<u>Esercizio</u> 2 p. 103

Si calcoli l'integrale:

$$\iiint_S \frac{x^2}{x^2+z^2}\, dx\, dy\, dz$$

f continua e limitata su ~~compatto~~ aperto misurabile

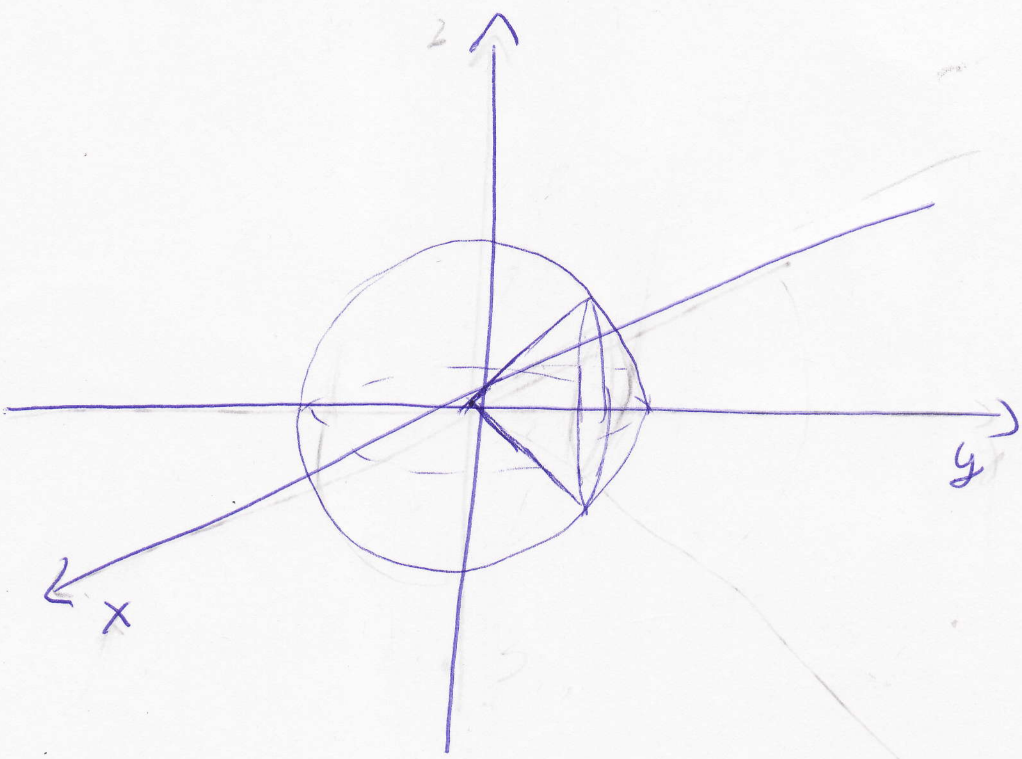
$$S = \{ x^2 + y^2 + z^2 < 2 \ , \ x^2 - y^2 + z^2 < 0 \ , \ y > 0 \}$$

<u>Oss</u> $C_1 : x^2 + y^2 + z^2 = 2$

→ circ. centro origine e raggio $\sqrt{2}$

$C_2 : x^2 - y^2 + z^2 = 0$

→ cono, con asse di simmetria ≡ asse y



Coord. cilindriche

$$\begin{cases} x = \rho \cos\theta \\ y = y \\ z = \rho \sin\theta \end{cases}$$

$$\boxed{0 \leq \theta < 2\pi}$$

→ Dove variano y e ρ

$C_1 : \quad y^2 + \rho^2 = 2$

$C_2 : \quad \rho^2 - y^2 = 0$

ρ varia da 0 a un massimo che si ha quando il cono $\cap$ la sfera

$$C_1 \cap C_2 \Rightarrow \begin{cases} y^2 = 2 - \rho^2 \\ y^2 = \rho^2 \end{cases} \Rightarrow \begin{cases} \rho^2 = 1 \\ y^2 = 1 \end{cases} \Rightarrow \rho = 1$$

$$\Rightarrow \boxed{0 < \rho < 1}$$

y varia dal cono C_2 : $y^2 = \rho^2 \Rightarrow y = \rho$

alla sfera $C_1 \Rightarrow y^2 = 2 - \rho^2 \Rightarrow y = \sqrt{2 - \rho^2}$

$$\Rightarrow \boxed{\rho < y < \sqrt{2 - \rho^2}}$$

• Quindi S diventa

$$\{ (\rho, y, \theta) \ / \ 0 \le \theta < 2\pi, \ 0 < \rho < 1, \ \rho < y < \sqrt{2-\rho^2} \}$$

• Sappiamo che lo Jacobiano è ρ

• $f(x, y, z) = \dfrac{x^2}{x^2 + z^2}$

$\Rightarrow f(\rho, y, \theta) = \dfrac{\rho^2 \cos^2\theta}{\rho^2} = \cos^2\theta$

$$\Rightarrow \iiint_S f \, dx \, dy \, dz =$$

$$\int_0^1 \rho \left(\int_\rho^{\sqrt{2-\rho^2}} \left(\int_0^{2\pi} \cos^2\theta \, d\theta \right) dy \right) d\rho =$$

$$= \underbrace{\left(\int_0^{2\pi} \cos^2\theta \, d\theta \right)}_{A} \cdot \underbrace{\left[\int_0^1 \rho \left(\int_\rho^{\sqrt{2-\rho^2}} dy \right) d\rho \right]}_{B}$$

$$A = \int_0^{2\pi} \cos^2\theta \, d\theta = \cancel{\tfrac{1}{2}} \pi$$

$$B = \int_0^1 \rho \left(y \right]_{\rho}^{\sqrt{2-\rho^2}} d\rho =$$

$$= \int_0^1 \rho \left(\sqrt{2-\rho^2} - \rho \right) d\rho =$$

$(2-\rho^2)^{1/2}$

$$= \int_0^1 \rho \sqrt{2-\rho^2}\, d\rho - \int_0^1 \rho^2\, d\rho$$

$$= \left[-\frac{1}{\cancel{2}} (2-\rho^2)^{3/2} \frac{\cancel{2}}{3} \right]_0^1 - \left[\frac{\rho^3}{3} \right]_0^1 =$$

$$= -\frac{1}{3} + \frac{2\sqrt{2}}{3} - \frac{1}{3} = \frac{2(\sqrt{2}-1)}{3}$$

$$\Rightarrow \iiint_S f = \frac{2}{3}\pi \left(\sqrt{2} - 1 \right)$$