Chapter 4

Heat equation

Let Ω be an open set in $\mathbb{R}^n$, we will study the *Heat equation*

$$u_t - \mu\Delta u = 0, \ x \in \Omega, \ t > 0 \tag{4.1}$$

and the *non-homogeneous Heat equation*

$$u_t - \mu\Delta u = f(x,t), \ x \in \Omega, \ t > 0 \tag{4.2}$$

with appropriate initial boundary conditions. The constant $\mu > 0$ models the heat diffusion.

A guiding principle is that any assertion about harmonic functions yields a analogous statement about solutions of the heat equation. However, we will follow a slight different approach to show different techniques.

4.1 Physical derivation

Assume Ω was occupied by the homogeneous media without heat source inside. Let $u(x,t)$ be the temperature at x and time t, $J(x,t)$ is the heat flux. For each sub-region G with smooth boundary ∂G and unit outer normal ν,

$$\int_{\partial G} J \cdot \nu \, dS$$

is the total heat flow out of G per unit time. According to *Fourier's law*,

$$J = -k\nabla_x u$$

with $k > 0$ the constant of heat conductivity. Therefore, we update the last equation by

$$-\int_{\partial G} k\frac{\partial u}{\partial \nu} \, dS.$$

On the other hand, the first principle of thermodynamics says that the absolute temperature $u(x,t)$ is proportional to the heat, therefore, we know

$$\int_G C_V u_t \; dx = -\int_{\partial G} k \frac{\partial u}{\partial \nu} \; dS,$$

where $C_V > 0$ is the specific heat constant. Now, by divergence theorem, one has

$$\int_G u_t - \mu \Delta u \; dx = 0$$

with $\mu = \frac{k}{C_V}$. This leads to heat equation. The f in non-homogeneous heat equation models the interior heat source.

4.2 Fundamental solution

4.2.1 Derivation

We observe that the Heat Equation is invariant under the transformation:

$$x \to \lambda x, \; t \to \lambda t, \; for \; \lambda \in \mathbb{R}.$$

Therefore, the ratio $\frac{|x|^2}{t}$ is an important variable for heat equation, and we shall look for a solution of the form

$$u(x,t) = v(\frac{r^2}{t}), \; r = |x|, \; t > 0. \tag{4.3}$$

Such class of solutions are called *self-similar solution.*

A quicker approach is to find a solution u of the special structure

$$u(x,t) = t^{-\alpha} v(\frac{|x|}{\sqrt{t}}).$$

Let $y = \frac{x}{\sqrt{t}}$, and we substitute it into (4.1) to obtain

$$\alpha v + \frac{1}{2} y \cdot Dv + \Delta v = 0 \tag{4.4}$$

which reduces into

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0$$

for $w(r) = v(|y|)$ where $r = |y|$. Now, $\alpha = \frac{n}{2}$ is the magic number so that the equation becomes

$$(r^{n-1} w')' + \frac{1}{2}(r^n w)' = 0.$$

Hence,

$$r^{n-1} w' + \frac{1}{2} r^n w = 0$$

if we require w and w' tends to 0 if $r \to \infty$. Therefore,

$$w = be^{-\frac{r^2}{4}}$$

for some constant b. Therefore,

$$u(x,t) = bt^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$$

solves (4.1).

This motivates the following deinition:

Definition 4.2.1 *The function*

$$E(x,t) = \begin{cases} (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, & (x \in \mathbb{R}^n,\ t > 0) \\ 0 & (x \in \mathbb{R}^n,\ t > 0), \end{cases}$$

is called the fundamental solution of the heat equation.

We note that $E(x,t)$ is smooth except the singular point $(0,0)$. In the next section, the fundamental solution is naturally derived when we apply Fourier transform to solve the initial value problem.

4.3 Fourier transform and intial value problem

4.3.1 Fourier transform

Let function $f(x)$ and $g(x)$ defined on $x \in \mathbb{R}^n$ are continuously differentiable and absolutely integrable, then the Fourier transform of $f(x)$ is defined as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n_x} f(x) e^{-ix\cdot\xi}\, dx \tag{4.5}$$

and the inverse Fourier transform of $g(\xi)$ is

$$\mathcal{F}^{-1}(g)(x) = \check{g}(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n_\xi} f(x) e^{ix\cdot\xi}\, d\xi. \tag{4.6}$$

We remark that the Fourier transform and its inverse are naturally generalized into the square integrable function space $L^2(\mathbb{R}^n)$. We recall that for a function $u(x) \in L^2(\mathbb{R}^n)$,

$$\|u\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} u^2(x)\, dx \right)^{\frac{1}{2}} < \infty.$$

We also recall that

$$f * g = \int_{\mathbb{R}^n} f(y) g(x-y)\, dy.$$

We list some properties of Fourier transform in the following theorem.

