Applied Partial Differential Equations (MathMods)
Exercise sheet 1

- Do the following exercises from Salsa's book [1]: **4.1, 4.2, 4.13(b), 4.15, 4.16.**

In addition, do the following exercises:

1. Solve the problem
$$\begin{aligned} \rho_t - c\rho_x &= 0, & x \le 0,\ t \ge 0, \\ \rho(x,0) &= 0, & x \le 0, \\ \rho(0,t) &= g(t), & t \ge 0, \end{aligned}$$
with $c > 0$ constant. Interpret the results in terms of traffic flow.
2. Solve the problem
$$\begin{aligned} \rho_t + c\rho_x &= 0, & x \ge 0,\ t \ge 0, \\ \rho(x,0) &= 0, & x \ge 0, \\ \rho(\alpha t,t) &= g(t), & \alpha > 0,\ t \ge 0, \end{aligned}$$
where $(x,t) \in \mathbb{R} \times [0,+\infty)$, $\alpha > 0$ and $c > 0$ are constant. Write the explicit solution and provide a geometric interpretation for it. What happens if $\alpha = c$? Interpret your solution in terms of traffic flow.
3. Study the characteristics and the solution to the linear equation
$$\begin{aligned} yu_x - xu_y &= 1, & (x,y) \in \mathbb{R}^2, \\ u(x,0) &= 0, & x \in \mathbb{R}. \end{aligned}$$
Find the system of ordinary differential equations and study the transformation $(\eta,\xi) \mapsto (x,y)$. Where is it invertible? Is the solution continuous at $x = 0$? If the initial data is now
$$u(x,0) = f(x),$$
and the solution $u = u(x,y)$ is continuous at $x = 0$, what can we say about the function $f(x)$? And about $f(0)$?
4. Solve the initial value problem
$$\begin{aligned} u_t + uu_x &= 1, \\ u(x,0) &= -\tfrac{1}{2}x. \end{aligned}$$
Find the characteristic curves (draw a picture) and give an explicit formula for the solution. Where does it exist? Does it exist for all $(x,t) \in \mathbb{R} \times [0,+\infty)$? Explain your answer.
5. Solve the linear equation
$$xu_y - yu_x = u,$$
under condition $u(x,0) = f(x)$, where f is a function of class C^1. Where is the solution valid? Classify the set of functions f for which

a *global* solution of class C^1 exists. (Global solution here means that the solution exists and it is of class C^1 for all $(x, y) \in \mathbb{R}^2$.)

6. Solve the Cauchy problem
$$u_x + u_y = u^4,$$
where $u(x, 0) = f(x)$ where f is of class C^1. Discuss the invertibility of the mapping and the range of existence of a C^1 solution.

7. For the quasi-linear Cauchy problem
$$u_y = xuu_x,$$
with initial data $u(x, 0) = x$, study the characteristics and discuss the invertibility of the mapping (this may not be possible explicitly). Write the solution to the Cauchy problem (implicitly).

8. Determine the solution to
$$u_x + (x + y)u_y = xy,$$
with data $u(x, 0) = x$. Again, study the system of ODEs, the invertibility of the mapping, etc.

References

[1] S. Salsa, *Partial differential equations in action. From modelling to theory*, Universitext, Springer-Verlag Italia, Milan, 2008.