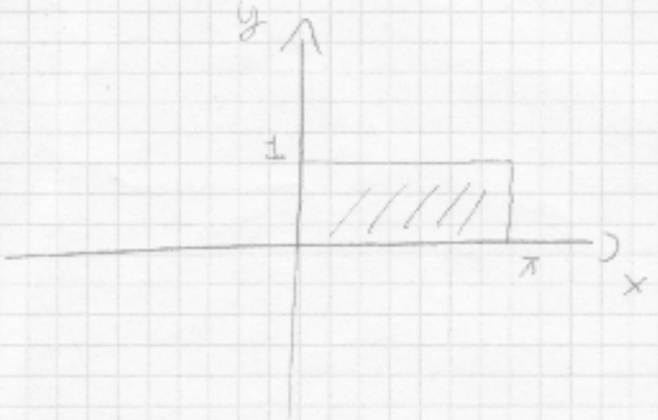
Esercizio 1

Risolvere mediante sep. variabili

$$
\begin{cases}
u_{xx} + u_{yy} = 0 & 0 < x < \pi \quad , 0 < y < 1 \\
u(0,y) = u(\pi,y) = 0 & 0 < y < 1 \\
u(x,0) = \sin 2x & \\
 & 0 < x < \pi \\
u(x,1) = \cancel{4}\sin 3x &
\end{cases}
$$

Problema di Laplace, ben posto, su

$\Omega \subseteq \mathbb{R}^2$, $\Omega = [0\pi] \times [0,1]$



Cerco una soluzione $u(x,y) = f(x)\, g(y)$

$u_{xx} = f''g$

$\Rightarrow$ il problema diventa

$u_{yy} = f g''$

$f''g = -f g''$

$\Rightarrow \dfrac{f''}{f} = -\dfrac{g''}{g}$

Cerco $\lambda \in \mathbb{R}$ t/c

$$\frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)} = -\lambda$$

Quindi devo risolvere

$$f''(x) + \lambda f(x) = 0$$

e $\quad g''(y) - \lambda g(y) = 0$

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1)

$$f''(x) + \lambda f(x) = 0$$

Uso la condizione $\quad u(0,y) = u(\pi,y) = 0$

$\Rightarrow f(0)\, g(y) = f(\pi)\, g(y) = 0 \quad \forall y$

$\Rightarrow f(0) = f(\pi) = 0$

$\longrightarrow$ Devo risolvere il prob. ai limiti

$$\begin{cases} f''(x) + \lambda f(x) = 0 & 0 < x < \pi \\ f(0) = f(\pi) = 0 \end{cases}$$

polinomio caratteristico

$$\alpha^2 + \lambda = 0 \quad \Rightarrow \quad \alpha_{1,2} = \pm\sqrt{-\lambda}$$

<u>Caso 1</u> $\lambda < 0 \Rightarrow \alpha_{1,2} = \pm\sqrt{-\lambda}$ reali

$$\Rightarrow f(x) = c_1 e^{-\sqrt{-\lambda}\,x} + c_2 e^{\sqrt{-\lambda}\,x}$$

impongo $f(0) = f(\pi) = 0$

$$\Rightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 e^{-\sqrt{-\lambda}\,\pi} + c_2 e^{\sqrt{-\lambda}\,\pi} = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ e^{-\sqrt{-\lambda}\,\pi} & e^{\sqrt{-\lambda}\,\ell} \end{vmatrix} = e^{\sqrt{-\lambda}\,\pi} - e^{-\sqrt{-\lambda}\,\pi} \neq 0$$

$\longrightarrow$ non ho autovalori e quindi autofunzioni (cioè sol. non nulle)

<u>Caso 2</u> $\lambda = 0$

$\longrightarrow f''(x) = 0$ + cond. al bordo

$\Rightarrow$ solo sol. nulla

<u>Caso 3</u> $\lambda > 0 \Rightarrow \alpha_{1,2} = \pm i\sqrt{\lambda}$

$$\Rightarrow f(x) = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$$

cond. al bordo

$$\Rightarrow \begin{cases} c_1 = 0 \\ c_1 \cos\sqrt{\lambda}\,\pi + c_2 \sin\sqrt{\lambda}\,\pi = 0 \end{cases}$$

$$a = \begin{vmatrix} 1 & 0 \\ \cos\sqrt{\lambda}\,\pi & \sin\sqrt{\lambda}\,\pi \end{vmatrix} = \sin\sqrt{\lambda}\,\pi$$

