Chapter 2

The first order quasi-linear PDEs

The first order quasi-linear PDEs have the following general form:

$$F(x, u, Du) = 0, \tag{2.1}$$

where $x = (x_1, x_2, \cdots, x_3) \in \mathbb{R}^n$, $u = u(x)$, Du is the gradient of u. Such equation often appears in physical applications, such as the famous transport equation, Burgers' equation and the Hamilton-Jacob equation

$$\alpha u_t + H(Du) = 0, \tag{2.2}$$

where $\alpha \geq 0$. The distinguished feature of this class is that one can solve it through a system of ODEs at least locally. Such a method is called *method of characteristic*. We will first introduce the general theory in two variables in first two sections, then we generalize it into general cases. We then discuss in details on the semi-linear case in two variables with some examples. The transport equation will be presented in fourth section with a direct approach.

2.1 Characteristic curves and integral surfaces

Consider the following first order quasi-linear PDE with two variables:

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \tag{2.3}$$

Here, $u = u(x, y)$. (2.3) can be rewritten in the following form

$$(a, b, c) \bullet (u_x, u_y, -1) = 0. \tag{2.4}$$

Keeping this equation in mind, the vector field (a, b, c) (called *characteristic direction*) therefore plays an essential role in solving (2.3). On some interval $I \subset \mathbb{R}$, this vector field determines the following parametric curves (called *characteristic curves*)

$$\gamma : x = x(t),\ y = y(t),\ z = z(t),\ t \in I \subset \mathbb{R},$$

through the following equation

$$\frac{dx}{a(x,y,z)} = \frac{dy}{b(x,y,z)} = \frac{dz}{c(x,y,z)},$$

i.e.

$$\begin{cases} \dfrac{dx}{dt} = a(x,y,z) \\ \dfrac{dy}{dt} = b(x,y,z) \\ \dfrac{dz}{dt} = c(x,y,z). \end{cases} \tag{2.5}$$

We call (2.5) the *characteristic equation* of (2.3). We see such curves, if exist, have the characteristic direction as tangent vectors. We remark that, from calculus, the integral surface $S : z = u(x,y)$, as the solution of (2.3), is tangent to the characteristic direction everywhere. The following theorem states the relationship between γ and S.

Theorem 2.1.1 *If there is a point $P = (x_0, y_0, z_0)$ of the characteristic curve γ lying on the integral surface $S : z = u(x,y)$, then $\gamma \subset S$.*

Proof: Assume that

$$\gamma : x = x(t),\ y = y(t),\ z = z(t),\ t \in I \subset \mathbb{R},$$

is a solution of (2.5) such that for some parameter $t = t_0$, it holds that

$$x_0 = x(t_0),\ y_0 = y(t_0),\ z_0 = z(t_0) = u(x_0, y_0).$$

Furthermore, we know that

$$P = (x_0, y_0, z_0) \in S.$$

Define

$$w = w(t) = z(t) - u(x(t), y(t)).$$

It is easy to check that $w(t)$ is the solution to the following initial value problem

$$\begin{cases} \dfrac{dw}{dt} = c(x,y,w+u) - u_x a(x,y,w+u) - u_y b(x,y,w+u), \\ w(t_0) = 0. \end{cases} \tag{2.6}$$

Clearly, $w \equiv 0$ is a solution of (2.6) since $z = u(x,y)$ is a solution of (2.3). Due to ODE theory, we know that (2.6) has a unique solution. Therefore, $w(t) \equiv 0$, namely, $z(t) = u(x(t), y(t))$, which means $\gamma \subset S$.

From the definition of S, we know that, for each point on S, there is a characteristic curve passing through. Therefore, S is the union of all characteristic curves. Now, it is not hard to see from argument we made above that we did prove the following result.

Theorem 2.1.2 *The general solution of a first-order, quasi-linear PDE (2.3) is*

$$f(\phi, \psi) = 0,$$

where f is an arbitrary function of $\phi(x, y, u)$ and $\psi(x, y, u)$ and ϕ = constant = c_1 and ψ = constant = c_2 are solution curves of

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)},$$

or equivalently, equation (2.5) replacing z with u.