Theorem 4.3.1 *(Properties of Fourier Transform). Assume $f, g \in L^2(\mathbb{R}^n)$.*

- *(i)* $\|f\|_{L^2(\mathbb{R}^n)} = \|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|\check{f}\|_{L^2(\mathbb{R}^n)}$;
- *(ii)* $\int_{\mathbb{R}^n} f\bar{g}\ dx = \int_{\mathbb{R}^n} \hat{f}\bar{\hat{g}}\ d\xi$,
- *(iii) For each multi-index α such that $D^\alpha f \in L^2(\mathbb{R}^n)$, $\widehat{D^\alpha f} = (i\xi)^\alpha \hat{f}$.*
- *(iv)* $\widehat{(f * g)} = (2\pi)^{\frac{n}{2}} \hat{f}\hat{g}$.
- *(v)* $f = \check{\hat{f}}$.

Example 4.3.2 *For $x \in \mathbb{R}$, find the Fourier transform for $f(x) = e^{-a|x|}$.*

Solution.

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-a|x|} e^{-ix\xi}\ dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-a|x|} (cos(x\xi) - isin(x\xi))\ dx \\
&= 2\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ax} cos(x\xi)\ dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{2a}{\xi^2 + a^2}.
\end{aligned}$$

Example 4.3.3 *For $\xi \in \mathbb{R}^n$ and $t > 0$, find the inverse Fourier transform for*

$$f(\xi) = (2\pi)^{-\frac{n}{2}} e^{-|\xi|^2 t}.$$

Solution.

$$\begin{aligned}
\check{f}(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-|\xi|^2 t} e^{ix\cdot\xi}\ d\xi \\
&= \prod_{k=1}^n \left(\frac{1}{2\pi} \int_{-\infty}^\infty e^{-t\xi_k^2 + ix_k\xi_k}\ d\xi_k \right) \\
&= \prod_{k=1}^n \left(\frac{1}{\pi} \int_0^\infty e^{-t\xi_k^2} cos(x_k\xi_k)\ d\xi_k \right) \\
&= \prod_{k=1}^n I(x_k).
\end{aligned}$$

By Euler's formula

$$\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2},$$

we know that

$$I(0) = \frac{1}{2\sqrt{\pi t}}.$$

Differentiating $I(x_k)$ once, and using integration by parts, one has

$$I' + \frac{x_k}{2t} I = 0,$$

therefore,

$$I(x_k) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x_k^2}{4t}}.$$

Finally, we obtain

$$\check{f}(x) = \mathcal{F}^{-1}((2\pi)^{-\frac{n}{2}} e^{-|\xi|^2 t}) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} = E(x,t).$$

4.3.2 Initial Value Problem

We now employ the Fourier transform to solve the following Cauchy problem for heat equation

$$\begin{cases} u_t - \Delta u = 0 \\ u(x,0) = \phi(x). \end{cases} \tag{4.7}$$

We perform Fourier transform in x, and denote

$$\hat{u}(\xi,t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(x,t) e^{-ix\cdot\xi}\, dx$$
$$\hat{\phi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \phi(x) e^{-ix\cdot\xi}\, dx,$$

we thus arrive at an initial value problem for the ODE about $\hat{u}(\xi,t)$

$$\begin{cases} \dfrac{d\hat{u}(\xi,t)}{dt} + |\xi|^2 \hat{u}(\xi,t) = 0 \\ \hat{u}(\xi,0) = \hat{\phi}(\xi). \end{cases} \tag{4.8}$$

