Applied Partial Differential Equations (MathMods)
First Midterm Exam: Solution

1. Use the method of characteristics to find a solution to

$$u_x + x(1+y)u_y = 4,$$
$$u(0,y) = 2y.$$

Find the characteristic system of ordinary differential equations (ODE) and solve it. Verify the invertibility of the maping $(\xi,\eta) \mapsto (\bar{x},\bar{y})$. Write explicitly the solution as $u = u(x,y)$ and verify that it is indeed solution to the problem. Is it a global solution (that is, is it defined for all $(x,y) \in \mathbb{R}^2$)? Explain your answer.

Solution: This is a linear equation of first order, with initial data on the curve $\mathcal{I} = \{(0,\xi) : \xi \in \mathbb{R}\}$, $u_{|\mathcal{I}} = 2\xi$. By the method of characteristics, for each fixed $\xi \in \mathbb{R}$ we consider the ODE system:

$$\frac{d\hat{x}}{d\eta} = 1, \qquad \hat{x}(0) = 0,$$
$$\frac{d\hat{y}}{d\eta} = \hat{x}(1+\hat{y}), \qquad \hat{y}(0) = \xi,$$
$$\frac{d\hat{u}}{d\eta} = 4, \qquad \hat{u}(0) = 2\xi.$$

Clearly, the solution to the first and third equations are $\hat{x}(\eta) = \eta$, and $\hat{u}(\eta) = 4\eta + 2\xi$, respectively. Substituting into the second equation we obtain:

$$(1+\hat{y})^{-1}\frac{d\hat{y}}{d\eta} = \eta \iff \hat{y}(\eta) = Ce^{\eta^2/2} - 1.$$

Since $\hat{y}(0) = \xi$ we get $\hat{y}(\eta) = (1+\xi)e^{\eta^2/2} - 1$. Varying $\xi \in \mathbb{R}$ we obtain a mapping $(\eta,\xi) \mapsto (\bar{x},\bar{y},\bar{u})(\eta,\xi) = (\eta,(1+\xi)e^{\eta^2/2} - 1, 4\eta + 2\xi)$. Since

$$\left|\begin{pmatrix} \bar{x}_\xi & \bar{x}_\eta \\ \bar{y}_\xi & \bar{y}_\eta \end{pmatrix}\right| = \left|\begin{pmatrix} 0 & 1 \\ e^{\eta^2/2} & (1+\xi)\eta e^{\eta^2/2} \end{pmatrix}\right| = -e^{\eta^2/2} \neq 0,$$

we can invert the mapping $(\eta,\xi) \mapsto (\bar{x},\bar{y})$. The result is $\eta = x$, $\xi = (1+y)e^{-\eta^2/2} - 1$. Upon substitution we obtain a solution

$$u(x,y) = 4x + 2((1+y)e^{-x^2/2} - 1).$$

To verify that this is a solution, note that $u(0,y) = 2y$ and, since $u_x = 4 - 2x(y+1)e^{-x^2/2}$, $u_y = 2e^{-x^2/2}$, then $u_x + x(1+y)u_y = 4$, as claimed. The solution is well-defined for all $(x,y) \in \mathbb{R}^2$. Therefore it is global.

2. Solve the quasi-linear equation

$$u_x + u_y = e^u,$$
$$u(x,0) = x.$$

Find the characteristic system of ODEs and solve it. Verify the invertibility of the corresponding map. Write the solution explicitly. Is the solution you found defined for all $(x,y) \in \mathbb{R}^2$? Draw the region of existence of the solution in the plane.

Solution: This is a quasi-linear equation of first order, with initial data on the curve $\mathcal{I} = \{(\xi, 0) : \xi \in \mathbb{R}\}$, $u_{|\mathcal{I}} = \xi$. By the method of characteristics, for each fixed $\xi \in \mathbb{R}$ we consider the ODE system:

$$\begin{aligned} \frac{d\hat{x}}{d\eta} &= 1, && \hat{x}(0) = \xi, \\ \frac{d\hat{y}}{d\eta} &= 1, && \hat{y}(0) = 0, \\ \frac{d\hat{u}}{d\eta} &= e^{\hat{u}}, && \hat{u}(0) = \xi. \end{aligned}$$