$\sin\sqrt{\lambda}\,\pi = 0 \iff \lambda_k = \dfrac{k^2\pi^2}{\pi^2} = k^2$ autovalori

~~$f_k(x) = c_1 \cos kx$ autofunzioni~~

$f_k(x) = c_2 \sin kx$ autosoluzioni

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2) Ora studio $g''(y) - \lambda g(y) = 0$ $\quad \lambda = k^2$

pol. caratteristico

$$\alpha^2 - k^2 = 0 \Rightarrow \alpha = \pm k$$

$\Rightarrow g(y) = a_k e^{ky} + b_k e^{-ky}$ $\quad k > 0$

Metto insieme 1 e 2 e trovo

$k>0$

$$u_k(x,y) = \left(a_k e^{ky} + b_k e^{-ky}\right)\sin kx$$

Anche la somma verifica le stesse equazioni e cond. al bordo

$\Rightarrow$

$$u(x,y) = \sum_{n\geq 1} \left(a_n e^{ny} + b_n e^{-ny} \right) \sin(nx)$$

Devo stabilire i coefficienti in base agli altri dati

Mancano da usare $\quad u(x,0) = \cancel{4} \sin 2x$

$$u(x,1) = 4 \sin 3x$$

$$u(x,0) = \sum_{n\geq 1} (a_n + b_n) \sin nx$$

$$u(x,1) = \sum_{n\geq 1} \left(a_n e^{n} + b_n e^{-n} \right) \sin nx$$

$$\sum_{n\geq 1} (a_n + b_n) \sin nx = \sin 2x$$

$$\Rightarrow \quad a_2 + b_2 = 1$$

$$a_n + b_n = 0 \quad \forall \; n \neq 2$$

$$\sum_{n\geq 1} \left(a_n e^{n} + b_n e^{-n} \right) \sin nx = 4 \sin 3x$$

$$\Rightarrow \quad a_3 e^{3} + b_n e^{-3} = 4$$

$$a_n e^{n} + b_n e^{-n} = 0 \quad \forall \; n \neq 3$$

Se $n \neq 2$, $n \neq 3$

$$\begin{cases} a_n + b_n = 0 \\ a_n e^{n} + b_n e^{-n} = 0 \end{cases} \Rightarrow \begin{cases} a_n = -b_n \\ a_n \left(+e^{n} \cancel{+} e^{-n} \right) = 0 \end{cases}$$

$\Rightarrow \quad a_n = b_n = 0$

Se $n = 2$

$$\begin{cases} a_2 + b_2 = 1 \\ a_2 e^{2} + b_2 e^{-2} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_2 = 1 - b_2 \\ e^{2} - b_2 \left(e^{2} - e^{-2} \right) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_2 = -1/(e^{4}-1) \\ b_2 = \dfrac{e^{2}}{e^{2}-e^{-2}} = \dfrac{e^{4}}{e^{4}-1} \end{cases}$$

Se $n = 3$

$$\begin{cases} a_3 + b_3 = 0 \\ a_3 e^{3} + b_3 e^{-3} = 4 \end{cases} \Rightarrow \begin{cases} a_3 = -b_3 \\ b_3 \left(-e^{3} + e^{-3} \right) = 4 \end{cases}$$

$$\begin{cases} a_3 = -4e^{3}/(1-e^{6}) \\ b_3 = \dfrac{4}{e^{-3}-e^{3}} = \dfrac{4e^{3}}{1-e^{6}} \end{cases}$$

Quinchi

$$u(x,y) = \left(\frac{-e^{2y}}{e^4-1} + \frac{e^4 e^{-2y}}{e^4-1} \right) \cdot \sin 2x +$$

$$+ \left(- \frac{4 e^3 e^{3y}}{1-e^3} + \frac{4 e^3 e^{-3y}}{1-e^3} \right) \sin 3x$$

Esercizio

1

$$\begin{cases} u_t - 3u_{xx} = 0 & 0<x<1 \quad t>0 \\ u(0,t) = u(1,t) = 0 & t>0 \\ u(x,0) = x & 0<x<1 \end{cases}$$

