y^2] \end{cases}$$

THE NORMAL FORM THEORY

- **Scope:**
  To use a smooth *nonlinear coordinate transformation*, in order to put the bifurcation equation in the simplest form.

- **Algorithm:**

Equations of motion: $\dot{\mathbf{x}} = \mathbf{J}\mathbf{x} + \mathbf{f}(\mathbf{x})$

Transformed Equations: $\dot{\mathbf{y}} = \mathbf{J}\mathbf{y} + \mathbf{g}(\mathbf{y})$

Near-identity transformation: $\mathbf{x} = \mathbf{y} + \mathbf{h}(\mathbf{y})$

where $\mathbf{x}$ (possibly) includes the dummy variables $\boldsymbol{\mu}$.

- Transformed equation in the **h**(**y**) unknown:

  By differentiating the nearly-identity transformation:

$$\dot{\mathbf{x}} = [\mathbf{I} + \mathbf{h}_{\mathbf{y}}(\mathbf{y})]\dot{\mathbf{y}}$$

  the equation of motion is transformed into:

$$[\mathbf{I} + \mathbf{h}_{\mathbf{y}}(\mathbf{y})][\mathbf{J}\mathbf{y} + \mathbf{g}(\mathbf{y})] = \mathbf{J}[\mathbf{y} + \mathbf{h}(\mathbf{y})] + \mathbf{f}(\mathbf{y} + \mathbf{h}(\mathbf{y}))$$

  or:

$$\mathbf{h}_{\mathbf{y}}(\mathbf{y})\mathbf{J}\mathbf{y} - \mathbf{J}\mathbf{h}(\mathbf{y}) = \mathbf{f}(\mathbf{y} + \mathbf{h}(\mathbf{y})) - [\mathbf{I} + \mathbf{h}_{\mathbf{y}}(\mathbf{y})]\mathbf{g}(\mathbf{y})$$

  which is a differential equation for **h**(**y**) for any given **g**(**y**). A series solution is often necessary.

- By letting:

$$\begin{pmatrix} \mathbf{f}(\mathbf{y}) \\ \mathbf{g}(\mathbf{y}) \\ \mathbf{h}(\mathbf{y}) \end{pmatrix} = \begin{pmatrix} \mathbf{f}_2(\mathbf{y}) \\ \mathbf{g}_2(\mathbf{y}) \\ \mathbf{h}_2(\mathbf{y}) \end{pmatrix} + \begin{pmatrix} \mathbf{f}_3(\mathbf{y}) \\ \mathbf{g}_3(\mathbf{y}) \\ \mathbf{h}_3(\mathbf{y}) \end{pmatrix} + \cdots$$

with the homogeneous polynomial of k-degree:

$$\mathbf{f}_k(\mathbf{y}) = \sum_{m=1}^{M_k} \alpha_{km} \mathbf{p}_{km}(\mathbf{y}), \quad \mathbf{g}_k(\mathbf{y}) = \sum_{m=1}^{M_k} \beta_{km} \mathbf{p}_{km}(\mathbf{y}), \quad \mathbf{h}_k(\mathbf{y}) = \sum_{m=1}^{M_k} \gamma_{km} \mathbf{p}_{km}(\mathbf{y})$$

- Zeroing the independent monomials:

$$\mathbf{L}_k \boldsymbol{\gamma}_k = \boldsymbol{\alpha}_k - \boldsymbol{\beta}_k$$

where (if **J** is diagonal):

$$\mathbf{L}_k = \mathrm{diag}[\Lambda_i], \qquad \Lambda_i := \sum_{j=1}^{N} (m_j \lambda_j - \lambda_i), \quad \sum_{j=1}^{N} m_j = k$$

- Resonance:

Since $\lambda_j = 0, 0, \cdots, \pm i\omega_1, \pm i\omega_2, \cdots,$ $\Lambda_i = 0$ for some i (i.e. $\mathbf{L}_k$ is singular); hence, $\boldsymbol{\beta}_k \neq \mathbf{0}$ must be taken for compatibility, and *resonant terms survive in the normal form!!!*

- **Example 1: System independent of parameters**

  - System *at* a Hopf bifurcation:

$$\begin{pmatrix} \dot{x} \\ \dot{\overline{x}} \end{pmatrix} = \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix} \begin{pmatrix} x \\ \overline{x} \end{pmatrix} + \begin{pmatrix} \alpha_{31}x^3 + \alpha_{32}x^2\overline{x} + \alpha_{33}x\overline{x}^2 + \alpha_{34}\overline{x}^3 \\ \overline{\alpha}_{31}\overline{x}^3 + \overline{\alpha}_{32}\overline{x}^2 x + \overline{\alpha}_{33}\overline{x}x^2 + \overline{\alpha}_{34}x^3 \end{pmatrix}, \quad x, \alpha \in \mathbb{C}$$

  - Normal form:

$$\dot{y} = i\omega y + \beta_{31}y^3 + \beta_{32}y^2\overline{y} + \beta_{33}y\overline{y}^2 + \beta_{34}\overline{y}^3$$

  - Near-identity transformation:

$$x = y + \gamma_{31}y^3 + \gamma_{32}y^2\overline{y} + \gamma_{33}y\overline{y}^2 + \gamma_{34}\overline{y}^3$$