The solutions for $\hat{x}$ and $\hat{y}$ (characteristic curves on the plane) are $\hat{x}(\eta) = \eta + \xi$, $\hat{y}(\eta) = \eta$. Solving the equation for $\hat{u}$ y separation of variables we obtain

$$e^{-\hat{u}} \frac{d\hat{u}}{d\eta} = 1 \iff -e^{-\hat{u}} = \eta + C.$$

Since $\hat{u}(0) = \xi$ we obtain $\hat{u}(\eta) = -\log(e^{-\xi} - \eta)$. The mapping $(\eta, \xi) \mapsto (\bar{x}, \bar{y}) = (\eta + \xi, \eta)$ is invertible everywhere, as

$$\left| \begin{pmatrix} \bar{x}_\xi & \bar{x}_\eta \\ \bar{y}_\xi & \bar{y}_\eta \end{pmatrix} \right| = \left| \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right| = 1 \neq 0,$$

Substituting we obtain a solution:

$$u(x, y) = -\log(e^{y-x} - y).$$

To verify that this is a solution, notice that $u(x, 0) = -\log e^{-x} = x$, and that $u_x = e^{y-x}/(e^{y-x} - y)$, $u_y = (1 - e^{y-x})/(e^{y-x} - 1)$, so that $u_x + u_y = 1/(e^{y-x} - y) = e^{u(x,y)}$. Observe, however, that although the mapping $(\eta, \xi) \mapsto (x, y)$ is everywhere invertible the solution is not defined in the whole plane: it is defined only for $e^{y-x} - y > 0$. The curve $\check{x}(y) = y - \log y$ bounds the domain on the right. The solutions exists for all $y \leq 0$, and for all $x < y - \log y$, when $y > 0$. (See figure.) Recall that the invertibility condition is necessary to obtain a global solution but *not* sufficient.

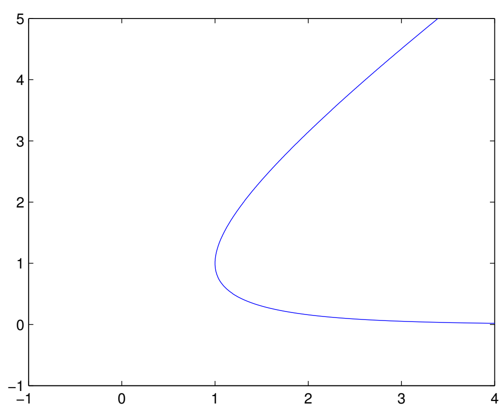

FIGURE 1. Plot of the curve $x = y - \log y$ in the plane. The solution exists in the open region left to the curve.

3. Consider the "bump" problem of traffic flow:

$$\rho_t + (\rho(1-\rho))_x = 0, \qquad x \in \mathbb{R},\ t > 0, \tag{1}$$

$$\rho(x,0) = \begin{cases} 0, & x < 0, \\ 1, & 0 \le x \le 1, \\ 0, & x > 1. \end{cases} \tag{2}$$

Find an entropic weak solution for all times. Write your solution explicitly. Why is it entropic? (*Hint:* Since the flux function is concave, the two discontinuities of $\rho(0,x)$ at $x = 0$ and $x = 1$ will give rise to a shock and a rarefaction wave, respectively. Note that the integrating factor of an equation of the form $y'(t) + ay(t) = f$ is $\mu(t) = \exp(\int^t a(s)\,ds)$. You will need to solve an equation of this form for the second shock.) Interpret your answer in terms of traffic flow.

Solution: Here the flux function $Q(\rho) = \rho(1-\rho)$ is concave, with $Q(0) = Q(1) = 0$. The characteristic velocity is $a(\rho) = Q'(\rho) = 1 - 2\rho$, so that the characteristics are straight lines with slopes $a(0) = 1$, or $a(1) = -1$. Since the flux function is concave, the initial jump at $x = 0$ is entropic, as $\rho_R = 1 > \rho_L = 0$. Therefore there is a shock curve generated at $(0,0)$ with speed $s_1 = [Q]/[\rho] = \frac{Q(1)-Q(0)}{1-0} = 0$. The shock curve has the form $\Sigma_1 = \{x = \hat{x}_1(t) = 0\}$.

