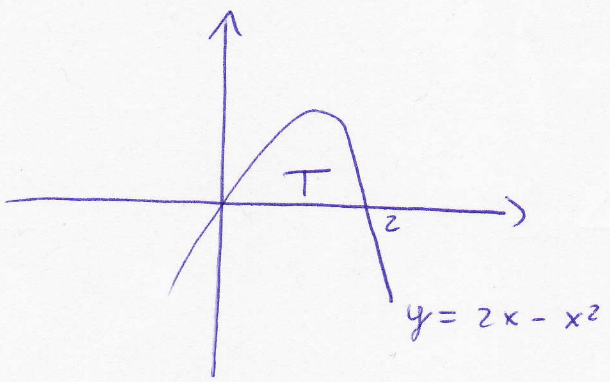
Esercizio p. 79 n 1

a) $\iint_T xy$ $\qquad T = \{(x,y) \in \mathbb{R}^2 / 0 \le y \le 2x - x^2\}$



$$\iint_T xy = \int_0^2 x\left(\int_0^{2x-x^2} y\,dy\right)dx = \int_0^2 x\left[\frac{y^2}{2}\right]_0^{2x-x^2} dx =$$

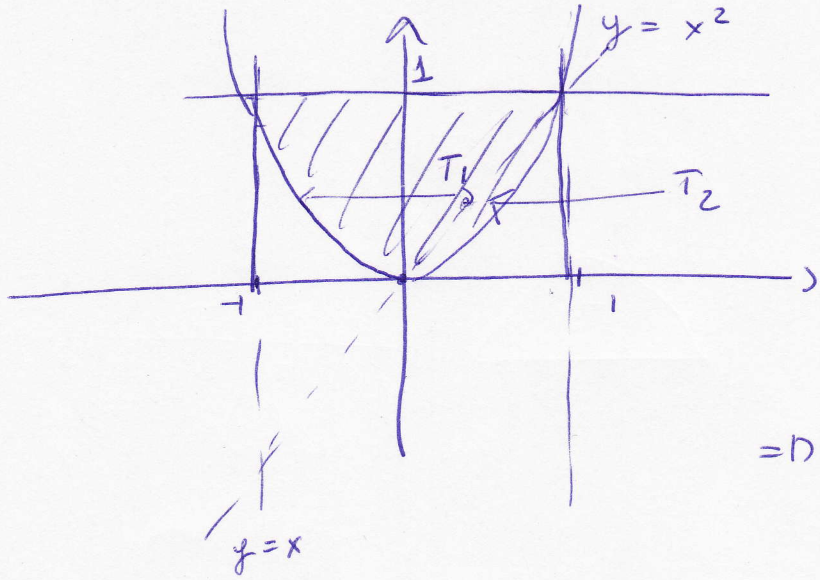
$$= \int_0^2 \frac{x}{2}\left(4x^2 + x^4 - 4x^3\right)dx = \int_0^2 \left(2x^3 + \frac{x^5}{2} - 2x^4\right)dx =$$

$$= \left[\frac{x^4}{2} + \frac{x^6}{12} - \frac{2x^5}{5}\right]_0^2 = 2^3 + \frac{2^4}{3} - \frac{2^6}{5} = \frac{8}{15}$$

Nota f è integrabile su T xchè f è continua

T $\underbrace{\text{compatto}}_{\text{(chiuso e limit. in } \mathbb{R}^2)}$ e misurabile

c) $\iint_T |y-x|$ $\qquad T = \{-1 \le x \le 1,\ x^2 \le y \le 1\}$



$y = x^2$

$\Rightarrow x = \pm \sqrt{y}$

Nota disc. in 0, altrimenti ok come prima

separo in due casi

$y - x > 0 \quad]\ T_1$

$y - x < 0 \quad]\ T_2$

Integrabile xché limitata su D mis. e continua tranne che in 0

$$\iint_{T_1} f(x,y) = \int_0^1 \int_{\sqrt{y}}^{y} (y-x)\,dx\,dy =$$

$$= \int_0^1 \left[yx - \frac{x^2}{2} \right]_{-\sqrt{y}}^{y} dy =$$

$$= \int_0^1 \left(y^2 - \frac{y^2}{2} + \overbrace{y\sqrt{y}}^{\to y^{3/2}} + \frac{y}{2} \right) dy =$$

$$= \left[\frac{y^3}{6} + \frac{2y^{5/2}}{5} + \frac{y^2}{4} \right]_0^1 = \frac{1}{6} + \frac{2}{5} + \frac{1}{4} = \cancel{\frac{10+24+15}{60}} = \cancel{\frac{49}{60}}$$