2.2 Cauchy problem

We now discuss the initial value problem for (2.3). We know from section 1.2 that, in order to have the well-posed Cauchy problem, one should not prescribe the initial data on the characteristic curves. The following theorem confirms this observation.

Theorem 2.2.1 *Suppose that $x_0(s)$, $y_0(s)$ and $u_0(s)$ are continuous differentiable functions of s in a closed interval $0 \le s \le 1$ and that a, b and c are functions of x, y and u with continuous first order partial derivatives with respect to their arguments in some domain D of (x, y, z)-space containing the initial curve*

$$\gamma : x = x_0(s),\ y = y_0(s),\ u = u_0(s), \tag{2.7}$$

where $s \in [0, 1]$, and satisfying the condition

$$\left| \begin{pmatrix} a(x_0(s), y_0(s), u_0(s)) & b(x_0(s), y_0(s), u_0(s)) \\ x_0'(s) & y_0'(s) \end{pmatrix} \right| \neq 0. \tag{2.8}$$

Then, there exists a unique solution $u = u(x, y)$ of (2.3) in the neighborhood of $C : x = x_0(s),\ y = y_0(s),\ s \in [0, 1]$, and the solution satisfies the initial condition

$$u_0(s) = u(x_0(s), y_0(s)),\ s \in [0, 1]. \tag{2.9}$$

Remark 2.2.2 The curve C, on which the Cauchy data is prescribed, is the projection of the initial curve γ onto the (x, y)-plane. The condition (2.8) is precisely equivalent to the curve C is not characteristic in the sense of section 1.2.

Roughly speaking, a proof of this theorem can be described as follows: One starts with a time $t = 0$ and solve the equation (2.5) with the initial data (2.7) which gives a unique solution

$$x = X(s, t),\ y = Y(s, t),\ z = Z(s, t),\ s \in [0, 1],\ t \in [0, T], \tag{2.10}$$

for some positive T, such that

$$(X(s, 0), Y(s, 0), Z(s, 0)) = (x_0(s), y_0(s), u_0(s)).$$

Now, with the condition (2.8), one can solve (s,t) in terms of x and y to obtain

$$s = \phi(x,y),\ t = \psi(x,y).$$

Substituting these into $z = Z(s,t)$, one has

$$u = Z(s,t) = Z(\phi(x,y), \psi(x,y)) \equiv u(x,y),$$

the solution of the Cauchy problem. Of course, in general, the solutions exist only in a neighborhood of the initial curve γ, and such a solution is call a *local* solution. The rigorous proof of this theorem involves the *implicit function theorem* and we shall not pursuit it here.

2.3 General cases

We now generalize the theory we established in the previous two sections to the following general first order quasi-linear PDEs:

$$\sum_{k=1}^{n} a_k(x_1, x_2, \cdots, x_n, u) u_{x_k} = c(x_1, x_2, \cdots, x_n, u),\ n \geq 2. \tag{2.11}$$

The corresponding system of ODEs for characteristic curves reads

$$\begin{cases} \dfrac{dx_k}{dt} = a_k(x_1, x_2, \cdots, x_n, z),\ k = 1, \cdots, n, \\ \dfrac{dz}{dt} = c(x_1, x_2, \cdots, x_n, z). \end{cases} \tag{2.12}$$

The corresponding Cauchy problem is thus to find the integral surface of (2.11) $z = u(x_1, x_2, \cdots, x_n)$ such that it pass through the following parametrized $(n-1)$ dimension initial surface S:

$$S : \begin{cases} x_k = f_k(s_1, s_2, \cdots, s_{n-1}),\ k = 1, \cdots, n, \\ z = u_0(s_1, s_2, \cdots, s_{n-1}). \end{cases} \tag{2.13}$$