In the region $a(-1) = -1 \le \rho \le a(0) = 1$ the function $a(\rho)$ is invertible with $g(\rho) = a^{-1}(\rho) = (1-\rho)/2$. The jump at $x = 1$ generates a rarefaction wave centered at $(x,t) = (1,0)$ given by

$$\rho(x,t) = g\Big(\frac{x-1}{t}\Big) = \frac{1}{2t}(t - x + 1).$$

It is defined in the region $-1 \le (x-1)/t \le 1$, for $t > 0$. (Another way to obtain the form of the rarefaction is to propose a self-similar solution of the form $\rho = v((x-1)/t)$. This leads to the equation

$$v'(\xi)(1 - \xi - 2v(\xi)) = 0,$$

where $\xi = (x-1)/t$, which holds whenever $2v(\xi) = 1-\xi$, that is, when $v(\xi) = g(\xi)$.) The rarefaction wave is bounded by the two characteristics $x_-(t) = -t + 1$ and $x_+(t) = t+1$, at $(x,t) = (1,0)$. Observe, however, that the characteristic $x = x_-(t)$ intersects the shock Σ_1 at time $t = 1$. Therefore, for later times the shock gets curved and there is a second shock generated at $(x,t) = (0,1)$. To obtain its form we use Rankine-Hugoniot conditions: $\rho_L = 0$ on the left, and $\rho_R = g((x-1)/t)$, for $t \ge 1$ (the rarefaction wave is on the right). Let's parametrize this shock curve as $\Sigma_2 = \{x = \hat{x}(t) : t \ge 1\}$. It satisfies $\hat{x}(1) = 0$. We compute

$$Q(\rho_R) = Q(g((\hat{x}-1)/t)) = Q((t - \hat{x} + 1)/2t) = \frac{1}{4t^2}(t - \hat{x} + 1)(t + \hat{x} - 1).$$

By Rankine-Hugoniot conditions the speed of the shock is

$$s_2 = \frac{d\hat{x}}{dt} = \frac{Q(\rho_R) - Q(\rho_L)}{\rho_R - \rho_L} = \frac{Q(g((\hat{x}-1)/t))}{g((\hat{x}-1)/t)} = \frac{1}{2t}(t + \hat{x} - 1).$$

This is an ODE for $\hat{x}$ with $\hat{x}(1) = 0$:

$$\frac{d\hat{x}}{dt} - \frac{1}{2t}\hat{x} = \frac{1}{2}\Big(1 - \frac{1}{t}\Big).$$

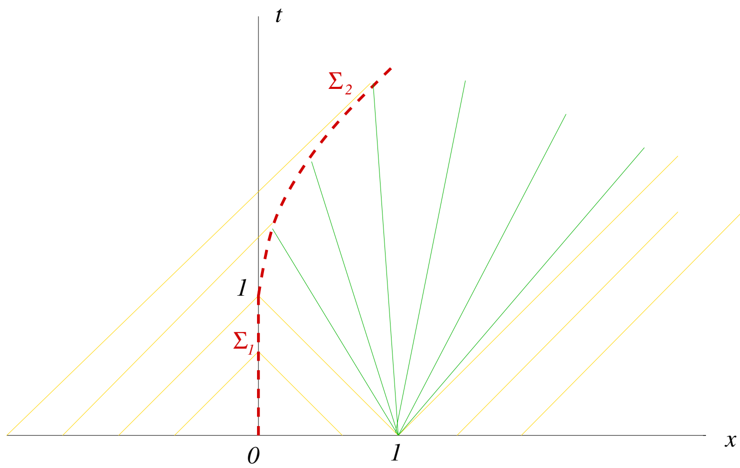

FIGURE 2. Sketch of the entropic solution to the bump problem. Characteristic lines are in yellow, the rarefaction region in green and the shock curves are the dotted curves in red.

