Chapter 1

Introduction to PDEs

1.1 Basic concepts and Examples

The *differential equations* are the class of equations involving derivatives of unknown functions. When the unknown function in the equation depends only on a single variable, the equation involves only the *ordinary derivatives* of the unknown function, we call it an *ordinary differential eqution*, or in short *ODE*. Quite often, the unknown functions depend on several independent variables, and the equations involving the *partial derivatives* of the unknown functions, then the equations are called the *partial differential equations*, or in short *PDEs*. In this course, the independent variables will always be formed by a *time variable* $t \geq 0$ and a *space variable* $x \in \mathbb{R}^n$, with n the spatial dimensions. In general, a PDE of the unknown function $u(x_1, x_2, \cdots, x_n)$ takes the form

$$F(x, u, Du, u_{x_1x_1}, u_{x_1x_2}, \cdots, u_{x_nx_n}, \cdots) = 0, \tag{1.1}$$

where $x = (x_1, x_2, \cdots, x_n)$, $Du = (u_{x_1}, u_{x_2}, \cdots, u_{x_n})$, and F is a function of the independent variable x and the unknown function u and finitely many partial derivatives of u. The equation (1.1) is called m-th *order*, if the *highest* order of the derivatives of u in (1.1) is m. An m-th order PDE of $u(x_1, x_2, \cdots, x_n)$ has a general form

$$F(x, u, Du, D^2u, \cdots, D^m u) = 0. \tag{1.2}$$

Examples of Partial differential Equations:

(a) Heat equation $u_t = u_{xx}$

(b) Linear transport eqution $u_t + cu_x = 0$, $c \in \mathbb{R}$

(c) Wave equation $u_{tt} - c^2 u_{xx} = 0$, $c > 0$

(d) Burgers' equation $u_t + uu_x = 0$

(e) Viscous Burgers' equation $u_t + uu_x = u_{xx}$

(f) Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$

We now consider the equation (1.2). In addition to its order, we also classify the equations by the relations between function F and u, as well as the derivatives of u. When the dependence of F on u and its derivatives is *linear*, we say (1.2) is linear, otherwise we say it is *nonlinear*. In the above examples, the (d) and (e) are nonlinear while the others are linear. In the class of nonlinear PDEs, if F is linear with respect to the highest (m-th) derivatives of u, i.e., the coefficients of the m-th order derivatives of u in (1.2) only depend on the independent variables x, u, and the derivatives of u up to $(m-1)$-th order, we call the equation is *quasi-liner*. In the quasi-linear case, if the coefficients of the highest derivatives of u are functions of independent variable x only, the equation is then called *semi-linear*. The general linear PDE of m-th order takes the following form:

$$P(x, D)u = f(x),$$

where, $P(x, D)$ is a general m-th order differential operator defined in (1.8) below. The equation is called *homogeneous* if $f(x) = 0$.

We now introduce the concept of *solution*. Again, consider the equation (1.2), and we restrict ourself on the domain $x \in \Omega \subset \mathbb{R}^n$. $u = \phi(x)$ is a solution of (1.2) on Ω, if $\phi(x)$ and all of its derivatives appears in (1.2) are continuous in Ω, and after substituting it into the equation, it makes (1.2) an identity. Such a solution, we will refer it as a *classical solution*.

The following theorem gives an important properties for solutions to a linear PDE.

Theorem 1.1.1 (Principle of superposition) *Let u_1 and u_2 be solutions to a given homogeneous linear PDE . Then for any constants λ and μ,*

$$\lambda u_1 + \mu u_2$$

is also a solution of that equation.

1.2 Well-posed problems

Just as what happened in ODEs, a PDE, if solvable, often has many solutions. However, as most equations have their physical or practical backgrounds, some conditions or constraints are necessary to determine a unique solution to the realistic problem. In general, there are two classes of "side" conditions: *the initial condition* and *the boundary conditions*. Some problems involve a mixture of initial and boundary conditions, called *initial-boundary value problems*.

We first start with the case of *evolutionary* PDEs, (or in which a time variable is involved). In this case, the initial conditions often involve the initial value of the unknown function, and the derivatives up to the next order of the highest time derivatives of the unknown function. The initial conditions (also called *initial data*) together with the PDE form an *intial value problem*, called *Cauchy problem*. The following example shows a typical Cauchy problem for a wave equation in one spatial dimension:

Example 1.2.1

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x). \end{cases} \tag{1.3}$$

When the problem is confined in a given domain, the constraints on the boundary are often needed for the problem. There are many different kinds of boundary conditions depending on the realistic applications. In this course, we will mainly discuss three typical boundary conditions. The first one is to give the value of the unknown function itself at the boundary of the domain. This type of boundary condition is called *Dirichlet condition*. The corresponding boundary value problem is thus called *Dirichlet problem*. The following example is a typical Dirichlet problem for a Poisson equation, describing the electronic field, with u the electronic potential and ρ the electronic density distribution.