Clearly, one has

$$\hat{u}(\xi,t) = \hat{\phi}(\xi) e^{-|\xi|^2 t},$$

therefore, we perform inverse Fourier transform

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}(\hat{u}(\xi,t)) \\ &= \mathcal{F}^{-1}(\hat{\phi}(\xi) e^{-|\xi|^2 t}) \\ &= \mathcal{F}^{-1}(\hat{\phi}(\xi)) * \mathcal{F}^{-1}((2\pi)^{-\frac{n}{2}} e^{-|\xi|^2 t}) \\ &= E(x,t) * \phi(x) \\ &= \int_{\mathbb{R}^n} E(x-y,t)\phi(y)\, dy. \end{aligned} \tag{4.9}$$

This also gives a natural way to derive the fundamental solution $E(x,t)$. We list some properties here for later applications:

Theorem 4.3.4 *(Properties of Fundamental solution) Let $E(x,t)$ be the fundamental solution of Heat equation.*

- *(i) For $\forall x, y \in \mathbb{R}^n$ and $t > 0$, $E(x-y,t) > 0$.*
- *(ii) For $\forall x, y \in \mathbb{R}^n$ and $t > 0$, $(\partial_t - \Delta)E(x-y,t) = 0$.*
- *(iii) For $\forall x \in \mathbb{R}^n$ and $t > 0$, $\int_{\mathbb{R}^n} E(x-y,t)\ dy = 1$.*
- *(iv) For any $\delta > 0$ and $\forall x \in \mathbb{R}^n$, $\lim_{t\to 0+} \int_{|y-x|>\delta} E(x-y,t)\ dy = 0$.*

Proof. By the expression of $E(x,t)$, the first two properties are obvious. For (iii), using substitution $y = x + \sqrt{4t}z$, we compute

$$\begin{aligned}\int_{\mathbb{R}^n} E(x-y,t)\ dy &= (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\ dy \\ &= (\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-|z|^2}\ dz \\ &= 1.\end{aligned}$$

Using the same substitution, we can prove (iv) as following

$$\begin{aligned}&\lim_{t\to 0+} \int_{|y-x|>\delta} E(x-y,t)\ dy \\ &= \lim_{t\to 0+} (\pi)^{-\frac{n}{2}} \int_{|z|>\frac{\delta}{\sqrt{4t}}} e^{-|z|^2}\ dz = 0.\end{aligned}$$

We remark that the formula (4.9) gives a formal solution to the intial value problem (4.7), where we assumed that the existence of Fourier transform of initial function $\phi(x)$, and we used the inverse transformation, this often requires $\phi(x)$ to be C^1 and absolutely integrable. However, under much weaker conditions on $\phi(x)$, we are able to prove that (4.9) does give the classical solution to (4.7).

Theorem 4.3.5 *If $\phi(x) \in C(\mathbb{R}^n)$ and there exist constants $M > 0$ and $A > 0$ such that*

$$|\phi(x)| \le Me^{A|x|^2},\ \forall x \in \mathbb{R}^n,$$

then

$$u(x,t) = E(x,t) * \phi(x)$$

is a C^∞ solution of (4.7) on the region

$$\Omega = \{(x,t)|x \in \mathbb{R}^n, 0 < t \le T\},\ for\ T < \frac{1}{4A}.$$

Proof. 1. We first show the continuity of $u(x,t)$. For any constants $a > 0$ and $t_0 \in (0,T)$, we define the set

$$V = \{(x,t) | |x| \le a,\ t_0 \le t \le T\}.$$

For $(x,t) \in V$, we see that

$$\begin{aligned} |u(x,t)| &\le M \int_{\mathbb{R}^n} E(x-y,t) e^{A|y|^2}\, dy \\ &\le cM \int_{\mathbb{R}^n} \exp\{A|y|^2 + \bar{A}|x-y|^2\}\, dy, \end{aligned}$$

where $c = (4\pi t_0)^{-\frac{n}{2}}$ and $\bar{A} = -\frac{1}{4T}$. We note that

$$A|y|^2 + \bar{A}|x-y|^2 = (A+\bar{A})|y - \frac{\bar{A}}{A+\bar{A}} x|^2 + \frac{A\bar{A}}{A+\bar{A}}|x|^2.$$

Therefore, for $A + \bar{A} < 0$, i.e., $T < \frac{1}{4A}$, we have

$$\begin{aligned} |u(x,t)| &\le M e^{\frac{A\bar{A}}{A+\bar{A}}|x|^2} \int_{\mathbb{R}^n} \exp\{(A+\bar{A})|y - \frac{\bar{A}}{A+\bar{A}} x|^2\}\, dy \\ &\le cM (\frac{-\pi}{A+\bar{A}})^{-\frac{n}{2}} e^{\frac{A\bar{A}}{A+\bar{A}}|x|^2}. \end{aligned}$$

Therefore, we know that the integral in (4.9) converges uniformly and absolutely on V, and so $u(x,t)$ is continuous on V. For $a > 0$ and $t_0 > 0$ arbitrary, $u(x,t)$ is continuous on Ω.