- Equation for the coefficients:

$$\begin{bmatrix} 2i\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2i\omega & 0 \\ 0 & 0 & 0 & -4i\omega \end{bmatrix} \begin{pmatrix} \gamma_{31} \\ \gamma_{32} \\ \gamma_{33} \\ \gamma_{34} \end{pmatrix} = \begin{pmatrix} \alpha_{31} \\ \alpha_{32} \\ \alpha_{33} \\ \alpha_{34} \end{pmatrix} - \begin{pmatrix} \beta_{31} \\ \beta_{32} \\ \beta_{33} \\ \beta_{34} \end{pmatrix}$$

- Solution:

$$\beta_{31} = \beta_{33} = \beta_{34} = 0, \quad \beta_{32} = \alpha_{32}$$

- Normal form:

$$\dot{y} = i\omega y + \alpha_{32} y^2 \overline{y}$$

The term $y^2\overline{y}$ is *resonant*, since $2\lambda_1 + \lambda_2 = \lambda_1$, i.e. $2(i\omega) + (-i\omega) = i\omega$

o Amplitude equation:

Changing the variables according to:

$$y(t) = A(t)\,\mathrm{e}^{i\omega t}, \quad A(t) \in \mathbb{C}$$

the normal form is transformed into:

$$[\dot{A}(t) + i\omega A(t)]\mathrm{e}^{i\omega t} = i\omega A(t)\,\mathrm{e}^{i\omega t} + \alpha_{32} A^2(t)\bar{A}(t)\,\mathrm{e}^{(2-1)i\omega t}$$

or:

$$\dot{A}(t) = \alpha_{32} A^2(t)\bar{A}(t)$$

This is called *Amplitude Modulation Equation* (AME). Since $\dot{A} = \mathrm{O}(A^3)$, the AME describes a slow modulation. Therefore, the change of variable filters the fast dynamics.

- **Example 2: Non-diagonalizable system**
  - o Double-zero bifurcation equations:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \alpha_{21} x_1^2 + \alpha_{22} x_1 x_2 + \alpha_{23} x_2^2 \\ \alpha_{24} x_1^2 + \alpha_{25} x_1 x_2 + \alpha_{26} x_2^2 \end{pmatrix}$$

  - o Normal Form:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \beta_{21} y_1^2 + \beta_{22} y_1 y_2 + \beta_{23} y_2^2 \\ \beta_{24} y_1^2 + \beta_{25} y_1 y_2 + \beta_{26} y_2^2 \end{pmatrix}$$

  - o Near-Identity transformation:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \gamma_{21} y_1^2 + \gamma_{22} y_1 y_2 + \gamma_{23} y_2^2 \\ \gamma_{24} y_1^2 + \gamma_{25} y_1 y_2 + \gamma_{26} y_2^2 \end{pmatrix} + \cdots$$

- By zeroing the coefficients of the three monomials in the two transformed equations:

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{pmatrix} \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \\ \hline \gamma_{24} \\ \gamma_{25} \\ \gamma_{26} \end{pmatrix} = \begin{pmatrix} \alpha_{21} - \beta_{21} \\ \alpha_{22} - \beta_{22} \\ \alpha_{23} - \beta_{23} \\ \hline \alpha_{24} - \beta_{24} \\ \alpha_{25} - \beta_{25} \\ \alpha_{26} - \beta_{26} \end{pmatrix}$$

$$\text{Since}\ \mathrm{Rank}[\mathbf{L}_2] = 4:$$

$$\mathcal{K}(\mathbf{L}_2) = \mathrm{span}\{\mathbf{u}_{21}, \mathbf{u}_{22}\}, \quad \mathbf{u}_{21} := (0,1,0,0,0,1)^T, \mathbf{u}_{22} = (0,0,1,0,0,0)^T$$

$$\mathcal{K}(\mathbf{L}_2^T) = \mathrm{span}\{\mathbf{v}_{21}, \mathbf{v}_{22}\}, \quad \mathbf{v}_{21} = (2,0,0,0,1,0)^T, \mathbf{v}_{22} = (0,0,0,1,0,0)^T$$

- Compatibility condition:

$$2(\alpha_{21} - \beta_{21}) + (\alpha_{25} - \beta_{25}) = 0, \qquad \alpha_{24} - \beta_{24} = 0$$

It is possible to take $\beta_{22} = \beta_{23} = \beta_{26} = 0$ and $\beta_{21} = 0$ *or* $\beta_{25} = 0$.

- By taking $\beta_{25} = 0, \beta_{21} = \alpha_{21} + \alpha_{25}/2$ the NF reads (*Takens normal form*):

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} (\alpha_{21} + \frac{1}{2}\alpha_{25}) y_1^2 \\ \alpha_{24} y_1^2 \end{pmatrix}$$

- By taking $\beta_{21} = 0, \beta_{25} = \alpha_{25} + 2\alpha_{21}$ the NF reads (*Bogdanov normal form)*:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_{24} y_1^2 + (\alpha_{25} + 2\alpha_{21}) y_1 y_2 \end{pmatrix}$$

<u>The Normal Form is not unique</u>!!