Example 1.2.2

$$\begin{cases} \Delta u = -4\pi\rho(x,y,z),\ (x,y,z) \in \Omega \subset \mathbf{R}^3, \\ u(x,y,z) = \phi(x,y,z),\ (x,y,z) \in \partial\Omega. \end{cases} \tag{1.4}$$

Here Δ is the well-known Laplace operator. In $\mathbb{R}^n$, it takes the form

$$\Delta = \sum_i^n \partial_i^2.$$

The second type is often called the *Neumann condition*, which assigns the normal derivative of the unknow function at the boundary. The third type is the *Robin condition*, which gives the nontrivial linear combination of the unknown function itself and its normal derivative at the boundary. The next example shows a Neumann problem of Laplace equation in 3 dimension.

Example 1.2.3

$$\begin{cases} \Delta(u^m) = 0,\ (x,y,z) \in \Omega \subset \mathbf{R}^3, \\ \nabla u \bullet \nu = \phi(x,y,z),\ (x,y,z) \in \partial\Omega, \end{cases}$$

where ν is the outer unit normal to $\partial\Omega$.

The following example is an initial-boundary value problem for heat equation involving the Robin boundary condition.

Example 1.2.4

$$\begin{cases} u_t - \Delta u = 0,\ (x,y,z) \in \Omega \subset \mathbf{R}^3,\ t > 0, \\ u - \frac{1}{2}\nabla u \bullet \nu = \phi(x,y,z),\ (x,y,xz) \in \partial\Omega,\ t > 0, \\ u(x,y,z,0) = u_0(x,y,z),\ (x,y,z) \in \Omega. \end{cases}$$

As the problems we encounter often arise from real applications, we thus expect each of our problem is uniquely solvable and the solution continuously depends on the relevant data. In this case, we say our problem is *well-posed*. However, it is very delicate to tell whether a problem is well-posed without any further techniques. The following example is due to J. Hadamard.

Example 1.2.5 *Consider the set of "initial-value" problems in the upper half-plane in* $\mathbf{R}^2$*, for* $n = 1, 2, \cdots$,

$$\begin{cases} \Delta u = 0, \ \ y > 0, \\ u(x,0) = 0, \ \ u_y(x,0) = \dfrac{\sin(nx)}{n}, x \in \mathbf{R}, \end{cases} \tag{1.5}$$

and

$$\begin{cases} \Delta u = 0, \ \ y > 0, \\ u(x,0) = 0, \ \ u_y(x,0) = 0, x \in \mathbf{R}. \end{cases} \tag{1.6}$$

The problem (1.5) has a unique solution

$$u^n(x,y) = \frac{1}{n^2}\sinh(ny)\sin(nx),$$

and the problem (1.6) has a unique soluton

$$u^0(x,y) = 0.$$

We note that, as $n \to \infty$*, the data of (1.5) tends uniformly to the data of (1.6). However, one has*

$$\lim_{n\to\infty} \sup |u^n(x,y) - u^0(x,y)| = \infty, \ \ y > 0$$

for each (x, y)*. Thus, arbitrarily small changes in the data lead to large changes in the solution, this is called* instability*.*

1.2.1 Characteristic and initial value problems

We see from last example that the Cauchy problem for a Laplace equation is not well-posed. This section is devoted to the study of how to set the initial value problem in a proper way so that the initial data is compatible with the structure of the PDE. The notion of *characteristic* plays an essential role in this context.

Let's consider a rather general linear PDE of order m:

$$P(x, D)u = g(x), \tag{1.7}$$

where

$$P(x, D) = \sum_{|\alpha|=m} a_\alpha(x)D^\alpha + \sum_{\alpha<m} a_\alpha(x)D^\alpha + a(x). \tag{1.8}$$

Here $x = (x_0, x_1, \cdots, x_n) \in \mathbf{R}^{n+1}$, α is a multi-index, $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n)$, where each α_i is a non-negative integer. $|\alpha| = \sum \alpha_i$, is the length of α. $D = (D_0, D_1, \cdots, D_n)$ is a differential operator where $D_i = \frac{\partial}{\partial x_i}$, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_0^{\alpha_0} \cdots \partial x_n^{\alpha_n}}.$$

We also define $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. We call

$$\sum_{|\alpha|=m} a_\alpha(x) D^\alpha$$

the *principal part* of the operator $P(x, D)$.