2. We now show that $u(x,t) \in C^\infty(\Omega)$ and it satisfies the Heat equation. This is because $E(x-y,t)$ is infinitely differentiable, with uniformly bounded derivatives, on V. For each multi-index $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n)$,

$$D^\alpha u(x,t) = \int_{\mathbb{R}^n} \phi(y) D^\alpha E(x-y,t)\, dy.$$

One can estimate this integral in a similar manner as in step 1 on V to prove its absolute and uniform convergence on V. Therefore, one proves that $u(x,t) \in C^\infty(\Omega)$. We now use the property of $E(x,t)$ to show that

$$(\partial_t - \Delta) u(x,t) = \int_{\mathbb{R}^n} \phi(y)(\partial_t - \Delta) E(x-y,t)\, dy = 0.$$

3. Finally, we prove that $u(x,t)$ verifies the initial condition, namely, we shall prove for any $x_0 \in \mathbb{R}^n$,

$$\lim_{(x,t)\to(x_0,0+)} u(x,t) = \phi(x_0).$$

Letting $v_0(x) = \phi(x) - \phi(x_0)$, using the property of $E(x,t)$, one has

$$u(x,t) = \int_{\mathbb{R}^n} \phi(x_0)E(x-y,t)\ dy + \int_{\mathbb{R}^n} v_0(y)E(x-y,t)\ dy$$
$$= \phi(x_0) + \int_{\mathbb{R}^n} v_0(y)E(x-y,t)\ dy.$$

Since $v_0(x)$ is continuous and $v_0(x_0) = 0$, for any fixed $\varepsilon > 0$ there exists δ such that when $|x - x_0| < \delta$, $|v_0(x)| < \varepsilon$. For $|x - x_0| > \delta$, there exists $B(\varepsilon) > 0$ large enough such that

$$|v_0(x)| \leq \varepsilon e^{B(\varepsilon)|x-x_0|^2}.$$

Therefore, for any $x \in \mathbb{R}^n$, one has

$$|\int_{\mathbb{R}^n} v_0(y)E(x-y,t)\ dy|$$
$$\leq \int_B (x_0,\delta)|v_0(y)|E(x-y,t)\ dy + \int_{\mathbb{R}^n\setminus B(x_0,\delta)} |v_0(y)|E(x-y,t)\ dy$$
$$\leq \varepsilon + \frac{\varepsilon}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp\{B(\varepsilon)|y|^2 - \frac{|x-y|^2}{4t}\}\ dy$$
$$\leq \varepsilon + \frac{\varepsilon}{(1-4Bt)^{\frac{n}{2}}} e^{\frac{B|x|^2}{1-4Bt}},$$

where we have chosen t so small such that $1 - 4Bt > 0$. Now, when t is sufficient small, we see that $|u(x,t) - \phi(x_0)| < 2\varepsilon$. Therefore, we verified that

$$\lim_{(x,t)\to(x_0,0+)} u(x,t) = \phi(x_0).$$

Remark 4.3.6 Some remarks are in order.

- 1. By the property of the fundamental solution $E(x,t)$, it is easy to see that

  $$u(x,t) = \int_{\mathbb{R}^n} E(x-y,t)\phi(y)\ dy \leq \sup_{y\in\mathbb{R}^n} \phi(y).$$

  Similarly, on finds the bound from below. Therefore, if $\phi(x)$ is bounded, then

  $$\inf_{y\in\mathbb{R}^n} \phi(y) \leq u(x,t) \leq \sup_{y\in\mathbb{R}^n} \phi(y) \tag{4.10}$$

- 2. From the theorem, it is obvious that the smaller A is, the larger existence time. If $\phi(x)$ is bounded, $A = 0$, then the solution is global in time.

