ANALISI MATEMATICA III del 24 luglio 2007 - ESERCIZIO N° 1

TESTO

Verificare il teorema di Gauss per il campo vettoriale

$$F(x,y,z) = (x+y+z,\ y^2,\ z^2)$$

e il dominio

$$D = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ z \geq 0,\ x^2+y^2+z^2 \leq 1\}$$

Risoluzione

Il dominio $D \subset \mathbb{R}^3$ corrisponde al primo ottante della sfera unitaria centrata nell'origine.
La formula da verificare é

$$\iiint_D \operatorname{div} F(x,y,z)\, dx\, dy\, dz = \iint_{\partial^+ D} \langle F, n_e \rangle\, d\sigma$$

La superficie ∂D al secondo membro é l'unione di quattro superfici regolari: $\partial D = S_1 + S_2 + S_3 + S_4$, dove

$S_1 = \{(x,y,z) \in \mathbb{R}^3 : x = 0,\ y \geq 0,\ z \geq 0,\ y^2+z^2 \leq 1\}$, $n = (-1,0,0)$

$S_2 = \{(x,y,z) \in \mathbb{R}^3 : y = 0,\ x \geq 0,\ z \geq 0,\ x^2+z^2 \leq 1\}$, $n = (0,-1,0)$

$S_3 = \{(x,y,z) \in \mathbb{R}^3 : z = 0,\ x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 1\}$, $n = (0,0,-1)$

$S_4 = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ z \geq 0,\ x^2+y^2+z^2 = 1\}$, n il raggio della sfera uscente.

Risulta

$$\iint_{S_1} \langle F, n_e \rangle\, d\sigma = \int_0^1 \int_0^{\pi/2} \rho(-\rho\cos\vartheta - \rho\,\mathrm{sen}\,\vartheta)\, d\rho\, d\vartheta =$$

$$= -\int_0^1 \rho^2\, d\rho \int_0^{\pi/2} (\cos\vartheta + \mathrm{sen}\,\vartheta)\, d\vartheta = -\frac{1}{3}\,2 = -\frac{2}{3}$$

$$\iint_{S_2} \langle F, n_e \rangle \, d\sigma = 0 \quad \text{in quanto } \langle F, n_e \rangle = 0 \text{ su } S_2 .$$

$$\iint_{S_3} \langle F, n_e \rangle \, d\sigma = 0 \quad \text{in quanto } \langle F, n_e \rangle = 0 \text{ su } S_3$$

Parametrizziamo ora S_4 in coordinate polari.

$$\begin{cases} x = \cos\varphi \cos\vartheta & 0 \le \vartheta \le \frac{\pi}{2} \\ y = \cos\varphi \,\mathrm{sen}\,\vartheta & \\ z = \mathrm{sen}\,\varphi & 0 \le \varphi \le \frac{\pi}{2} \end{cases}$$

$$\det \begin{pmatrix} i & j & k \\ -\cos\varphi \,\mathrm{sen}\,\vartheta & \cos\varphi \cos\vartheta & 0 \\ -\mathrm{sen}\,\varphi \cos\vartheta & -\mathrm{sen}\,\varphi \,\mathrm{sen}\,\vartheta & \cos\varphi \end{pmatrix} =$$

$$= \left(\cos^2\varphi \cos\vartheta, \cos^2\varphi \,\mathrm{sen}\,\vartheta, \ \mathrm{sen}\,\varphi \cos\varphi \,\mathrm{sen}^2\vartheta + \mathrm{sen}\,\varphi \cos\varphi \cos^2\vartheta\right)$$

$$= \left(\cos^2\varphi \cos\vartheta, \ \cos^2\varphi \,\mathrm{sen}\,\vartheta, \ \mathrm{sen}\,\varphi \cos\varphi\right)$$

che è un vettore normale ad S_4 che punta verso l'esterno della sfera. Di conseguenza

$$\iint_{S_4} \langle F, n_e \rangle \, d\sigma = \int_0^{\pi/2}\int_0^{\pi/2} \Big[(\cos\varphi\cos\vartheta + \cos\varphi \,\mathrm{sen}\,\vartheta + \mathrm{sen}\,\varphi)\cos^2\varphi\cos\vartheta + \cos^2\varphi\,\mathrm{sen}^2\vartheta \cdot \cos^2\varphi\,\mathrm{sen}\,\vartheta + \mathrm{sen}^2\varphi \cdot \mathrm{sen}\,\varphi\cos\varphi\Big] d\vartheta\, d\varphi =$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \left(\cos^3\varphi\cos^2\vartheta + \cos^3\varphi\,\mathrm{sen}\,\vartheta\cos\vartheta + \mathrm{sen}\,\varphi\cos^2\varphi\cos\vartheta + \cos^4\varphi\,\mathrm{sen}^3\vartheta - \cos^2\varphi\,\mathrm{sen}^2\varphi\,\mathrm{sen}^3\vartheta + \mathrm{sen}^3\varphi\cos\varphi\right) d\vartheta\, d\varphi$$