From the *principle of superposition*, we know that the solutions of (1.7) can be obtained from a particular solution of (1.7) and a general solution of the following *homogeneous* equation:

$$P(x, D) = 0. \tag{1.9}$$

Assume that equation (1.9) is defined in a neighborhood of a smooth n-dimensional surface S given by $f(x) = 0$. The initial value problem (or, Cauchy problem) for (1.9) consists of assigning u and its derivatives of order $\leq (m-1)$ on S, and it is required to solve (1.9) in a neighborhood of S. For instance, in Example 1.2.1, the initial surface is $S = t = 0$, and the initial conditions are $u(x, 0) = \phi(x)$, and $u_t(x, 0) = \psi(x)$, and we shall try to solve the equation

$$u_{tt} - c^2 u_{xx} = 0$$

in the upper half plane $\{t > 0\}$.

We now proceed to solve (1.9) as follows. First, we shall make proper change of coordinates in a neighborhood of S to distinguish the *time-like* direction so that we know in which direction we will solve our problem. This can be done by introducing the new independent variables $y_0, y_1, \cdots, y_n$, where $y_1, \cdots, y_n$ are independent coordinates on S, and $y_0 = f(x)$; i.e., S corresponds to $y_0 = 0$. Note that u is given on S, all the derivatives of u with respect to the new variables $y_1, \cdots, y_n$ on S.

By *chain rule*, we have, for $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = m$,

$$D^\alpha u = \frac{\partial^\alpha u}{\partial x_0^{\alpha_0} \cdots \partial x_n^{\alpha_n}} = \frac{\partial^m u}{\partial y_0^m} \left(\frac{\partial y_0}{\partial x_0}\right)^{\alpha_0} \cdots \left(\frac{\partial y_0}{\partial x_n}\right)^{\alpha_n} + \cdots ,$$

where the last dots represent derivatives of u with respect to y_0 of orders $< m$, together with derivatives of u with respect to y_i, $i = 1, \cdots, n$, and are thus all *known quatities*. The equation (1.9) becomes

$$\sum_{|\alpha|=m} a_\alpha(x) \left(\frac{\partial y_0}{\partial x_0}\right)^{\alpha_0} \cdots \left(\frac{\partial y_0}{\partial x_n}\right)^{\alpha_n} \frac{\partial^m u}{\partial y_0^m} + \cdots , \tag{1.10}$$

where the dots are quantities known on S. Now it is clear that, if we want to solve this equation for $\frac{\partial^m u}{\partial y_0^m}$, it is necessary and sufficient that

$$\sum_{|\alpha|=m} a_\alpha(x) \left(\frac{\partial y_0}{\partial x_0}\right)^{\alpha_0} \cdots \left(\frac{\partial y_0}{\partial x_n}\right)^{\alpha_n} \neq 0,$$

or equivalently, that

$$\sum_{|\alpha|=m} a_\alpha(x)(\nabla f(x))^\alpha = \sum_{|\alpha|=m} a_\alpha(x) \left(\frac{\partial f(x)}{\partial x_0}\right)^{\alpha_0} \cdots \left(\frac{\partial f(x)}{\partial x_n}\right)^{\alpha_n} \neq 0, \forall x \in S. \tag{1.11}$$

When (1.11) fails to hold, the initial-value problem would be unreasonable, and in this case we say S is a *characteristic surface* of $P(x, D)$. Formally, we have the following defintion.

Definition 1.2.6 *The surface $S = \{x : f(x) = 0\}$ is said to be* characteristic *at a point $p \in S$ for the operator $P(x, D)$ defined in (1.8) if*

$$\sum_{|\alpha|=m} a_\alpha(x)(\nabla f(x))^\alpha|_{x=p} = 0.$$

S is a characteristic surface for $P(x, D)$ if it is characteristic at each point of S. The equation

$$\sum_{|\alpha|=m} a_\alpha(x)\sigma^\alpha = 0, \tag{1.12}$$

with $\sigma = (\sigma_0, \sigma_1, \cdots, \sigma_n)$, is called the characteristic equation *for the operator $P(x, D)$ in (1.8).*

By these terms, we know that a surface S is characteristic at $p \in S$, for the operator (1.8), provided that the normal vector to S at p satisfies the characteristic equation (1.12). We remark that if $f(x) = 0$ is a characteristic surface for the operator (1.8), (1.11) shows that the differential equation (1.9) imposes an additional restrictions on the data; namely, the *known quantities*, denoted by dots in (1.11) must vanish. Similar statement can be made to the equation (1.7).