To proceed, we could first find the characteristic curves passing through a point on S with parameter $s = (s_1, \cdots, s_{n-1})$ to obtain the solution of (2.12)

$$\begin{cases} x_k = X_k(s_1, s_2, \cdots, s_{n-1}, t),\ k = 1, \cdots, n, \\ z = Z(s_1, s_2, \cdots, s_{n-1}, t), \end{cases} \tag{2.14}$$

which satisfies the initial datum given in (2.13). Now, similar to (2.8), we propose the condition

$$J = \det \begin{pmatrix} a_1 & a_2 & \cdots & a_n(s) \\ \frac{\partial f_1}{\partial s_1} & \frac{\partial f_2}{\partial s_1} & \cdots & \frac{\partial f_n}{\partial s_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial s_{n-1}} & \frac{\partial f_2}{\partial s_{n-1}} & \cdots & \frac{\partial f_n}{\partial s_{n-1}} \end{pmatrix} \Bigg|_S \neq 0, \tag{2.15}$$

which is equivalent to that the initial surface is not characteristic. Under this condition (2.15), one can solve from (2.14) for $s = (s_1, s_2, \cdots, s_{n-1})$ and t in terms of $x_1, \cdots, x_n$, and then upon to a substitution, the solution $u = Z(x_1, \cdots, x_n)$ to (2.11) with initial condition (2.13) is obtained.

2.4 Some examples

In this section, we will show some examples on how to apply the method of characteristic established so far to solve some Cauchy problems on first order quasi-linear PDEs.

Example 2.4.1 Consider the semi-linear equation

$$a(x, y)u_x + b(x, y)u_u = f(x, y, u), \tag{2.16}$$

where the functions a, b, and f are smooth functions. In this case, we could simply solve the following system of ODEs

$$\frac{dx}{dt} = a(x, y), \ \frac{dy}{dt} = b(x, y),$$

which determines a family of curves $(x(t), y(t))$ on (x, y)-plane. This family of curves are actually the projection of the characteristic curves on to (x, y)-plane. Then we solve the equation

$$\frac{dz}{dt} = f(x(t), y(t), z)$$

to obtain the characteristic curve γ: $(x(t), y(t), z(t))$.

Example 2.4.2 We now solve the following Cauchy problem

$$\begin{cases} u_t + vu_x = 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{2.17}$$

where $v > 0$ is a constant, and u_0 is a regular function. First of all, one easily verifies that the initial curve

$$\gamma : t = 0, \ x = \tau, \ z = u_0(x)$$

is not characteristic, as the determinant defined in (2.8) is 1.

In view of Example 2.4.1, we solve the following

$$\begin{cases} \dot{x}(s) = v, \\ \dot{t}(s) = 1, \end{cases} \tag{2.18}$$

where " $\cdot$ " indicates the derivative with respect to the parameter s. By imposing the initial conditions at $s = 0$,

$$\begin{cases} x(0) = x_0, \\ t(0) = t_0, \end{cases}$$

we determine a unique solution for (2.18) in the explicit form

$$\begin{cases} x(s) = vs + x_0, \\ t(s) = s + t_0. \end{cases}$$

Since the initial condition is assigned on $t = 0$, which implies (x_0, t_0) is a point of the curve and $t_0 = 0$. So the characteristic curve may be rewritten, eliminating the parameter s, as

$$x(t) = vt + x_0. \tag{2.19}$$

This is a line in (x, t)-plane intersecting $\{t = 0\}$ at x_0 with the slope $\dfrac{1}{v}$ or, equivalently *speed* v.

Now let $\phi(t) = u(x(t), t)$ be a solution of (2.17). Along the characteristic curve (2.19) we have:

$$\dot{\phi}(t) = \dot{x}(t)u_x(x(t), t) + u_t(x(t), t) = vu_x(x(t), t) + u_t(x(t), t) = 0,$$

i.e., *the solution of (2.17) is constant along the characteristics (2.19).* So you can resolve the ordinary differential equation for ϕ:

$$\phi(t) = \phi(0),$$

i.e.

$$u(vt + x_0, t) = u(x(t), t) = u(x(0), 0) = u_0(x_0).$$

To determine the value of solution u at a generic point in (x, t)-plane, one simply determines the point x_0 of intersection between the characteristic passing through (x, t) and the axis $\{t = 0\}$, that is, from $x = vt + x_0$ to obtain

$$x_0 = x - vt.$$

Ultimately, the solution of (2.17) is given by

$$u(x, t) = u_0(x - vt).$$

Example 2.4.3 In this example, we solve the following Cauchy problem

$$\begin{cases} x\dfrac{\partial u}{\partial x} + 2y\dfrac{\partial u}{\partial u} + \dfrac{\partial u}{\partial z} = 3u, \\ u(x, y, 0) = \phi(x, y). \end{cases} \tag{2.20}$$

