Applied Partial Differential Equations (MathMods)
Exercise sheet 6

Laplace and Poisson equations (continued):

- Do the following exercises from Salsa's book [1]: **3.4, 3.5, 3.10, 3.11, 3.15, 3.16.**

In addition, do the following exercises:

1. Solve the Laplace equation $\Delta u = u_{xx} + u_{yy} = 0$ in the disk $D = \{r^2 = x^2 + y^2 < a^2\}$ with the boundary condition $u = \sin^3\theta$ on $r = a$. (*Hint:* Use the identity $\sin^3\theta = 3\sin\theta - 4\sin 3\theta$.)
2. Solve the Laplace equation in the disk $D = \{r < a\}$ with the boundary condition
$$\frac{\partial u}{\partial r} + \alpha u = f(\theta),$$
where $\alpha > 0$ and f is an arbitrary function. Write the answer in terms of the Fourier coefficients of f.
3. Prove the uniqueness of the solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ to the Robin problem
$$\Delta u = 0, \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial n} + \alpha u = 0, \quad \text{at } \partial\Omega,$$
where $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is an open, bounded set with smooth boundary $\partial\Omega$ and $\alpha > 0$ is a positive constant.
4. Solve the Neumann problem in the half-plane in two dimensions: $\Delta u = 0$ in $y > 0$, with $u_y = h(x)$ on $y = 0$, and with u bounded as $x^2 + y^2 \to +\infty$. (*Hint*: Consider the problem satisfied by $v = u_y$.)
5. Let $\Omega \subset \mathbb{R}^d$, be open, with $d \geq 2$. Let $u \in C^2(\Omega)$, with $x \in \Omega$. Show that
$$\Delta u(x) = \lim_{r\to 0^+} \frac{2d}{r^2}\left(\frac{1}{\omega_d}\int_{|\eta|=1} u(x + r\eta)\, dS_\eta - u(x)\right).$$
This formula yields another proof of the mean value property for harmonic functions. (*Hint:* Consider the Taylor expansion of second order of u around the point x. Verify that $\int_{|\eta|=1} \eta_j\, dS_\eta = 0$, for each $1 \leq j \leq d$, and that $\int_{|\eta|=1} \eta_j\eta_k\, dS_\eta = 0$, if $j \neq k$. Compute $\int_{|\eta|=1} \eta_j^2\, dS_\eta$ for every $1 \leq j \leq d$.)
6. Let $B_1 = \{x \in \mathbb{R}^d : |x| < 1\}$ be the unit ball with center at the origin. Show that there exists a positive constant $C > 0$, depending only on the dimension $d \geq 2$, such that
$$\max_{B_1} |u| \leq C\left(\max_{B_1} |f| + \max_{\partial B_1} |g|\right),$$

where $f \in C(\overline{B_1})$, $g \in C(\partial B_1)$, and u is the solution to

$$\begin{aligned} \Delta u &= f, \quad \text{in } B_1, \\ u &= g, \quad \text{at } \partial B_1. \end{aligned}$$

(*Hint:* Use Poisson's formula for the ball.)

References

[1] S. Salsa, *Partial differential equations in action. From modelling to theory*, Universitext, Springer-Verlag Italia, Milan, 2008.