Multiply by the integrating factor $1/\sqrt{t}$ we obtain $t^{-1/2}\hat{x} = t^{1/2} + t^{-1/2} + C$, with C constant. Since $\hat{x}(1) = 0$ we obtain

$$\Sigma_2 = \{(\hat{x}(t), t) \,:\, \hat{x}(t) = t + 1 - 2\sqrt{t},\ t \geq 1\}.$$

Observe that for $t \geq 1$, the shock curve Σ_2 never intersects the other characteristic bounding the rarefaction: $x_+(t) = t + 1$. The solution is thus given by:

$$\rho(x,t) = \begin{cases} 0, & x < 0, \text{ or, } x \geq 1 + t, \\ 1, & 0 < x < 1 - t, \\ \frac{1}{2t}(t - x + 1), & 1 - t \leq x \leq 1 + t, \end{cases}$$

for all $x \in \mathbb{R}$, and for $0 < t < 1$, and by,

$$\rho(x,t) = \begin{cases} 0, & x < t + 1 - 2\sqrt{t}, \text{ or, } x \geq 1 + t, \\ \frac{1}{2t}(t - x + 1), & t + 1 - 2\sqrt{t} \leq x \leq 1 + t, \end{cases}$$

for all $x \in \mathbb{R}$, and for $t \geq 1$. (See sketch of the solution.)

The solution is bounded for all $(x,t) \in \mathbb{R} \times (0, +\infty)$. Moreover, there is only one discontinuity (shock curve), $\Sigma = \Sigma_1 \cup \Sigma_2$. The characteristics seem to “enter” the shock at Σ_1: indeed, $a(\rho_R) = a(1) = -1 < s_1 = 0 < a(\rho_L) = 1$. Thus, Lax entropy condition is satisfied. At Σ_2 we have $a(\rho_R) = 1 - 2g((\hat{x} - 1)/t)$, and thus,

$$a(\rho_R) = \frac{1}{t}(\hat{x}(t) - 1) = 1 - \frac{2}{\sqrt{t}} < s_2 = \frac{d\hat{x}}{dt} = \frac{1}{2t}(t + \hat{x}(t) - 1) = 1 - \frac{1}{\sqrt{t}} < a(\rho_L) = a(0) = 1.$$

These inequalities are satisfied for all $t \geq 1$. Hence, the solution is entropic.

The interpretation in terms of traffic flow is as follows: the initial condition is the “bump”, that is, maximum density in a short piece of the road (between $x = 0$ and $x = 1$) with no cars before or after. (Think of a street, blocked at both ends by police officers letting some kids cross the street.) At the same time ($t = 0$), the two ends are set free. The cars to arrive at $x = 0$ at time $t = 0$ start to form a shock wave with a velocity $s_1 \leq 0$ (in this example, $s_1 = 0$). The cars ahead at $x = 1$

start to move continuously (rarefaction). After some positive time $t \geq 1$, the shock wave starts to move with positive (non-constant) speed, in the direction ahead of the road, but it never reaches the cars who were at $x = 1$ at time $t = 0$.

4. Consider the following equation in one spatial dimension

$$u_t = Du_{xx} - \alpha u, \tag{3}$$

with $\alpha > 0$ constant, $D > 0$, in the domain $x \in [0,1]$, $t > 0$. This equation models the temperature distribution with heat loss along a bar of unit length. Suppose we impose homogeneous boundary conditions:

$$u(0,t) = u(1,t) = 0, \qquad t > 0. \tag{4}$$

(a) The equilibrium temperatures are the time-independent solutions to

$$\begin{aligned} Du_{xx} - \alpha u &= 0, \\ u(0) = u(1) &= 0. \end{aligned} \tag{5}$$

Find all solutions $u = u(x)$ to (5).

(b) Solve the boundary value problem (3) and (4) subject to initial conditions of the form $u(x,0) = x$ by the method of separation of variables. Make sure you analyze all the eigenvalues of the problem. (You may take for granted the uniform convergence of the series.)

(c) Study the temperature solution u obtained in (b) for large times $t \to +\infty$ and compare it to what you found in (a).

Solution: **(a)** Here $u_{xx} = (\alpha/D)u$. Since $\alpha/D > 0$ the solution is of the form $u(x) = Ae^{(\alpha/D)x} + Be^{-(\alpha/D)x}$. But, from the boundary conditions $u(0) = u(1) = 0$ $\Rightarrow A + B = 0$, $A(e^{2\alpha/D} - 1) = 0 \Rightarrow A = B = 0$. The only stationary solution is the trivial one: $u(x) = 0$.