On the initial surface

$$S : x = s_1,\ y = s_2,\ z = 0,\ w = \phi(s_1, s_2),$$

we compute

$$J = \det \begin{pmatrix} x & 2y & 1 \\ \dfrac{\partial x}{\partial s_1} & \dfrac{\partial y}{\partial s_1} & \dfrac{\partial z}{\partial s_1} \\ \dfrac{\partial x}{\partial s_2} & \dfrac{\partial y}{\partial s_2} & \dfrac{\partial z}{\partial s_2} \end{pmatrix} = \det \begin{pmatrix} s_1 & 2s_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = -1 \neq 0.$$

Therefore, we expect the problem (2.20) has a unique solution near S. We solve the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = x, \ \dfrac{dy}{dt} = 2y, \\ \dfrac{dz}{dt} = 1, \ \dfrac{dw}{dt} = 3w, \\ (x, y, z, w)|_{t=0} = (s_1, s_2, 0, \phi(s_1, s_2)), \end{cases}$$

to obtain

$$x = s_1 e^t, \ y = s_2 e^{2t}, \ z = t, \ w = \phi(s_1, s_2) e^{3t}.$$

From the first three equations in the above, we have

$$t = z, \ s_1 = xe^{-z}, \ s_2 = ye^{-2z}.$$

Therefore, the solution of the problem (2.20) is

$$w = u(x, y, z) = \phi(xe^{-z}, ye^{-2z}) e^{3z}.$$

Example 2.4.4 Solve the following problem

$$\begin{cases} tu_t + xu_x = cu, \ x \in \mathbb{R}, \ t \geq 0, \\ u(x, 1) = f(x), \end{cases} \tag{2.21}$$

where c is a constant.

For this problem, we note that the coefficients (x, t, c) is singular at $(0, 0, c)$ for $c \neq 0$, where the equation does not make sense except for $u \equiv 0$. The initial curve is

$$\gamma : \ x = s, \ t = 1, \ z = f(s),$$

which is not characteristic if $s \neq 0$. Solving the system of ODEs

$$\frac{dx}{d\tau} = x, \ \frac{dt}{d\tau} = t, \ \frac{dz}{d\tau} = cz,$$

one obtains the characteristic curves

$$x(\tau, s) = se^{\tau}, \ t(\tau, s) = e^{\tau}, \ z(\tau, s) = f(s) e^{c\tau}.$$

Therefore,

$$s = \frac{x}{t}, \tau = \log t$$

and the solution is

$$u(x,t) = z = f(\frac{x}{t})t^c.$$

We note that, in the particular case where $f \equiv$ *constant*, we see from the above formula that if $c > 0$ we have solution $u(x,t)$ for all $t \geq 0$. On the other hand, if $c < 0$, we see that $u(x,t) \to \infty$ as t tends to zero. In the latter case, we say the solution *blows up* at $t = 0$.

We now use this example to explain what will happen if the initial curve is characteristic. Now, we fix $c = 1$, and choose the initial curve

$$\gamma_1 : x(0,\tau) = \tau,\ t(0,\tau) = \frac{\tau}{\alpha},\ z(0,\tau) = \frac{\beta}{\alpha}\tau$$

for $s = 0$, and $\alpha \neq 0$, β are arbitrary constants. One easily checks that both $u(x,t) = \frac{\beta}{\alpha}x$, and $u(x,t) = \beta t$ are solutions for the equation with the new initial data. We found infinitely many solutions in this case.

Example 2.4.5 We now consider the following problem

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = u_0(x). \end{cases} \tag{2.22}$$

In view of example 2.4.1 and 2.4.2, one easily find the characteristic curves as the solution of

$$\frac{dx}{dt} = u(x,t),\ x(0) = x_0.$$

Also, we know that the solution $u(x,t)$ is constant along each characteristic curves. Therefore, the characteristic curves are the family of straight lines

$$x(t) = x_0 + tu_0(x_0).$$

So, the solution of u at each point (x,t) is thus determined by the characteristic line from certain x_0 on $\{t = 0\}$ passing through (x,t) where $u(x,t) = u_0(x_0)$.