$$\iint_{T_2} f(x,y) = \int_0^1 \left(\int_{x^2}^{x} (x-y)\,dy \right) dx =$$

$$= \int_0^1 \left[xy - \frac{y^2}{2} \right]_{x^2}^{x} dx = \int_0^1 \left(x^2 - \frac{x^2}{2} - x^3 + \frac{x^4}{2} \right) dx =$$

$$= \left[\frac{x^3}{6} - \frac{x^4}{4} + \frac{x^5}{10} \right]_0^1 = \frac{1}{6} - \frac{1}{4} + \frac{1}{10}$$

$$\Rightarrow \iint_T f(x,y) = \frac{1}{6} + \frac{2}{5} + \cancel{\frac{1}{4}} + \frac{1}{6} - \cancel{\frac{1}{4}} + \frac{1}{10} =$$

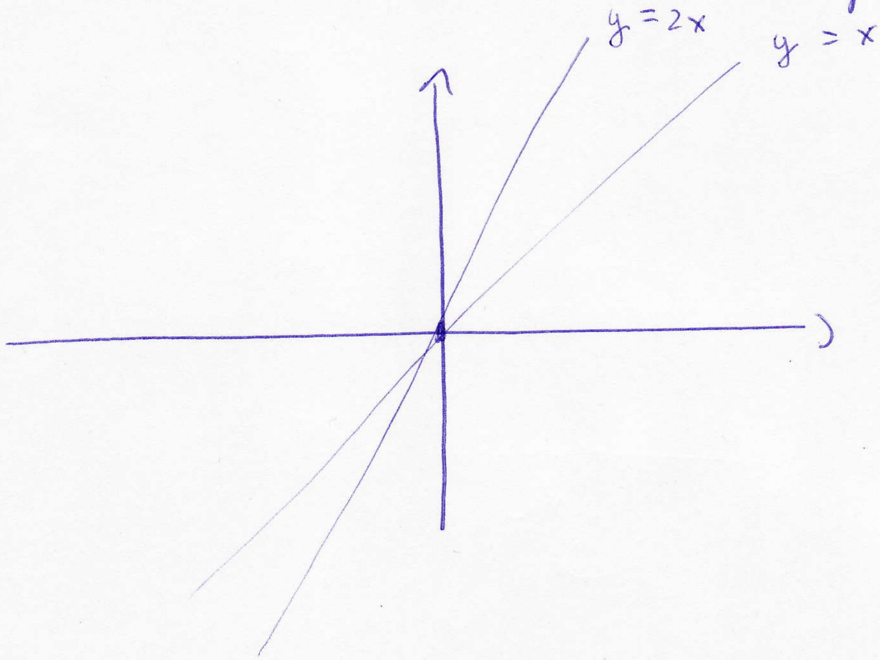
$$= \frac{10 + 24 + 10 + 6}{60} = \frac{50}{60} = \frac{5}{6}$$

Esercizio p. 81 n 7

Si calcoli $\iint_T xy \sin(xy)\,dx\,dy$

$T = \{ (x,y) \in \mathbb{R}^2 \,/\, x \leq y \leq 2x ,\ \frac{2}{x}\pi \leq y \leq \frac{3\pi}{x} ,\ x > 0 \}$