(b) By separation of variables: $u(x,t) = \psi(x)\theta(t)$. Then $\psi(x)\theta'(t) = D\psi''(x)\theta(t) - \alpha\psi(x)\theta(t)$. Thus,

$$\frac{\theta'(t)}{\theta(t)} = D\frac{\psi''(x)}{\psi(x)} - \alpha = \lambda,$$

with λ constant. Let's solve $\psi'' = (\lambda + \alpha)\psi/D$. We have three cases. (i): $(\lambda + \alpha)/D = 0$. Then $\psi'' = 0$ and $\psi = Ax+B$. By the boundary conditions: $A = B = 0$. (ii): $\mu := (\lambda + \alpha)/D > 0$. The solution is of the form $\psi = Ae^{\mu x} + Be^{-\mu x}$. Again, $\psi(1) = \psi(0) = 0$ imply that $A = B = 0$. The only-trivial case is (iii): $(\lambda + \alpha)/D = -\mu^2$. The solutions have the form $\psi(x) = A\sin\mu x + B\cos\mu x$. Since $\psi(1) = \psi(0) = 0$ then $B = 0$, $\mu = n\pi$, $n \in \mathbb{N}$, and A arbitrary. Therefore, $\lambda_n := -(\alpha + Dn^2\pi^2)$, $\psi_n(x) = \sin(n\pi x)$, for each n.

Now solve $\theta'(t)/\theta(t) = \lambda_n$, to obtain $\theta_n(t) := A_n e^{-(\alpha + Dn^2\pi^2)t}$. We define the solutions:

$$u_n(x,t) := \psi_n(x)\theta_n(t) = A_n e^{-(\alpha + Dn^2\pi^2)t}\sin(n\pi x).$$

Each $u_n(x,t)$ is a solution to the equation, with the boundary conditions. We propose the series solution:

$$u(x,t) = \sum_{n=1}^{+\infty} u_n(x,t) = \sum_{n=1}^{+\infty} A_n e^{-(\alpha + Dn^2\pi^2)t}\sin(n\pi x).$$

It can be proved that the convergence is uniform just as we did in the lecture for the heat equation. The coefficients A_n are computed through the Fourier series expansion of the initial condition $u(x,0) = x = \sum A_n \sin(n\pi x)$. Thus,

$$A_n = 2\int_0^1 x\sin(n\pi x)\,dx = -\frac{2}{n\pi}(x\cos(n\pi x))|_{x=0}^{x=1} = \frac{2(-1)^{n+1}}{n\pi}.$$

The solution is

$$u(x,t) = \sum_{n=1}^{+\infty} \frac{2(-1)^{n+1}}{n\pi} e^{-(\alpha+Dn^2\pi^2)t}\sin(n\pi x).$$

Clearly, $u(0,t) = u(1,t) = 0$ for all t; $u(x,0) = x$ and it satisfies the equation by unform convergence.

(c) Since the series converges uniformly

$$\lim_{t\to+\infty} u(x,t) = \sum_{n=1}^{\infty} \lim_{t\to+\infty} u_n(x,t) = 0,$$

because $e^{-(\alpha+Dn^2\pi^2)t} \to 0$ as $t \to +\infty$ for each n. Thus, the solution tends to the equilibrium solution found in (a).

5. Consider the heat equation

$$u_t - u_{xx} = 0, \qquad x \in [0,L],\ t > 0. \tag{6}$$

Let $u_1(x,t)$ be the solution to (6) with boundary data $u_1(0,t) = f_1(t)$, $u_1(L,t) = h_1(t)$ for all $t > 0$, and subject to the initial condition $u_1(x,0) = g_1(x)$ for all $x \in [0,L]$. Let $u_2(x,t)$ be the solution to (6) with boundary data $u_2(0,t) = f_2(t)$, $u_2(L,t) = h_2(t)$ for all $t > 0$, and subject to the initial condition $u_2(x,0) = g_2(x)$ for all $x \in [0,L]$. Suppose that $f_1(t) \le f_2(t)$, $h_1(t) \le h_2(t)$ for all $t > 0$, and $g_1(x) \le g_2(x)$ for all $x \in [0,L]$. Prove that $u_1 \le u_2$ in the set $(x,t) \in [0,L]\times[0,+\infty)$.