We remark that this process cannot go far in general. Suppose there are two point $a < b$ on x-axis such that $u_0(a) > u_0(b)$. We know that the characteristic line issuing from $(a,0)$ will overtake the characteristic line from $(b,0)$ at

$$t = \frac{b-a}{u_0(a) - u_0(b)}.$$

In this case, we could not assign a value for $u(x,t)$ at this intersection point, which corresponds to two different values. Actually, the solution *blows up* in this case in its derivative. More precisely, we carry out the following calculations. Let $w = u_x$, which satisfies

$$w_t + uw_x = -w^2.$$

Therefore, along the characteristic line, we have

$$\begin{cases} \dot{w} = -w^2 \\ w(x_0, 0) = u_0'(x_0). \end{cases}$$

It is now clear that

$$w(x_0 + tu_0(x_0), t) = \frac{u_0'(x_0)}{1 + u_0'(x_0)t}.$$

Therefore, w tends to infinity in finite time if $u_0'(x_0) < 0$.

2.5 Linear Transport Equation

The linear transport equation

$$u_t + b \bullet Du = 0, \ x \in \mathbb{R}^n, \ t > 0 \tag{2.23}$$

arises from many important applications, such as particle mechanics, kinetic theory. Here $b = (b_1, b_2, \cdots, b_n)$ is a constant vector. Of course, one can easily solve this PDE using the characteristic method. Here, we will use a direct approach instead. Actually, the (2.23) can be rewritten as

$$(Du, u_t) \bullet (b, 1) = 0,$$

which means the function $u(x, t)$ is constant in the direction of $(b, 1)$ on the (x, t)-space. Therefore, if we know the value of u at any point of the straight line through (x, t) with the direction $(b, 1)$ in (x, t)-space, we know the value of $u(x, t)$.

Now let us consider the initial-value problem

$$\begin{cases} u_t + b \bullet Du = 0, \ x \in \mathbb{R}^n, \ t > 0, \\ u(x, 0) = g(x), \ x \in \mathbb{R}^n, \end{cases} \tag{2.24}$$

where $g(x)$ is a given function. Fix a point (x, t), the line through (x, t) in the direction ofd $(b, 1)$ is given in the parametric form with the parameter s as

$$(x + sb, t + s), \ s \in \mathbb{R}.$$

The line hits the initial plane $\{t = 0\}$ when $s = -t$ at the point $(x - bt, 0)$. We know that u is constant on the line, we thus obtain

$$u(x, t) = g(x - tb), \ x \in \mathbb{R}^n, t \geq 0. \tag{2.25}$$

We see from the above that if (2.24) has a smooth solution, then it is given by (2.25). Conversely, if g is continuous differentiable, the (2.25) is the unique solution of (2.24). We also remark that the solution in the form of (2.25) reads as the information traveling in a constant velocity b, such a solution is thus called *traveling wave* solution.

Remark 2.5.1 When $g(x)$ is not $\mathcal{C}^1$, problem (2.24) does not admit a smooth solution. However, even when g is not continuous, the formula (2.25) does provide a reasonable candidate for a solution. We may informally declare that (2.25) is a *weak solution* of (2.24). This makes sense even if g is discontinuous and so is u. This notion is very useful in nonlinear PDEs.

Next, we consider the non-homogeneous problem

$$\begin{cases} u_t + b \bullet Du = f(x,t), \ x \in \mathbb{R}^n, \ t > 0, \\ u(x,0) = g(x), \ x \in \mathbb{R}^n. \end{cases} \tag{2.26}$$

As in the homogeneous case, we set

$$z(s) = u(x+sb, t+s), \ s \in \mathbb{R},$$

which satisfies

$$\dot{z}(s) = Du(x+sb, t+s) \bullet b + u_t(x+sb, t+s) = f(x+sb, t+s).$$

Therefore,

$$\begin{aligned} u(x,t) - g(x-bt) = z(0) - z(-t) &= \int_{-t}^{0} \dot{z}(s)\ ds \\ &= \int_{-t}^{0} f(x+sb, t+s)\ ds \\ &= \int_{0}^{t} f(x+(s-t)b, s)\ ds, \end{aligned} \tag{2.27}$$

which gives the solution

$$u(x,t) = g(x-tb) + \int_0^t f(x+(s-t)b, s)\ ds. \tag{2.28}$$

2.6 Problems

Problem 1. Solve the following problems (x, y, $z \in \mathbb{R}$, $t > 0$)