Cauchy-Dirichlet per calore → Ben posto per $t>0$

$u(x,t) = f(x)\,g(t)$

$$\frac{g'(t)}{g(t)} = 3\,\frac{f''(x)}{f(x)} = -\lambda$$

1. $f''(x) + \frac{\lambda}{3} f(x) = 0$

con il dato ho $f(0) = f(1) = 0$

→ non trovo sol. per $\lambda \leq 0$

per $\lambda > 0$

$$f_k(x) = \bar{b}_k \sin k\pi x \qquad \lambda_k = 3k^2\pi^2$$

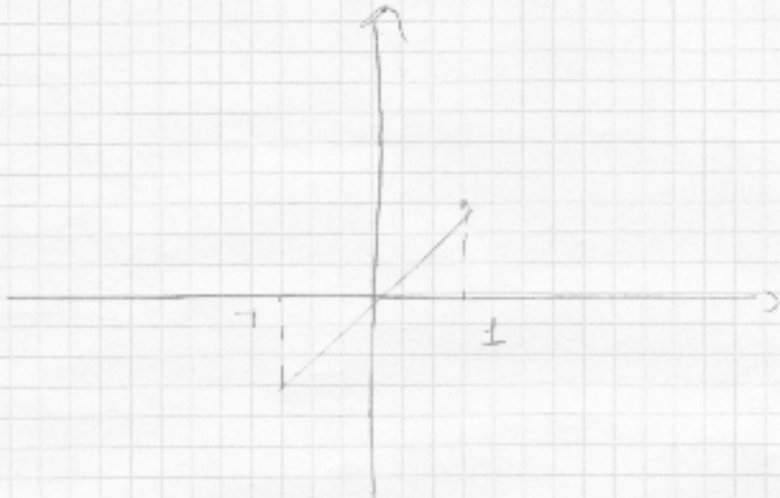
2. $g'(t) = -3k^2\pi^2\, g(t)$

$$\Rightarrow g_k(t) = \gamma_k e^{-3k^2\pi^2 t}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n e^{-3n^2\pi^2 t} \sin(n\pi x)$$ (2)

$\longrightarrow$ è serie di seni $\Rightarrow$ dispari

per usare il dato iniziale lo ~~pro~~ prolungo dispari in $[-1,1]$ e lo sviluppo in serie di Fourier.



$$f(x) = x \qquad x \in [-1,1]$$

$$= \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T}$$

$T = 1 \qquad a_n = 0$ perché f dispari

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x) \sin\frac{n\pi x}{T}\, dx =$$

$$= \int_{-1}^{1} x \sin n\pi x \, dx =$$

$$= \left[-x \frac{\cos n\pi x}{n\pi} \right]_{-1}^{1} + \int_{-1}^{1} \frac{\cos n\pi x}{n\pi}\, dx =$$

③

$$= -\frac{\cos n\pi}{n\pi} - \frac{\cos(-n\pi)}{n\pi} + \left[\frac{\sin n\pi x}{n^2\pi^2}\right]_{-1}^{1} =$$

$$= \frac{-2}{n\pi}\cos n\pi + \frac{\sin n\pi}{n^2\pi^2} - \frac{\sin(-n\pi)}{n^2\pi^2} =$$

$$= \frac{-2}{n\pi}(-1)^n + \frac{2}{n^2\pi^2}\sin n\pi =$$

$$= \frac{-2}{n\pi}(-1)^n$$

$$\Rightarrow \boxed{f(x) = \sum_{n\geq 1}\frac{2(-1)^{n+1}}{n\pi}\cdot\sin n\pi x}$$

$$\Rightarrow u(x,0) = \sum_{n\geq 1} b_n \sin(n\pi x) = x$$

$$\Rightarrow u(x,t) = \sum_{n\geq 1}\frac{2}{n\pi}(-1)^{n+1} e^{-3n^2\pi^2 t}\sin(n\pi x)$$

Esercizio

Mediante l'uso della separazione delle variabili risolvere

$$\begin{cases} u_t - 2u_{xx} = 0 & 0 < x < \pi \quad t > 0 \\ u(x,0) = \cos^4 x & 0 < x < \pi \\ u_x(0,t) = u_x(\pi,t) = 0 & t > 0 \end{cases}$$

Eq. del calore

dato iniziale + Dato di Neumann

$\Rightarrow$ problema ben posto su $\Omega = [0,\pi] \times (0,+\infty)$

Cerco

$$u(x,t) = f(x) \cdot g(t)$$

$$u_t = f g' \qquad u_{xx} = f'' g \quad \Rightarrow \quad u_t - 2u_{xx} = 0 \qquad f(x) g'(t) = 2 f''(x) g(x)$$

$$\Rightarrow \frac{2 f''(x)}{f(x)} = \frac{g'(t)}{g(t)} = -2\lambda \qquad \lambda \in \mathbb{R}$$

