Applied Partial Differential Equations (MathMods)
Exercise sheet 3

Fully nonlinear first order equations:

- Do the following exercises from Salsa's book [1]: **4.4, 4.5, 4.7, 4.10.**

In addition, do the following exercises:

1. Find an entropic solution to Burgers' equation
$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0, \tag{1}$$
with initial condition
$$u(x,0) = u_0(x) = \begin{cases} 1, & x < -1 \\ 0, & -1 \le x \le 0, \\ 2 & 0 < x. \end{cases}$$
Sketch the characteristics, shock paths and rarefaction waves in the $x - t$ plane.
2. Find an entropic solution to Burgers equation (1) with initial condition
$$u(x,0) = u_0(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \le x \le 1 \\ 0, & x > 1. \end{cases}$$
Sketch the characteristics, shock paths and rarefaction waves in the $x - t$ plane.
3. Find an entropic solution to Burgers equation (1) with initial condition
$$u(x,0) = u_0(x) = \begin{cases} 2, & x < 0 \\ 1, & 0 < x \le 2 \\ 0, & x > 2. \end{cases}$$
Sketch the characteristics, shock paths and rarefaction waves in the $x - t$ plane. (*Hint:* The two shocks merge into one shock at some point.)
4. Consider the following traffic model
$$\rho_t + (\rho(1-\rho))_x = 0,$$
with initial condition
$$\rho(x,0) = \begin{cases} \frac{1}{2}, & x < 0, \\ 0, & x > 0. \end{cases}$$
Find a weak solution. Is your solution continuous? Is it entropic? (Argue why or why not.) Interpret your solutions in terms of traffic flow. What is the meaning of the initial condition?

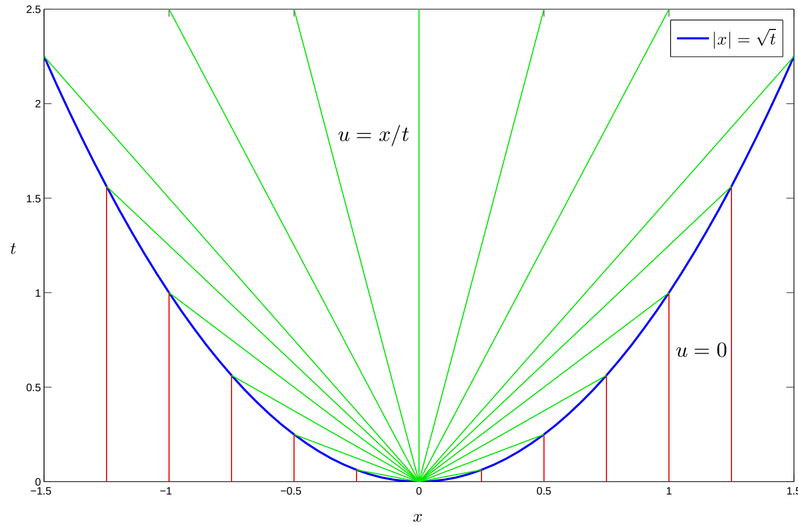

FIGURE 1. Sketch of $N = N(x,t)$ in the (x,t)-plane, defined in (2). The autosimilar rarefaction wave is limited on the left and on the right by the curves $x = \pm\sqrt{t}$.

5. Consider the following "solution" to the Burger's equation (1), known as N*-wave*,

$$N(x,t) = \begin{cases} x/t, & |x| < \sqrt{t}, \\ 0, & |x| > \sqrt{t}. \end{cases} \tag{2}$$

An sketch of the solution can be found in figure 1.

(a) Prove that (2) satisfies Rankine-Hugoniot condition on each discontinuity, for the Burgers' flux function $f(u) = u^2/2$.

(b) Show that the Lax entropy condition is satisfied on every discontinuity.

(c) Explain why, however, N *is not* an entropic weak solution to the Cauchy problem (1) with $u(x,0) = 0$ (The unique entropy solution is the solution identically zero.) (*Hint:* Is the N-wave a bounded solution?).

REFERENCES

[1] S. SALSA, *Partial differential equations in action. From modelling to theory*, Universitext, Springer-Verlag Italia, Milan, 2008.