In the following, we assume σ is a unit normal vector given at a point of S, i.e.,

$$\sum_{k=0}^{n} \sigma_k^2 = 1. \tag{1.13}$$

Now, we discuss several examples.

Example 1.2.7 *The* 3*-dimensional wave equation:*

$$u_{x_0x_0} - \sum_{k=1}^{3} u_{x_kx_k} = 0.$$

(Here, one usually denotes x_0 as t) The characteristic equation is

$$\sigma_0^2 - \sum_{k=1}^{3} \sigma_k^2 = 0,$$

which together with (1.13) gives $\sigma_0 = \pm\frac{1}{\sqrt{2}}$. Therefore, a surface is characteristic for the 3-D wave equation if and only if its normal makes an angle of $\frac{\pi}{4}$ with respect to the x_0 axis.

Example 1.2.8 *Consider the $(n+1)$-dimensional Laplace equation*

$$\sum_{k=0}^{n} u_{x_k x_k} = 0.$$

Here the characteristic equation reads

$$\sum_{k=0}^{n} \sigma_k^2 = 0$$

which is incompatible with (1.13). Therefore, there are no (real) characteristics for Laplace equation.

Example 1.2.9 *Consider the following first-order linear equation*

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y).$$

The characteristic equation is

$$a\sigma_0 + b\sigma_1 = 0.$$

Solving this together with (1.13) gives

$$(\sigma_0, \sigma_1) = \pm\frac{1}{\sqrt{a^2+b^2}}(b, -a).$$

Therefore, the characteristic curves are solutions of the following system

$$\begin{cases} \dot{x} = a(x,y) \\ \dot{y} = b(x,y). \end{cases}$$

This example is very useful in the next Chapter.

1.3 Classifications of second order semilinear PDEs

In this section, we further discuss the types of PDEs, for which the different types of equations often require different methods to resolve. As the decisive coefficients of quisa-linear PDEs depend on solutions, we will discuss semi-linear equations. Roughly speaking, the classifications of the semi-linear PDEs depend on how informations propagate. If the information carried by solution propagates at a finite speed, we call it *hyperbolic*; if the information propagates at an infinite speed, we call it *parabolic*; if there is no (real) speed for the information to travel with, we call it *elliptic*. From the knowledge in the previous section, we know the latter often links to the problem without reasonable time-like direction, or the problem is static.

1.3.1 Equations with several variables

To be precise, for $n \geq 2$, we will now discuss only the second order semi-linear PDEs of the following form

$$\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + F(x,u,Du) = 0, \tag{1.14}$$

where, $x = (x_1, x_2, \cdots, x_n) \in \Omega$ with Ω an open subset in $\mathbb{R}^n$, $a^{ij} = a^{ji}$ as we expect $u_{x_i x_j} = u_{x_j x_i}$. The *linear principal part* of this equation at a fixed point $p \in \Omega$ is

$$\sum_{i,j=1}^{n} a^{ij}(p)u_{x_i x_j}(p), \tag{1.15}$$

which corresponds to a quadratic form

$$\sum_{i,j=1}^{n} a^{ij}(p)\xi_i\xi_i = 0,\ \forall \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n \tag{1.16}$$

This is the left hand side of the characteristic equation of (1.14). By linear algebra, we know that the numbers of positive, negative and zero (real) eigenvalues of the matrix $A = (a^{ij}(p))$ is invariant under the transformation of the form $P^T AP$ for any invertible $n \times n$ matrix P. there is an orthogonal matrix O such that

$$O^T AO = diag\{\lambda_1, \cdots, \lambda_n\}, \tag{1.17}$$

and the transformation $\xi = Oy$ changes the quadratic form into *diagonal form*

$$Q(y) = \sum_{i=1}^{n} \lambda_i y_i^2. \tag{1.18}$$

The way to transform $Q(\xi)$ into its diagonal form is not unique. Upon re-scalings, we see there is an invertible matrix U such that the transformation $\xi = Uy$ changes $Q(\xi)$ into its *standard form*

$$Q(y) = \sum_{i=1}^{n} \chi_i y_i^2. \tag{1.19}$$

Here, $\chi_i = 1$, -1, or 0. (By Shur's lemma of linear algebra, we know that the numbers of positive, negative and zero (real) eigenvalues of the matrix $A = (a^{ij}(p))$ is invariant under the transformation of the form $P^T AP$ for any invertible $n \times n$ matrix P.) If we fix such a U, the transformation

$$y = P^T x$$

will transfer the equation (1.14) into its *standard form*:

$$\sum_{i=1}^{n} \chi_i u_{x_i x_i} + \cdots = 0, \tag{1.20}$$

where the dots terms contain at most first order derivatives.