Solution: Let $\Omega = (0,L)$, and $\Omega_T = \Omega \times (0,T]$, with $T > 0$ arbitrary. Let $v(x,t) := u_1(x,t) - u_2(x,t)$. Then clearly v satisfies $v_t - v_{xx} = 0$ in Ω_T, with $v(0,t) = f_1(t) - f_2(t) \le 0$, $v(L,t) = h_1(t) - h_2(t) \le 0$, for all $t \in (0,T]$; and $v(x,0) = g_1(x) - g_2(x) \le 0$, for all $x \in [0,L]$. Whence we have $v \le 0$ at $\Gamma_T := \bar\Omega_T \backslash \Omega_T$. By the maximum principle,

$$\max_{\bar\Omega_T} v = \max_{\Gamma_T} v \le 0.$$

Therefore, $u_1 \le u_2$ for all $(x,t) \in \bar\Omega_T$. Since $T > 0$ is arbitrary, the inequality holds for all $x \in [0,L]$, and all $t \in [0,+\infty)$.

6. Use the energy method to show that the solution $u \in C^2(\Omega_T)$ to the initial-boundary value problem

$$\begin{aligned} u_t - \Delta u &= f, && (x,t) \in \Omega_T, \\ u &= g, && (x,t) \in (\Gamma_1 \times (0,T]) \cup (\Omega \times \{t=0\}), \\ u + \alpha\nabla u \cdot \hat n &= h, && (x,t) \in \Gamma_2 \times (0,T], \end{aligned}$$

is unique. Here $\Omega \subset \mathbb{R}^d$ with $d \ge 1$ is bounded, open with smooth boundary $\partial\Omega$; $\Omega_T = \Omega \times (0,T]$ is the parabolic cylinder with fixed $T > 0$; $\Gamma_T = \bar\Omega_T \backslash \Omega_T$ is the parabolic boundary, with $\Gamma_1 \cup \Gamma_2 = \partial\Omega$; $\hat n$ is the outer unit normal to $\partial\Omega$ and $\alpha > 0$ is constant. f, g, h are continuous known functions.

Solution: Let us assume there are two solutions $u_1, u_2 \in C^2(\Omega_T)$ and let $u := u_1 - u_2$. Then u is a solution to the homogeneous problem

$$\begin{aligned} u_t - \Delta u = 0, &\qquad (x,t) \in \Omega_T, \\ u = 0, &\qquad (x,t) \in (\Gamma_1 \times (0,T]) \cup (\Omega \times \{t=0\}), \\ u + \alpha \nabla u \cdot \hat{n} = 0, &\qquad (x,t) \in \Gamma_2 \times (0,T]. \end{aligned}$$

Multiply the equation by u and integrate in $x \in \Omega$ for each time $t \in (0,T]$ fixed. By Green's formula, and by the boundary conditions $u = 0$ at Γ_1, $u + \alpha \nabla u \cdot \hat{n} = 0$ at Γ_2 we obtain

$$\begin{aligned} \tfrac{1}{2} \frac{d}{dt} \int_\Omega u^2 \, dx = \int_\Omega u u_t \, dx &= \int_\Omega u \Delta u \, dx \\ &= -\int_\Omega |\nabla u|^2 \, dx + \int_{\partial\Omega} u \nabla u \cdot \hat{n} \, dS_x, \\ &= -\int_\Omega |\nabla u|^2 \, dx + \int_{\Gamma_1} u \nabla u \cdot \hat{n} \, dS_x + \int_{\Gamma_2} u \nabla u \cdot \hat{n} \, dS_x, \\ &= -\int_\Omega |\nabla u|^2 \, dx - \alpha \int_{\Gamma_2} (\nabla u \cdot \hat{n})^2 \, dS_x \le 0, \end{aligned}$$

because $\alpha > 0$. Therefore the energy $E(t) = \int_\Omega u(x,t)^2 \, dx$ is a decreasing function of $t \in (0,T]$, and

$$0 \le E(t) \le E(0) = \int_\Omega u(x,0)^2 \, dx = 0,$$

inasmuch as $u(x,0) = 0$ for all $x \in \Omega$. Thus, $E(t) = 0$ for all $t \in [0,T]$, yielding $u(x,t) = 0$ for all $(x,t) \in \Omega_T$. The solution is unique.