utilizzando un cambiamento di variabili che trasformi T in un rettangolo

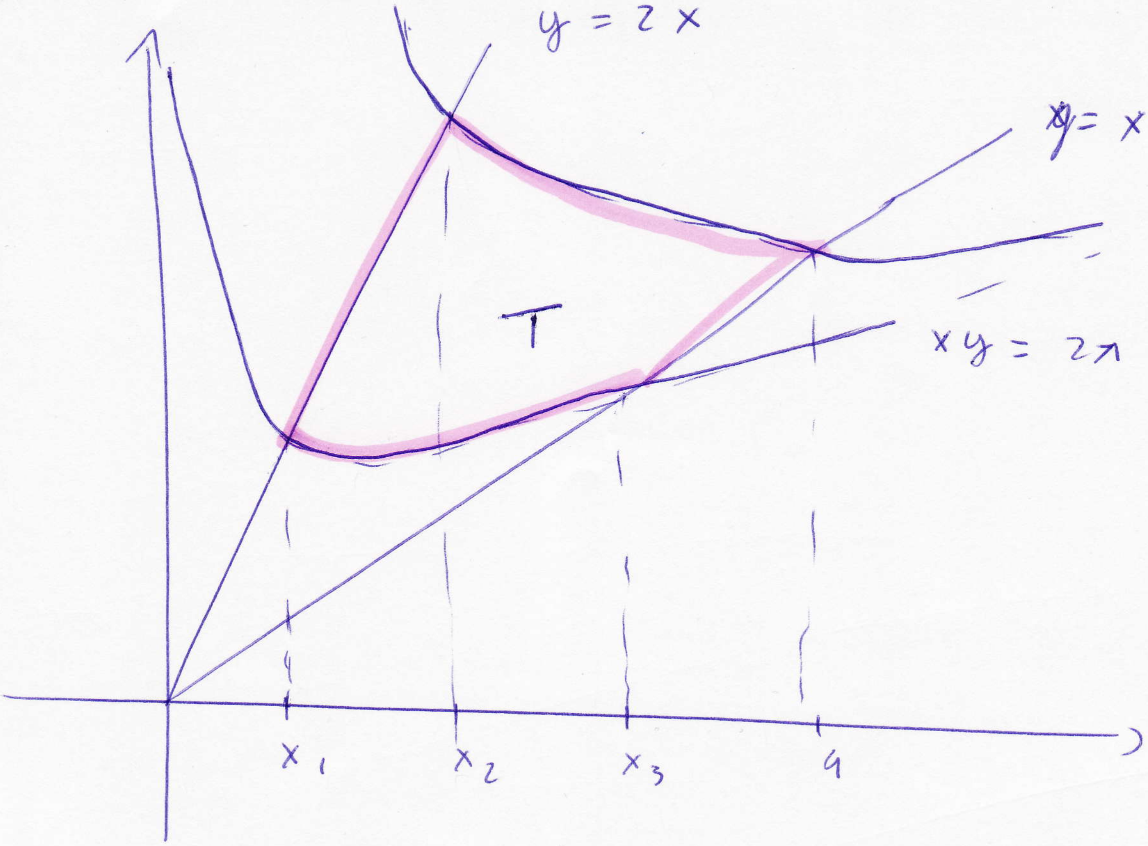


$\frac{2}{x}\pi = y \Rightarrow xy = 2\pi$ iperbole

$y = \frac{3\pi}{x} \Rightarrow xy = 3\pi$ iperbole

Cerco intersezioni con $x = y$

$y = 2x$



$$\begin{cases} xy = 2\pi \\ x = y \end{cases} \Rightarrow x^2 = 2\pi \Rightarrow x_3 = \sqrt{2\pi}$$

$$\begin{cases} xy = 2\pi \\ y = 2x \end{cases} \Rightarrow x_1 = \sqrt{\pi}$$

$$\begin{cases} xy = 3\pi \\ x = y \end{cases} \Rightarrow x_4 = \sqrt{3\pi}$$

$$\begin{cases} xy = 3\pi \\ 2y = x \end{cases} \Rightarrow x_2 = \sqrt{\frac{3}{2}\pi}$$

Oss $\quad x \le y \le 2x \Rightarrow 1 \le \frac{y}{x} \le 2$

$$\frac{2\pi}{x} \le y \le \frac{3\pi}{x} \Rightarrow 2\pi \le xy \le 3\pi$$

Uso la trasformazione $T: D \to T(D)$

$(u,v) \to (x,y)$

$$\begin{cases} u = xy \\ v = y/x \end{cases} \qquad D = [2\pi, 3\pi] \times [1,2]$$

$$\Rightarrow \begin{cases} x = \sqrt{u/v} = \varphi(u,v) \\ y = \sqrt{u \cdot v} = \psi(u,v) \end{cases}$$

T deve essere

(i) $T \in C^1(D)$ con deriv. parziali limitate

(ii) $\left|\frac{\partial(\varphi,\psi)}{\partial(u,v)}\right| \neq 0$ in D

$$\left|\frac{\partial(\varphi,\psi)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{1}{2v\sqrt{u/v}} & \frac{-u}{2v^2\sqrt{u/v}} \\ \frac{v}{2\sqrt{uv}} & \frac{u}{2\sqrt{uv}} \end{vmatrix} = \frac{1}{2v} \neq 0 \text{ in D}$$

$\Rightarrow$ l'integrale diventa:

$f(x,y) = xy\sin(xy)$

$f(u,v) = u \cdot \sin u$

$$\iint_T f(x,y)\,dx\,dy =$$

$$= \iint_D f(u,v) \cdot \left|\frac{\partial(\varphi,\psi)}{\partial(u,v)}\right| du\,dv =$$

$$= \int_1^2 \left(\int_{2\pi}^{3\pi} \frac{u}{2v} \sin u \, du \right) dv =$$

$$= \int_1^2 \frac{1}{2}\left[(-u\cos u)_{2\pi}^{3\pi} - \int_{2\pi}^{3\pi} \cos u \, du \right] dv = \int_1^2 \frac{5\pi}{2v}\, dv = \boxed{\frac{5\pi}{2}\log 2}$$