Now, we are able to give a precise classification for (1.14):

Definition 1.3.1 *For (1.14) we have the following classifications:*

- *If all the eigenvalues of A have the same sign(all positive or all negative), we say (1.14) is* elliptic *at p.*
- *If A has zero eigenvalues, we call (1.14)* parabolic *at p.*
- *If $(n-1)$ eigenvalues of A have the same sign different from the other one, we say (1.14) is* hyperbolic *at p. If both the number of positive eigenvalues and the number of negative eigenvalues of A are greater than 1, and A had no zero eigenvalue, we call (1.14)* super hyperbolic*.*

Example 1.3.2 *The following equation is super hyperbolic:*

$$u_{x_1x_1} + u_{x_2x_2} - u_{x_3x_3} - u_{x_4x_4} = 0.$$

If the equation (1.14) is elliptic (or parabolic, or hyperbolic respectively) at each point in Ω, we say (1.14) is elliptic (or parabolic, or hyperbolic respectively). If (1.14) is of different types on different points of Ω, we call (1.14) of *mixed type*.

Example 1.3.3 *Tricomi equation*

$$yu_{xx} + u_{yy} = 0,$$

is of mixed type on any region including points on the x-axis.

Unfortunately, when the independent variables are more than 2, there are examples showing that no matter how small the region Ω is, there does not exist a single change of variables such that the equation (1.14) is transfered into a single type. However, for only two independent variables, under mild conditions on the coefficients of the equation, it is possible to make a change of variables such that the equation (1.13) was of the same type on the whole region Ω (very small sometime).

1.3.2 The case of two variables

Consider the semi-linear PDE of the second order with independent variables x and y

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = F(x,y,u,u_x,u_y), \tag{1.21}$$

where F, a, b and c are smooth functions with respect to their arguments. We also assume that a, b and c are not all zero at any point on the working region Ω. According to the classifications in last sub-section, equation (1.21) is classified as the sign of the following determinant:

$$d := \det A = \left| \begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix} \right| = a(x,y)c(x,y) - b^2(x,y). \tag{1.22}$$

We then have the following cases:

- If $d > 0$, equation (1.21) is *elliptic*. A typical example of elliptic equations is the Laplace equation
$$u_{xx} + u_{yy} = 0,$$
where $d = 1$.

- If $d < 0$, equation (1.21) is *hyperbolic*. A typical example of hyperbolic equation is the wave equation
$$u_{tt} - c^2 u_{xx} = 0, \qquad c > 0,$$
where $d = -c^2$.

- If $d = 0$ and this matrix A is not identically zero, the equation (1.21) is *parabolic*. A typical example of parabolic equation is the Heat equation
$$u_t - \alpha u_{yy} = 0, \qquad \alpha > 0.$$

We now describe how to transfer (1.21) into standard form. The linear principal part of (1.21) is
$$L_0 u = a u_{xx} + 2b u_{xy} + c u_{yy}. \tag{1.23}$$
For $(x, y) \in \Omega$, and any invertible smooth change of variables
$$\xi = \xi(x, y),\ \ \eta = \eta(x, y),\ \ \frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0, \tag{1.24}$$
it is easy to compute that the linear principal part of (1.21) becomes
$$L_0 u = a^* u_{\xi\xi} + 2b^* u_{\xi\eta} + c^* u_{\eta\eta}, \tag{1.25}$$
where
$$\left\{\begin{array}{l} a^* = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 \\ b^* = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y \\ c^* = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2. \end{array}\right. \tag{1.26}$$
Of course, we require the second order differentiability of ξ and η. It is now clear that if $a^* = c^* = 0$ and $b^* \neq 0$, the equation (1.25) is of hyperbolic type, and it has a simple form
$$L_0 u = 2b^* u_{\xi\eta},$$
which is called *second standard form* of hyperbolic equations. In this case, we know that ξ and η are solutions to the following equation
$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = 0. \tag{1.27}$$
This is exact the characteristic equation of L_0. $\phi(x, y) = constant$ defines implicitly a family of curves on xy-plane as $y = y(x)$ (or, $x = x(y)$ if necessary), which satisfies the following ODE
$$a(\frac{dy}{dx})^2 - 2b\frac{dy}{dx} + c = 0. \tag{1.28}$$

We note that (1.27) is a fully nonlinear PDE for ϕ while (1.28) is an (nonlinear) ODE which is easier to solve. As long as $\|\nabla\phi\| \neq 0$, these two equations are equivalent in the sense that $\phi(x, y) = constant$ is an implicit solution to (1.28).

