Equazioni lineari di Eulero

$$x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_0 y = g(x)$$

$a_0, \dots a_{n-1} \in \mathbb{R}$

negli intervalli $(0,+\infty)$ oppure $(-\infty, 0)$ si riconduce a un'eq. a coefficienti costanti sostituendo $x = e^t$ per $t \in (0,+\infty)$

$x = -e^t$ per $t \in (-\infty, 0)$

Oss per $x > 0$, integrali particolari del tipo $y = x^\alpha$.

Esercizio (Pagani Salsa, p. 231 n 13 (c))

trovare l'integrale generale di

$$x^3 y''' + x y' - y = \frac{\log x}{x} \qquad x > 0$$

$\Rightarrow$ uso la sostituzione $x = e^t$

Con la sostituzione $x = e^t$, $t = \log x$

$z(t) = y(e^t)$ avremo $\boxed{y(x)} = z(\log x)$

$y(x) = z(\log x)$

$$\frac{d}{dx} y(x) = z' \cdot \frac{1}{x} = \frac{d}{dt} z(\log x) \cdot \frac{1}{x}$$

$$\frac{d^2}{dx} y(x) = z'' \cdot \frac{1}{x} \cdot \frac{1}{x} + z' \left(-\frac{1}{x^2}\right)$$

$$= z'' \cdot \frac{1}{x^2} - z' \frac{1}{x^2} = (z'' - z') \frac{1}{x^2}$$

$$\frac{d^3}{dx} y = \left(z''' \cdot \frac{1}{x} - z'' \frac{1}{x}\right) \frac{1}{x^2} +$$

$$-2 (z'' - z') \frac{1}{x^3} =$$

$$= (z''' - 3z'' + 2z') \frac{1}{x^3}$$

$\Rightarrow x^3 y''' + x y' - y = \frac{\log x}{x}$ diventa

$$x^3 (z''' - 3z'' + 2z') \frac{1}{x^3} + x \left(z' \cdot \frac{1}{x}\right) - z = t e^{-t} \qquad (*)$$

- $z''' - 3z'' + 3z' - z = 0$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$(\lambda - 1)^3 = 0 \quad \Rightarrow \lambda = 1 \quad \text{molt. } 3$$

$$z_0(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$$

$$y_0(x) = z(\log x) =$$

$$= c_1 x + c_2 x \log x + c_3 x (\log x)^2$$

$\longrightarrow$ devo risolvere il problema non omogeneo

uso la $\boxed{\text{variazione delle costanti}}$

cerco la soluzione di ⊛ nella forma

$$z(t) = c_1(t)\, e^t + c_2(t)\, t\, e^t + c_3(t)\, t^2 e^t$$

devo trovare $c_1(t)$, $c_2(t)$, $c_3(t)$:

- calcolo

$$z'(t) = c_1' e^t + c_1 e^t + c_2' t e^t + c_2 (e^t + t e^t) +$$
$$+ c_3'(t)\, t^2 e^t + c_3 (2t e^t + t^2 e^t)$$

condizione 1

$$\boxed{c_1' e^t + c_2' t e^t + c_3' t^2 e^t = 0}$$

$$\Rightarrow z'(t) = c_1 e^t + c_2 (1+t) e^t + c_3 (2t + t^2) e^t$$

- ~~de~~ calcolo

$$z''(t) = c_1' e^t + c_1 e^t + c_2'(1+t) e^t + c_2 \left(e^t + (1+t) e^t\right) +$$
$$+ c_3' (2t + t^2) e^t + c_3 \left((2+2t) e^t + (2t+t^2) e^t\right)$$

condizione 2

$$\boxed{c_1' e^t + c_2' (1+t) e^t + c_3' (2t+t^2) e^t = 0}$$

$\Rightarrow z''(t) = c_1 e^t + c_2 (2+t) e^t + c_3 (2+4t+t^2) e^t$

- calcolo

$z'''(t) = c_1' e^t + c_1 e^t + c_2' (2+t) e^t + c_2 (e^t + (2+t) e^t) +$

$+ c_3' (2+4t+t^2) e^t + c_3 ((4+2t) e^t + (2+4t+t^2) e^t)$

sostituisco z, z', z'', z'' in

$$z''' - 3z'' + 3z' - z = t \cdot e^t$$

$\big[c_1' e^t + c_2' (2+t) e^t + c_3' (2+4t+t^2) e^t +$

$+ c_1 e^t + c_2 (3+t) e^t + c_3 (6+6t+t^2) e^t\big] +$

$+3 \big[c_1 e^t + c_2 (1+t) e^t + c_3 (2t+t^2) e^t\big] +$

$- \big[c_1 e^t + c_2 t e^t + c_3 t^2 e^t\big] +$

$-3 \big[c_1 e^t + c_2 (2+t) e^t + c_3 (2+4t+t^2) e^t\big] = t e^{-t}$

$\rightarrow$ i termini con c_1, c_2, c_3 si cancellano

rimane:

