Applied Partial Differential Equations (MathMods)
Exercise sheet 4

The heat equation:

- Do the following exercises from Salsa's book [1]: **2.1, 2.2, 2.3, 2.8, 2.12, 2.14, 2.18.**

In addition, do the following exercises:

1. Solve the heat equation in $\mathbb{R}^d$ *with convection*:
$$u_t - \Delta u + c \cdot \nabla u = 0,$$
where $x \in \mathbb{R}^d$, $t > 0$, $c = (c_1, \dots c_d)^\top \neq 0$ is a constant vector, and with initial condition $u(x, 0) = g(x)$. (*Hint:* Find an appropriate change of variables that transforms the equation into the heat equation. Apply the formula for the global Cauchy problem seen during the lecture.)

2. Supposing that $u \in C^2(\mathbb{R}^d \times (0, +\infty))$ is a solution to the heat equation
$$u_t - \Delta u = 0, \qquad x \in \mathbb{R}^d,\ t > 0.$$
   (a) Show that $u_\lambda(x, t) = u(\lambda x, \lambda^2 t)$ is also a solution to the heat equation for each $\lambda \in \mathbb{R}$.
   (b) Use (a) to prove that
$$v(x, y) = x \cdot \nabla u + 2tu_t,$$
   is also a solution.

2. Let
$$g(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$
Show that the solution to
$$\begin{aligned} u_t - u_{xx} &= 0, \qquad x \in \mathbb{R},\ t > 0, \\ u(x, 0) &= g(x), \quad x \in \mathbb{R}, \end{aligned}$$
is given by
$$u(x, t) = \frac{1}{2}\left(1 + \phi(x/\sqrt{4t}\,)\right),$$
where
$$\phi(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-t^2}\, dt.$$
The function ϕ is called the *error function*.

3. Solve the one-dimensional heat equation
$$u_t - u_{xx} = 0,$$
for $x > 0$ and $t > 0$, with initial condition $u(x, 0) = g(x)$, g bounded, such that $g(0) = 0$, and with boundary condition $u(0, t) = 0$ for all $t > 0$. (*Hint:* Consider the odd extension of g to the whole real line, given by $g(x) = -g(-x)$, for $x < 0$. With this new initial condition apply the solution to the global Cauchy problem that we proved in the lecture. Be careful with the boundary condition.)

4. Assuming that each of the functions $u_1(y,t),\dots,u_d(y,t)$, with $d \geq 2$, is a solution to the one-dimensional heat equation, $u_t = u_{yy}$, show that the function
$$v(x,t) = \prod_{j=1}^{d} u_j(x_j,t),$$
is a solution to the heat equation in dimension d, that is, $v_t - \Delta v = 0$.
5. Let $\epsilon > 0$. Let $u \in C^2(\mathbb{R}\times(0,+\infty))$, with $u > 0$, be a solution to
$$u_t - \epsilon u_{xx} = 0, \qquad x \in \mathbb{R},\ t > 0.$$
Prove that
$$v(x,t) = -\frac{2\epsilon u_x}{u}$$
satisfies the *viscous Burgers equation:*
$$v_t + v v_x = \epsilon v_{xx},$$
for $x \in \mathbb{R}$, $t > 0$. This transformation is known as the *Hopf-Cole transformation*. It is remarkable because it transforms a nonlinear equation into a linear one.
6. Let $\Omega \subset \mathbb{R}^d$ be open, bounded with smooth boundary $\partial\Omega$. Let $u \in C^1(\overline{\Omega}\times(0,T))$, with fixed $T > 0$ be a solution to
$$u_t - \Delta u = 0, \qquad \text{en}\,\Omega\times(0,T),$$
which, in addition, satisfies the boundary condition: $u = 0$ in $\Gamma_1 \subset \partial\Omega$ for all $t \in (0,T)$; and, $\partial u/\partial n = \nabla u\cdot\hat{n} = 0$ in $\Gamma_2 \subset \partial\Omega$, with $\partial\Omega = \Gamma_1\cup\Gamma_2$. Show that
$$\rho(t) = \|u(t)\|_{L^2(\Omega)} = \int_\Omega u(x,t)^2\,dx,$$
is a non-increasing function of $t \in (0,T)$.
7. Let $\Omega \subset \mathbb{R}^d$ be bounded, open with smooth boundary. Let $T > 0$, and $\Omega_T = \Omega\times(0,T]$, $\Gamma_T = \overline{\Omega_T}\setminus\Omega_T$. Show that if $u \in C^2(\Omega_T)\cap C(\Gamma_T)$ is a solution to the heat equation in Ω_T then
$$\min_{\Gamma_T} u \leq u(x,t) \leq \max_{\Gamma_T} u,$$
for all $(x,t) \in \Omega_T$.
8. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set with smooth boundary $\partial\Omega$. Prove that if there is a solution $u \in C^2(\bar{\Omega}\times[0,T))$ with fixed $T > 0$, to the non-homogeneous heat equation with initial and Neumann conditions,
$$\begin{aligned} u_t - \Delta u &= h(x,t), \quad x \in \Omega,\ T > t > 0,\\ u(x,0) &= f(x), \quad x \in \Omega,\\ \nabla u\cdot\hat{n} = \frac{\partial u}{\partial n} &= g(t), \qquad x \in \partial\Omega,\ T > t > 0, \end{aligned}$$
then it is unique. (*Hint*: You may apply the energy method.)

References

[1] S. Salsa, *Partial differential equations in action. From modelling to theory*, Universitext, Springer-Verlag Italia, Milan, 2008.