From the theory of classification in last sub-section and (1.27)–(1.28), we see that (1.28) will not give any real nontrivial solution to (1.27) in the elliptic region. While in the hyperbolic region, (1.28) gives two distinct families of real characteristic curves. Finally, in the parabolic set of points, (1.28) gives only one family of real characteristic curves.

We now show how to utilize the characteristic curves to transfer (1.21) into standard form. We will do so case by case.

First of all, assume that $d < 0$ in Ω, so (1.21) is hyperbolic. We solve (1.28) to obtain two distinct families of characteristic directions

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}. \tag{1.29}$$

Integrating them, we find two families of characteristic curves

$$\phi_1(x, y) = c_1, \;\; \phi_2(x, y) = c_2.$$

When $\phi_{ix}^2 + \phi_{iy}^2 \neq 0$ for $i = 1, 2$, (1.29) implies that

$$\frac{\partial(\phi_1, \phi_2)}{\partial(x, y)} \neq 0.$$

Therefore, after the change of variables

$$\xi = \phi_1(x, y), \;\; \eta = \phi_2(x, y),$$

one obtains (1.25) with $a^* = c^* = 0$ and $b^* \neq 0$. (1.21) becomes

$$u_{\xi\eta} + \tilde{F}(\xi, \eta, u, u_\xi, u_\eta) = 0,$$

which, after a further change of variables into s and t such that

$$\xi = \frac{1}{2}(s + t), \;\; \eta = \frac{1}{2}(s - t),$$

becomes the standard form

$$u_{ss} - u_{tt} + F_1(s, t, u, u_s, u_t) = 0,$$

for some smooth F_1.

Secondly, we assume that $d > 0$ so that (1.21) is elliptic. We know that it is impossible to obtain real-valued characteristic curves from (1.28). However, we are able to solve (1.28) for a complex-valued solution

$$\phi(x, y) = \phi_1(x, y) + i\phi_2(x, y) = constant,$$

where ϕ_1 and ϕ_2 are real functions, and $i = \sqrt{-1}$ is the imaginary unit. It is not hard to show that if ϕ_x and ϕ_y don't vanish at the same time, the following transformation

$$\xi = \phi_1(x, y), \;\; \eta = \phi_2(x, y),$$

satisfies

$$\frac{\partial(\phi_1, \phi_2)}{\partial(x, y)} \neq 0.$$

Substituting $\xi + i\eta$ into (1.27), separating the real and imaginary part, one discovers $a^* = c^* \neq 0$ and $b^* = 0$. Therefore, (1.21) takes the following standard form

$$u_{\xi\xi} + u_{\eta\eta} + G(\xi, \eta, u, u_\xi, u_\eta) = 0,$$

for some smooth function G.

Finally, we assume that $d = 0$ and thus (1.21) is parabolic. It is clear that $ac = b^2$, and one can assume $a > 0$ and $c > 0$ without loss of the generality since $b \neq 0$. From (1.28) we obtain

$$\frac{dy}{dx} = \sqrt{\frac{c}{a}}.$$

Solve this equation, we have the family $\phi(x, y) = c_3$. Since $\|\nabla\xi\| \neq 0$, $\xi = \phi(x, y) \not\equiv$ *constant*. Choose appropriate $\eta = \eta(x, y)$ such that

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0.$$

For instance, if $\eta = x$, one has

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = -\xi_y \neq 0.$$

Otherwise, if $\xi_y = 0$, equation (1.27) implies $\xi_x = 0$, therefore $\xi \equiv$ *constant*, a contradiction. For some isolated points where $\xi_y = 0$, one choose Ω to exclude these points.

Now, under such a transformation, (1.21) becomes standard form

$$u_{\eta\eta} + G_1(\xi, \eta, u, u_\xi, u_\eta) = 0$$

for some smooth function G_1.

In the following, we show some examples.

Example 1.3.4 *Discuss the type of the following equation*

$$u_{xx} + y u_{yy} = 0 \tag{1.30}$$

and change it into standard form.

Solution: We compute $d = y$. Therefore, this equation is elliptic in upper half plane $y > 0$, is hyperbolic in lower half-plane $y < 0$, and is parabolic on x-axis. The ODE associate to the characteristic equation is

$$(\frac{dy}{dx})^2 + y = 0.$$

Case 1: When $y = 0$, one substitute $y = 0$ into (1.30), one gets the standard form

$$u_{xx} = 0.$$

The characteristic curve in this case is the integral curve of $\dfrac{dy}{dx} = 0$, and that is x-axis since $y = 0$.