$$\boxed{c_1' e^t + c_2' (2+t) e^t + c_3' (2+4t+t^2) e^t = t e^{-t}}$$

condizione 3

Metto a sistema le 3 condizioni

$$\begin{cases} c_1' e^t + c_2' t e^t + c_3' t^2 e^t = 0 \\ c_1' e^t + c_2' (1+t) e^t + c_3' (2t+t^2) e^t = 0 \\ c_1' e^t + c_2' (2+t) e^t + c_3' (2+4t+t^2) e^t = t e^{-t} \end{cases}$$

$$\Downarrow$$

$$\begin{cases} c_1' + c_2' t + c_3' t^2 = 0 \\ c_1' + c_2' (1+t) + c_3' (2t+t^2) = 0 \\ c_1' + c_2' (2+t) + c_3' (2+4t+t^2) = t e^{-2t} \end{cases}$$

$(2)-(1) \Rightarrow \boxed{c_2'} + 2c_3' t = 0 \Rightarrow c_2' = -2 c_3' t$

$(1) \Rightarrow \boxed{c_1'} = -c_2' t - c_3' t^2 = +2c_3' t^2 - c_3' t^2 = c_3' t^2$

$(3) \Rightarrow c_3' t^2 - 2t c_3' (2+t) + c_3' (2+4t+t^2) = t e^{-2t}$

$\Rightarrow \cancel{c_3' t^2} - \cancel{4t c_3'} - \cancel{2t^2 c_3'} + 2c_3' + \cancel{4t c_3'} + \cancel{t^2 c_3'} = t e^{-2t}$

$\Rightarrow 2 c_3' = t e^{-2t}$

<u>Quindi</u>

$$\begin{cases} c_1' = \frac{t^3}{2} e^{-2t} \\ c_2' = -t^2 e^{-2t} \\ c_3' = \frac{t}{2} e^{-2t} \end{cases}$$

Ora calcolo $c_1(t)$, $c_2(t)$, $c_3(t)$

$$c_3(t) = \int \frac{t}{2} e^{-2t} dt = \dots = -\frac{e^{-2t}}{4}\left(t+\frac{1}{2}\right) + k_3$$

$$c_2(t) = -\int t^2 e^{-2t} = \dots = \frac{e^{-2t}}{2}\left(t^2+t+\frac{1}{2}\right) + k_2$$

$$c_1(t) = \int \frac{t^3}{2} e^{-2t} dt = \dots = \frac{e^{-2t}}{4}\left(-t^3-\frac{3}{2}t^2-\frac{3}{2}t-\frac{3}{4}\right) + k_1$$

Ora uso $z(t) + c_1(t)e^t + c_2(t)te^t + c_3(t)t^2e^t$ e trovo la soluzione di ⊛

$$z(t) =$$

$$= \frac{e^{-t}}{4}\left(-t^3-\frac{3}{2}t^2-\frac{3}{2}t-\frac{3}{4}\right) + k_1 e^t +$$

$$+ \frac{e^{-t}}{2} t \left(t^2+t+\frac{1}{2}\right) + k_2 t e^t +$$

$$- \frac{e^{-t}}{4} t^2 \left(t+\frac{1}{2}\right) + k_3 t^2 e^t$$

Infine trovo la soluzione $y(x)$ del problema iniziale ricordando che $t = \log x$:

$$y(x) = -\frac{\cancel{x}}{4}\left(-(\log x)^3 - \frac{3}{2}(\log x)^2 - \frac{3}{2}(\log x) - \frac{3}{4}\right) +$$

$$+ k_1 x - \frac{x}{2}\log x\left((\log x)^2 + \log x + \frac{1}{2}\right) + k_2 x \log x +$$

$$+ \frac{x(\log x)^2}{4}\left(\log x + \frac{1}{2}\right) + k_3 x (\log x)^2 .$$

(Potrei scrivere raccogliendo!)