Devo risolvere

$$f''(x) + \lambda f(x) = 0$$

$$g'(t) + 2\lambda g(t) = 0$$

1) Risolvo $f''(x) + \lambda f(x) = 0$ ②

Uso il dato $u_x(0,t) = u_x(\pi, t) = 0 \quad t > 0$

$\Rightarrow f'(0) g(t) = f'(\pi) g(t) = 0 \quad \forall t > 0$

$\Rightarrow f'(0) = f'(\pi) = 0$

$\Rightarrow$ Risolvo $\begin{cases} f''(x) + \lambda f(x) = 0 \\ f'(0) = f'(\pi) = 0 \end{cases}$

Polinomio caratteristico $\alpha^2 + \lambda = 0$

$\Rightarrow \alpha_{1,2} = \pm\sqrt{-\lambda}$

- per $\lambda < 0 \longrightarrow$ solo sol. nulla
- per $\lambda = 0$ ho $f''(x) = 0$

$\Rightarrow f'(x) = a$

$\Rightarrow f(x) = ax + b$

uso $f'(0) = f'(\pi) = 0 \Rightarrow a = 0$

$\Rightarrow f(x) = b$ (cost. reale)

- per $\lambda > 0$ $f(x) = c_1 \sin\sqrt{\lambda}\, x + c_2 \cos\sqrt{\lambda}\, x$

$f'(x) = c_1\sqrt{\lambda}\cos\sqrt{\lambda}\, x - c_2\sqrt{\lambda}\sin\sqrt{\lambda}\, x$

$\Rightarrow \begin{cases} c_1\sqrt{\lambda} = 0 \\ c_1\sqrt{\lambda}\cos\pi\sqrt{\lambda} - c_2\sqrt{\lambda}\sin\pi\sqrt{\lambda} = 0 \end{cases}$

$$\Delta = \begin{vmatrix} 1 & 0 \\ \cos\pi\sqrt{\lambda} & -\sin\pi\sqrt{\lambda} \end{vmatrix} = -\sin\pi\sqrt{\lambda}$$

$\Delta = 0 \iff \lambda_k = k^2$ autovalori

$\Rightarrow f_k(x) = c_k \cos kx$ autosoluzioni

<u>Oss</u> se in f_k includo $k=0$ completo

anche $f_k(x) = c_k$ cioè il caso ottenuto

per $\lambda = 0$.

Ora risolvo $g'(t) + 2k^2 g(t) = 0$

$$\frac{d}{dt} g(t) = -2k^2 g(t)$$

$$\int \frac{1}{z}\, dg = \int -2k^2\, dt$$

$$\Rightarrow \log|g| = -2k^2 t + \text{cost}$$

$$\Rightarrow g_k(t) = b_k e^{-2k^2 t}$$

Quindi ottengo

$$u_k(x,t) = a_k \cos kx \, e^{-2k^2 t}$$

$$u(x,t) = \sum_{n \geq 0} a_n e^{-2n^2 t} \cos nx =$$

$$= \frac{a_0}{2} + \sum_{n \geq 1} a_n e^{-2n^2 t} \cos nx$$

Devo usare $u(x,0) = \cos^4 x \qquad 0 < x < 2$

per trovare a_n, $n \geq 0$

<u>Oss</u> so che $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

Quindi

$$\cos^4 x = \cos^2 x \cdot \cos^2 x =$$

$$= \frac{1}{4}\left[(\cos 2x)^2 + 2\cos 2x + 1\right] =$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8}(1 + \cos 4x) =$$

$$= \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

$$u(x,0) = \frac{a_0}{2} + \sum_{n \geq 1} a_n \cos nx = \cos^4 x$$

$$\Rightarrow \frac{a_0}{2} = \frac{3}{8} \qquad a_2 = \frac{1}{2} \qquad a_4 = \frac{1}{8}$$

$$a_n = 0 \quad \forall n \neq \{0, 2, 4\}$$

$$\Rightarrow \boxed{u(x,y) = \frac{3}{8} + \frac{1}{2} e^{-8t} \cos 2x + \frac{1}{8} e^{-32t} \cos 4x}$$