Case 2: In the hyperbolic region where $y < 0$, we solve the ODE and found

$$\xi = x + 2\sqrt{-y} = c_1, \ \eta = x - 2\sqrt{-y} = c_2,$$

and thus the transformation

$$\xi = x + 2\sqrt{-y}, \ \eta = x - 2\sqrt{-y},$$

transfers, with some standard calculations, into standard form

$$u_{\xi\eta} + \frac{1}{2(\xi - \eta)}(u_\xi - u_\eta) = 0, \ y < 0.$$

The characteristic curves in this case are two branches of the parabola $y = -\frac{1}{4}(x - C)^2$, where C is an arbitrary constant. The one with positive slope is corresponding to $\xi = \mathit{constant}$ while the one with negative slope is corresponding to $\eta = \mathit{constant}$. Both branches are tangent to x-axis.

Case 3: In the elliptic region $y > 0$, the ODE will become a pair of conjugate complex equations

$$\frac{dy}{dx} \pm i\sqrt{y} = 0.$$

Solving the one with plus sign (you can choose either one), we obtain

$$x - 2i\sqrt{y} = c.$$

The real and imaginary parts are

$$\xi = x, \ \eta = 2\sqrt{y},$$

which is our desired transformation. After some calculations, we arrive at

$$u_{\xi\xi} + u_{\eta\eta} - \frac{1}{\eta}u_\eta = 0, \ y > 0.$$

Example 1.3.5 *Discuss the type of the following equation*

$$yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0, \tag{1.31}$$

and find the general solution when $x \neq y$.

Solution: To solve the problem, we first compute

$$d = xy - \frac{1}{4}(x+y)^2 = -\frac{1}{4}(x-y)^2 \leq 0.$$

Therefore, when $x = y$, the equation(1.31) is parabolic; when $x \neq y$, the equation (1.31) is hyperbolic. In the latter case, we substitute the parameters into (1.28) to find either

$$\frac{dy}{dx} = 1$$

or

$$\frac{dy}{dx} = \frac{x}{y}.$$

Therefore, the two families of characteristic curves are

$$y - x = c_1, \text{ and } y^2 - x^2 = c_2.$$

Which tells us, if we perform the transformation

$$\xi = y - x, \ \eta = y^2 - x^2,$$

the Jacobian satisfies

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = 2(x-y) \neq 0$$

in hyperbolic region. Furthermore, the equation (1.31) takes the second standard form

$$u_{\xi\eta} + \frac{1}{\xi}u_\eta = 0,$$

which is equivalent to

$$(\xi u_\eta)_\xi = 0.$$

Now, we integrate the above equation with respect to ξ,

$$\xi u_\eta = f(\eta),$$

where f is any integrable function. Therefore,

$$\begin{aligned} u &= \frac{1}{\xi}\int f(\eta)\, d\eta + g(\xi) \\ &= g(y-x) + \frac{1}{y-x}h(y^2-x^2), \end{aligned}$$

where g and h are C^2 functions. This formula gives the general solution of (1.31) on the hyperbolic region where $x \neq y$.

1.4 Problems

Problem 1. For the following equations, identify which are linear, semi-linear, quasi-liner, or fully nonlinear, and their orders.

- (a) $u_{tt} - \Delta u = 0$;
- (b) $u_t + u_{xxxx} = 0$;
- (c) $div(|Du|^{p-2}Du) = 0$;
- (d) $div\left(\dfrac{Du}{(1+|Du|^2)}\right) = 0$;
- (e) $u_t + divF(u) = 0,\ F : \mathbb{R} \to \mathbb{R}^n$;
- (f) $u_t - \Delta(u^\gamma) = 0$;
- (g) $u_t - \Delta u = f(u)$;
- (h) $\begin{cases} u_t + u \cdot Du - \Delta u + Dp = 0, \\ divu = 0;\ u \in \mathbb{R}^3, x \in \mathbb{R}^3 \end{cases}$;
- (i) $u_t + divF(u) = 0,\ F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$.

Problem 2. Let $\Omega = \{(x,y) : 0 < x < 1, 0 < y < 1\}$, and consider the following boundary value problem

$$\begin{cases} u_{xx} - u_{yy} = 0,\ (x,y) \in \Omega, \\ u(x,0) = f_1(x),\ u(x,1) = f_2(x), \\ u(0,y) = g_1(y),\ u(1,y) = g_2(y). \end{cases}$$

where f_1, f_2, g_1 and g_2 are given functions. Is this problem well-posed?