Per calcolare il flusso uscente da S_4 resta da calcolare l'ultimo integrale.
Per prepararci a ciò, bisogna calcolare

$$\int_0^{\pi/2} \cos^2 t \, dt = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \cos^3 t \, dt = \int_0^{\pi/2} \cos t \, (1 - \operatorname{sen}^2 t) \, dt = \int_0^{\pi/2} \cos t \, dt - \int_0^{\pi/2} \cos t \operatorname{sen}^2 t \, dt$$

$$= 1 - \frac{1}{3} \operatorname{sen}^3 t \Big|_{t=0}^{t=\pi/2} = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\int_0^{\pi/2} \operatorname{sen} t \cos t \, dt = \frac{1}{2} \operatorname{sen}^2 t \Big|_{t=0}^{t=\pi/2} = \frac{1}{2}$$

$$\int_0^{\pi/2} \operatorname{sen} t \cos^2 t \, dt = -\frac{1}{3} \cos^3 t \Big|_{t=0}^{t=\frac{\pi}{2}} = \frac{1}{3}$$

$$\int_0^{\pi/2} \operatorname{sen}^3 t \, dt = \frac{2}{3} \quad \text{come per } \cos^3 t$$

$$\int_0^{\pi/2} \operatorname{sen}^2 t \cos^2 t \, dt = \frac{1}{4} \int_0^{\pi/2} \operatorname{sen}^2 (2t) \, dt = \frac{1}{8} \int_0^{\pi/2} \operatorname{sen}^2 (2t) \, d(2t) =$$

$$= \frac{1}{8} \int_0^{\pi} \operatorname{sen}^2 s \, ds = \frac{1}{8} \, \frac{\pi}{2} = \frac{\pi}{16}$$

$$\int_0^{\pi/2} \cos t \operatorname{sen}^3 t \, dt = +\frac{1}{4} \operatorname{sen}^4 t \Big|_{t=0}^{t=\frac{\pi}{2}} = +\frac{1}{4}$$

Ritornando al calcolo del flusso uscente da S_4 si ha

$$\iint_{S_4} \langle F, n_e \rangle \, d\sigma = \frac{2}{3} \cdot \frac{\pi}{4} + \frac{2}{3} \, \frac{1}{2} + \frac{1}{3} + \frac{\pi}{4} \, \frac{2}{3} - \frac{\pi}{16} \, \frac{2}{3} + \frac{1}{4} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{6} + \frac{1}{3} + \frac{1}{3} + \frac{\pi}{6} - \frac{\pi}{24} + \frac{\pi}{8} = \frac{5}{12}\pi + \frac{2}{3}$$

Per quanto riguarda il primo membro dell'identità da verificare risulta

$$\operatorname{div} F = 1 + 2y + 2z$$

Mediante le coordinate polari

$$\begin{cases} x = \rho \cos\varphi \cos\vartheta \\ y = \rho \cos\varphi \,\operatorname{sen}\vartheta \\ z = \rho \,\operatorname{sen}\varphi \end{cases} \qquad \begin{array}{l} 0 \le \rho \le 1 \\ 0 \le \vartheta \le \pi/2 \\ 0 \le \varphi \le \frac{\pi}{2} \end{array}$$

$$\det \begin{pmatrix} -\rho\cos\varphi\,\operatorname{sen}\vartheta & \rho\cos\varphi\cos\vartheta & 0 \\ -\rho\,\operatorname{sen}\varphi\cos\vartheta & -\rho\,\operatorname{sen}\varphi\,\operatorname{sen}\vartheta & \rho\cos\varphi \\ \cos\varphi\cos\vartheta & \cos\varphi\,\operatorname{sen}\vartheta & \operatorname{sen}\varphi \end{pmatrix} =$$

$$= \left(+\rho\,\operatorname{sen}^2\varphi\,\operatorname{sen}\vartheta + \rho\cos^2\varphi\,\operatorname{sen}\vartheta\right)\rho\cos\varphi\,\operatorname{sen}\vartheta +$$

$$+ \rho\cos\varphi\cos\vartheta\left(+\rho\,\operatorname{sen}^2\varphi\cos\vartheta + \rho\cos^2\varphi\cos\vartheta\right) =$$

$$= \rho^2\,\operatorname{sen}^2\vartheta\cos\varphi + \rho^2\cos\varphi\cos^2\vartheta = \rho^2\cos\varphi \ge 0.$$

Si ha, pertanto,

$$\iiint_D \operatorname{div} F \,dx\,dy\,dz = \int_0^1\int_0^{\pi/2}\int_0^{\pi/2} \rho^2\cos\varphi\left(1 + 2\rho\cos\varphi\,\operatorname{sen}\vartheta + 2\rho\,\operatorname{sen}\varphi\right) d\rho\, d\vartheta\, d\varphi =$$

$$= \int_0^1\int_0^{\pi/2}\int_0^{\pi/2} \left(\rho^2\cos\varphi + 2\rho^3\cos^2\varphi\,\operatorname{sen}\vartheta + 2\rho^3\,\operatorname{sen}\varphi\cos\varphi\right) d\rho\, d\vartheta\, d\varphi =$$

$$= \frac{\pi}{2}\,\frac{1}{3} + \frac{1}{2}\,\frac{\pi}{4} + \frac{2}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{2}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{4}\right) = \frac{5}{12}\pi$$

In definitiva si ha

$$\iiint_D \operatorname{div} F \; dx\,dy\,dz = \frac{5}{12}\pi$$

$$\iint_{\partial^+ D} \langle F, n_e \rangle \, d\sigma = -\frac{2}{3} + \frac{5}{12}\pi + \frac{2}{3} = \frac{5}{12}\pi$$

e la verifica è compiuta.