- (a) $u_t + uu_x = 1, \ u(x,0) = h(x);$
- (b) $u_x + u_y = u, \ u(x,0) = \cos(x);$
- (c) $xu_y - yu_x = u, \ u(x,0) = h(x);$
- (d) $x^2u_x + y^2u_y = u^2, \ u(x,y)|_{y=2x} = 1;$
- (e) $xu_x + yu_y + u_z = u, \ u(x,y,0) = h(x,y);$

- (f) $\sum_{k=1}^{n} x_k u_{x_k} = 3u,\ u(x_1, \cdots, x_{n-1}, 1) = h(x_1, \cdots, x_{n-1});$
- (g) $uu_x - uu_y = u^2 + (x+y)^2,\ u(x,0) = 1;$
- (h) $2xyu_x + (x^2+y^2)u_y = 0,\ u = exp\left(\dfrac{x}{x-y}\right),\ on\ x+y=1;$
- (i) $xu_x + yu_y = u + 1,\ u(x,y) = x^2\ on\ y = x^2.$

Problem 2. Solve the Cauchy problem

$$\begin{cases} u_t + b \bullet Du + cu = 0,\ x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = g(x),\ x \in \mathbb{R}^n. \end{cases}$$

where $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constant.

Problem 3. For the following problem

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \dfrac{1}{1+x^2}, \end{cases}$$

find the solution and the time and the point where the solution blows up first.

Problem 4. If u is the $\mathcal{C}^1$ solution of the following equation

$$a(x,y)u_x + b(x,y)u_y = -u$$

on the closed unit disk $\Omega = \{(x,y) : x^2 + y^2 \le 1\}$. Suppose

$$a(x,y)x + b(x,y)y > 0,\ on\ x^2 + y^2 = 1,$$

prove that $u \equiv 0$.

Problem 5. Show that the solution of the equation

$$yu_x - xu_y = 0$$

containing the curve $x^2 + y^2 = a^2,\ u = y$, does not exit.

Problem 6. Solve the following Cauchy problems:

- (a) $x^2 u_x - y^2 u_y = 0,\ u \to e^x\ as\ y \to \infty;$
- (b) $yu_x + xu_y = 0,\ u = \sin(x)\ on\ x^2 + y^2 = 1;$
- (c) $-xu_x + yu_y = 1,\ in\ 0 < x < y,\ u(x) = 2x\ on\ y = 3x;$

- (d) $2xu_x + (x+1)u_y = y,\ in\ x > 0,\ u(1,y) = 2y;$
- (e) $xu_x - 2yu_y = x^2 + y^2\ in\ x > 0,\ y > 0,\ u = x^2\ on\ y = 1;$

Problem 7. Show that $u_1 = e^x$, and $u_2 = e^{-y}$ are solutions of the nonlinear equation

$$(u_x + u_y)^2 - u^2 = 0,$$

but that their sum $(e^x + e^{-y})$ is not a solution of the equation.

Problem 8. Solve the following equations:

- (a) $(y+u)u_x + (x+u)u_y = x + y,$
- (b) $xu(u^2 + xy)u_x - yu(u^2 + xy)u_y = x^4.$

Problem 9. Solve the equation

$$xz_x + yz_y = z,$$

and find the curves which satisfy the associated characteristic equations and intersect the helix $x^2 + y^2 = a^2,\ z = b\tan^{-1}(y/x)$.

Problem 10. Find the family of curves which represent the general solution of the PDE

$$(2x - 4y + 3u)u_x + (x - 2y - 3u)u_y = -3(x - 2y).$$

Determine the particular member of the family which contains the line $u = x$ and $y = 0$.

Problem 11. Find the solution of the equation

$$yu_x - 2xyu_y = 2xu$$

with the condition $u(0,y) = y^3$.

Problem 12. Find the solution surface of the equation

$$(u^2 - y^2)u_x + xyu_y + xu = 0,\ u = y = x,\ x > 0.$$