Problem 3. Find the type of the following equation

$$3u_{xx} - 2u_{xy} + 2u_{yy} - 2u_{yz} + 3u_{zz} + 5u_y - u_x + 10u = 0.$$

Transfer it into its standard form.

Problem 4. Identify the types of the following equations:

- (a) $xu_{xx} + 2yu_{xy} + yu_{yy} = 0$;
- (b) $u_{xx} + (x-y)^3 u_{yy} = 0$;
- (c) $yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$;

- (d) $\sin(x)u_{xx} - 2\cos(x)u_{xy} - (1+\sin(x))u_{yy} = 0$;
- (e) $e^z u_{xy} - u_{xx} = \log(x^2 + y^2 + z^2 + 1)$.
- (f) $7u_{xx} - 10u_{xy} - 22u_{yz} + 7u_{yy} - 16u_{xz} - 5u_{zz} = 0$.

Problem 5. Transfer the following equations into standard form.

- (a) $x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0$;
- (b) $u_{xx} + xyu_{yy} = 0$;
- (c) $u_{xx} - 2\cos(x)u_{xy} - (3+\sin^2(x))u_{yy} - yu_y = 0$;
- (d) $y^2 u_{xx} - e^{\sqrt{2x}} x u_{xy} + u_x = 0,\ x > 0$.

Problem 6. Determine the type of Tricomi equation

$$u_{xx} + xu_{yy} = 0,$$

and then transfer it into standard form.

Problem 7. Transfer the following equation

$$u_{xx} + yu_{yy} + \frac{1}{2}u_y = 0$$

into standard form. Find the general solution.

Problem 8. Show that for any second order hyperbolic or elliptic PDEs with two variables and constant coefficients, one can always combine the change of variables (1.24) and the the following transformation of unknown function

$$u = v\exp(\lambda\xi + \mu\eta)$$

to obtain a form of

$$v_{\xi\xi} \pm v_{\eta\eta} + cv = f.$$

Problem 9. Based on the problem 8, classify the following equations and transfer them into a standard form without first order derivatives.

- (a) $u_{xx} + 4u_{xy} + 3u_{yy} + 3u_x - u_y + 2u = 0$;
- (b) $u_{xx} + 2u_{xy} + u_{yy} + 5u_x + 3u_y + u = 0$;
- (c) $u_{xx} - 6u_{xy} + 12u_{yy} + 4u_x - u = \sin(xy)$.

Problem 10. Make the change of unknown function $u = v + w$ with v the new unknown functions, such that the following problems have homogeneous boundary conditions. Where (a) has Neumann boundary condition, (b) has Dirichlet boundary condition, while (c) has one Neumann condition and one Robin condition.

$$\text{(a)} \begin{cases} u_{tt} - c^2 u_{xx} = 0,\ 0 < x < +\infty,\ t > 0, \\ u_x(0,t) = g(t),\ t \geq 0, \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x),\ 0 \leq x < +\infty; \end{cases}$$

$$\text{(b)} \begin{cases} u_{tt} - c^2 u_{xx} = 0,\ 0 < x < l,\ t > 0, \\ u(0,t) = \mu(t),\ u(l,t) = \nu(t),\ t \geq 0, \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x),\ 0 \leq x < l; \end{cases}$$

$$\text{(c)} \begin{cases} u_{tt} - c^2 u_{xx} = 0,\ 0 < x < l,\ t > 0, \\ -u_x(0,t) = \mu(t),\ u_x(l,t) + u(l,t) = \nu(t),\ t \geq 0, \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x),\ 0 \leq x < l. \end{cases}$$

Problem 11. Determine w such that the change of unknown function $u = vw$ changes the equation

$$u_t - u_{xx} + a u_x + b u = f(x,t)$$

into

$$v_t - v_{xx} = f_1(x,t).$$

Problem 12. Assume u is a solution of the heat equation

$$u_t - a^2 u_{xx} = 0,$$

with the form

$$u(x,t) = \tilde{u}\left(\frac{x}{\sqrt{t}}\right).$$

Derive the ODE for $\tilde{u}$, and then solve the following problem

$$\begin{cases} u_t - a^2 u_{xx} = 0,\ 0 < x < +\infty,\ t > 0, \\ u(0,t) = 0,\ t \geq 0, \\ u(x,0) = u_0,\ 0 \leq x < +\infty, \end{cases}$$

where u_0 is a constant.