Applied Partial Differential Equations (MathMods)
Exercise sheet 7

Wave equation

- Do the following exercises from Salsa's book [1]: **5.2, 5.3, 5.5, 5.10, 5.16, 5.17, 5.18.**

In addition, do the following exercises:

1. (a) Prove that the solution to the second order equation $u_{xy} = 0$, is given by $u(x, y) = F(x) + G(y)$, with F and G arbitrary functions.
   (b) Use the change of variables $\xi = x+ct$ and $\eta = x-ct$ to show that the equation $u_{tt} - c^2 u_{xx} = 0$ gets transformed into its canonical form $u_{\eta\xi} = 0$. Use (a) to obtain D'Alembert's formula again.
2. Let $f, g \in C_0^1(\mathbb{R}; \mathbb{R})$, that is, of class C^1 and with compact support: $f = g \equiv 0$ outside a bounded interval $|x| \leq R$, for some $R > 0$. Show that the solution $u \in C^2(\mathbb{R} \times (0, +\infty); \mathbb{R})$ to

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \qquad x \in \mathbb{R},\, t \geq 0, \\ u(x, 0) &= f(x), \\ u_t(x, 0) &= g(x), \end{aligned}$$

   is of compact support in the variable x (with a different R, of course), for each $t > 0$ fixed. (*Hint:* Use D'Alembert's formula.) Since the solution has the form $u(x, t) = F(x + ct) + G(x - ct)$, show that the functions $F, G : \mathbb{R} \to \mathbb{R}$ hace compact support only if

$$\int_{-\infty}^{+\infty} g(y)\, dy = 0.$$

3. Define the linear wave operator in one dimension as

$$Lu := u_{tt} - c^2 u_{xx}.$$

   (a) Prove that $L(u_t v_t + c^2 u_x v_x) = 0$ for any pair of solutions u, v of the homogeneous wave equation ($Lu = Lv = 0$).
   (b) Suppose that u is a solution to $Lu = 0$ in $(x, t) \in \mathbb{R} \times (0, +\infty)$, subject to the initial conditions

$$\begin{aligned} u(x, 0) &= f(x), \quad x \in \mathbb{R}, \\ u_t(x, 0) &= g(x), \end{aligned}$$

   where f and g are functions of class C^2 and with *compact support*. Prove that the *total energy*,

$$E(t) = E_{cin}(t) + E_{pot}(t),$$

is constant. Here the potential and kinetic energies are defined as

$$E_{cin}(t) = \tfrac{1}{2}\int_{-\infty}^{+\infty} u_t^2(x,t)\,dx, \quad \text{and} \quad E_{pot}(t) = \tfrac{1}{2}\int_{-\infty}^{+\infty} c^2 u_x^2(x,t)\,dx,$$

respectively. (*Hint:* Use (a) to compute dE/dt. Apply D'Alembert's formula and use the fact that both f and g, and all their derivatives are zero outside a compact interval.)

(c) Under the same assumptions in (b), prove the principle of *equipartition of energy*: there exists $T > 0$ such that $E_{cin}(t) = E_{pot}(t)$ for all $t \geq T$.

4. Apply Duhamel's principle to solve

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= x^2, && x \in \mathbb{R},\ t \geq 0,\\ u(x,0) &= x, && x \in \mathbb{R},\\ u_t(x,0) &= 0, && x \in \mathbb{R}. \end{aligned}$$

5. Solve the following problem:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= xt, && x \in \mathbb{R},\ t \geq 0,\\ u(x,0) &= e^x, && x \in \mathbb{R},\\ u_t(x,0) &= \sin x, && x \in \mathbb{R}. \end{aligned}$$

with $c \neq 0$ constant.

6. Solve the homogeneous wave equation

$$u_{tt} - c^2 \Delta u = 0,$$

for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t > 0$ with initial conditions $u(x,0) = 0$, $u_t(x,0) = x_2$. Verify that the solution is correct.

7. Consider the homogeneous wave equation in $\mathbb{R}^3$,

$$u_{tt} - c^2 \Delta_x u = 0,$$

with $t \geq 0$, $x \in \mathbb{R}^3$.

(a) Show that any solution with *spherical symmetry* (that is, $u = u(r,t)$, where $r = |x|$) has the form

$$u = \frac{1}{r}\left(F(r+ct) + G(r+ct)\right).$$

(b) Show that, if the initial data are $u(x,0) = 0$, $u_t(x,0) = g(r)$, where g is an *even* function then the solution is

$$u(r,t) = \frac{1}{2cr}\int_{r-ct}^{r+ct} \rho g(\rho)\,d\rho.$$

(c) Suppose that g is given by:

$$g(r) = \begin{cases} 1, & \text{for } 0 < r < a,\\ 0, & \text{for } r > a, \end{cases}$$

with $a > 0$. Find the solution explicitly. (*Hint:* Use (b) to determine u in the different regions bounded by the cones $r = a \pm ct$ in space-time.) Show that u is discontinuous in $(0, a/c)$ (this is because of the *focusing effect* of the discontinuity of u_t in $t = 0$, $|x| = a$).

8. Consider the non-homogeneous wave equation:
$$u_{tt} - \Delta u = 1,$$
where $u = u(x, y, z, t)$ (that is, in $\mathbb{R}^3$), with initial conditions
$$u_{|t=0} = \sqrt{x^2 + y^2 + z^2}, \qquad u_{t|t=0} = x^2 + y^2 + z^2.$$
   (a) Express the laplacian in spherical coordinates
   $$(x, y, z) = (r \cos\phi \sin\theta, r \sin\phi \sin\theta, r \cos\theta),$$
   with $r \geq 0$, $\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$.
   (b) Find a solution depending only on $r = |\bar{x}|$, $\bar{x} = (x, y, z) \in \mathbb{R}^3$. (*Hint:* Reduce the problem to the non-homogeneous wave equation in one dimension for $r > 0$, $t > 0$. Use Green-Lagrange formula and analyze the cases $r \geq t$ and $r < t$.)
   (c) Is the solution unique? (*Hint:* Use Duhamel's principle to prove that any other solution depends only on r as well; apply uniqueness of the solution to the one-dimensional wave equation and conclude.)
   (d) Discuss the differentiability of the solution at the curve $t = |\bar{x}|$.
   (e) Now, find the solution directly. First find *one* solution to:
   $$u_{tt} - \Delta_x u = 1,$$
   $$u_{|t=0} = u_{t|t=0} = 0.$$
   (*Hint:* Apply Duhamel's principle.) Then, find the solution to
   $$u_{tt} - \Delta_x u = 0,$$
   $$u_{|t=0} = \sqrt{x^2 + y^2 + z^2},$$
   $$u_{t|t=0} = x^2 + y^2 + z^2,$$
   using Kirchhoff's formula and computing the surface integrals. The sum of the particular solution and the homogeneous solution must be identical to the solution you obtained in (b).

References

[1] S. Salsa, *Partial differential equations in action. From modelling to theory*, Universitext, Springer-Verlag Italia, Milan, 2008.