Applied Partial Differential Equations (MathMods)
Exercise sheet 5

Laplace and Poisson equations:

- Do the following exercises from Salsa's book [1]: **3.1, 3.2, 3.6, 3.8, 3.9.**

In addition, do the following exercises:

1. Let
$$u(x) = \frac{1}{|x|}, \qquad x \in \mathbb{R}^3.$$
   (a) Show that $\Delta u = 0$ in $\mathbb{R}^3 \backslash \{0\}$.
   (b) Compute
   $$I = \int_{|x|=\rho} \frac{\partial u}{\partial n}\, dS.$$
   Does this result contradict the fact that $\int_{\partial\Omega} \frac{\partial u}{\partial n}\, dS = 0$ for each harmonic function u in Ω?
   (c) Consider the problem:
   $$\begin{aligned} \Delta u &= 0, \qquad \text{in } |x| > 1, \\ u &= 1, \qquad \text{at } |x| = 1. \end{aligned}$$
   Find, at least, two solutions.
   (d) If, in addition, we impose the condition $u(x) \to 0$ as $|x| \to +\infty$, use the maximum principle to show that such solution is unique.
2. Let $G(x, y)$ be the Green function for the ball $B_R(0) \subset \mathbb{R}^d$, $d \geq 2$, with $R > 0$. Prove that
$$\frac{\partial G}{\partial n_y}(x, y) := \nabla_y G(x, y) \cdot \hat{n} = \frac{|x|^2 - R^2}{\omega_d R |x - y|^d},$$
for any $x \in B_R(0)$, $y \in \partial B_R(0)$, where $\hat{n}$ is the unit outer normal at $\partial B_R(0)$. The function
$$K(x, y) = -\frac{\partial G}{\partial n_y}(x, y) = \frac{R^2 - |x|^2}{\omega_d R |x - y|^d},$$
is known as the *Poisson kernel* for the ball $B_R(0) \subset \mathbb{R}^d$.
3. Let $\Omega = \{a_1 < |x| < a_2\} \subset \mathbb{R}^d$, $d \geq 2$, where $a_2 > a_1 > 0$. Suppose u is harmonic in Ω. Let us define:
$$I(\rho) = \frac{1}{\omega_d \rho^{d-1}} \int_{|x|=\rho} u(x)\, dS_x, \qquad \rho \in (a_1, a_2).$$
Show that $I(\rho)$ has the form
$$I(\rho) = \begin{cases} \alpha + \beta \rho^{2-d}, & d \geq 3, \\ \alpha + \beta \log \rho, & d = 2, \end{cases}$$
by finding the constants α and β as functions of $I(a_1)$ and $J_1 := \int_{|x|=a_1} \frac{\partial u}{\partial n}\, dS_x$. Use the fact that
$$\int_{\partial(a_1<|x|<\rho)} u(x) \frac{\partial \Phi(x,0)}{\partial n_x} - \Phi(x,0) \frac{\partial u(x)}{\partial n_x}\, dS_x = 0,$$

where $\Phi = \Phi(x, y)$ is the fundamental solution:

$$\Phi(x,y) = \begin{cases} -\dfrac{1}{2\pi} \log |x-y|, & d = 2, \\[2ex] \dfrac{1}{\omega_d (d-2)} \dfrac{1}{|x-y|^{d-2}}, & d \geq 3, \end{cases}$$

for $x, y \in \mathbb{R}^d$, $x \neq y$.

4. Consider the problem:

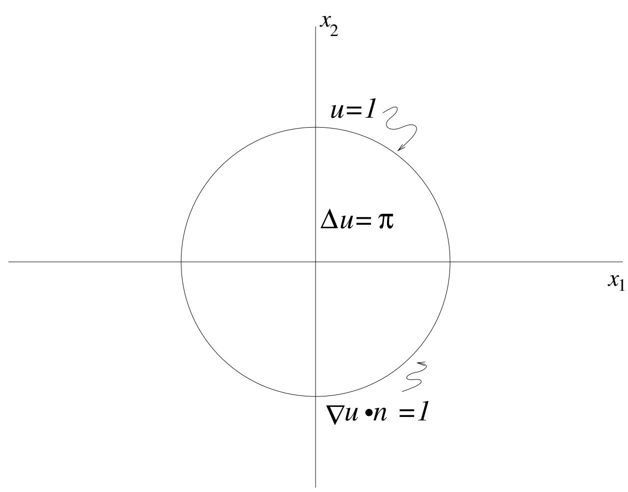
$$\Delta u = \pi, \qquad x \in \Omega,$$

with $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1\}$, subject to boundary conditions of the form

$$u = 1, \qquad \text{at } x_2 = +\sqrt{1 - x_1^2}, \ |x_1| < 1,$$

$$\frac{\partial u}{\partial n} = \nabla u \cdot \hat{n} = 1, \qquad \text{at } x_2 = -\sqrt{1 - x_1^2}, \ |x_1| \leq 1,$$

where $\hat{n}$ is the outer unit normal to the disc.



If there is a solution $u \in C^2(\bar{\Omega})$, is it unique? (*Hint:* Use the energy method.)

References

[1] S. Salsa, *Partial differential equations in action. From modelling to theory*, Universitext, Springer-Verlag Italia, Milan, 2008.