- 3. It also clear that even if $\phi(x)$ is replaced by measurable function, while other conditons keep the same, $u(x,t)$ given in (4.9) is still the C^∞ solution to (4.7), and verifies the initial conditions at all the continuous points of ϕ.

- 4. If $\phi(x) \geq 0$ and $\phi \not\equiv 0$, the $u(x,t) > 0$ for any $x \in \mathbb{R}^n$ and $t > 0$. This is called the *infinite propogation speed* for disturbances. If the initial temperature is nonnegative and positive somewhere, the temperature is positive everywhere at any later time.

4.3.3 Nonhomogeneous Problem

We now derive a general formula for the following initial-value problem

$$\begin{cases} u_t - \Delta u = f(x,t) \ x \in \mathbb{R}^n, \ t > 0 \\ u(x,0) = 0. \end{cases} \tag{4.11}$$

For this purpose, we note that

$$u(x,t;s) = \int_{\mathbb{R}^n} E(x-y,t-s) f(y,s) \ dy$$

solves the following initial value poblem

$$\begin{cases} u_t(x,t;s) - \Delta_x u(x,t;s) = 0, \ x \in \mathbb{R}^n, \ t \, s, \\ u(x,s;s) = f(x,s). \end{cases} \tag{4.12}$$

Of course, $u(x,t;s)$ does not solve (4.11), however, the *Duhamel's Principle* allows us to build a solution of (4.11) from the solutions of (4.12), by integrating with respect to s. More precisely, consider

$$\begin{aligned} u(x,t) &= \int_0^t u(x,t;s) \ ds \\ &= \int_0^t \int_{\mathbb{R}^n} E(x-y,t-s) f(s,y) \ dyds, \ x \in \mathbb{R}^n, \ t > 0. \end{aligned} \tag{4.13}$$

The following Theorem confirms that (4.13) gives the solution to (4.11).

Theorem 4.3.7 *Assume $f \in C_1^2((0,\infty) \times \mathbb{R}^n)$ has compact support. Deinfe u by (4.13), then*

- *(i) $u \in C_1^2((0,\infty) \times \mathbb{R}^n)$,*
- *(ii) $u_t - \Delta u = f(x,t)$,*
- *(iii) For each $x_0 \in \mathbb{R}^n$, $\lim_{(x,t) \to (x_0, 0+)} u(x,t) = 0$.*

Proof.

1. We first change variables to write

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} E(y,s) f(x-y,t-s) \ dyds.$$

We note that $E(y,s)$ is smooth near $s = t > 0$ and f has compact support, then

$$\begin{aligned} u_t(x,t) = &\int_0^t \int_{\mathbb{R}^n} E(y,s) f_t(x-y,t-s) \ dyds \\ &+ \int_{\mathbb{R}^n} E(y,s) f(x-y,0) \ dy, \end{aligned}$$

$$u_{x_i}(x,t) = \int_0^t \int_{\mathbb{R}^n} E(y,s) f_{x_i}(x-y, t-s)\ dyds,$$
$$u_{x_i x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} E(y,s) f_{x_i x_j}(x-y, t-s)\ dyds.$$

Thus, $u(x,t) \in C_1^2((0,\infty) \times \mathbb{R}^n)$.

2. We not compute

$$\begin{aligned} & u_t - \Delta u \\ &= \int_0^t \int_{\mathbb{R}^n} E(y,s)[(\partial_t - \Delta_x) f(x-y,t-s)]\ dyds + \int_{\mathbb{R}^n} E(y,s) f(x-y,0)\ dy \\ &= I_\varepsilon + J_\varepsilon + K, \end{aligned}$$

where,

$$\begin{aligned} I_\varepsilon &= \int_\varepsilon^t \int_{\mathbb{R}^n} E(y,s)[(-\partial_s - \Delta_y) f(x-y,t-s)]\ dyds, \\ J_\varepsilon &= \int_0^\varepsilon \int_{\mathbb{R}^n} E(y,s)[(-\partial_s - \Delta_y) f(x-y,t-s)]\ dyds, \\ K &= \int_{\mathbb{R}^n} E(y,s) f(x-y,0)\ dy. \end{aligned}$$

Now,

$$|J_\varepsilon| \le (\|f_t\|_{L^\infty} + \|D^2 f\|_{L^\infty}) \int_0^\varepsilon \int_{\mathbb{R}^n} E(y,s)\ dyds \le C\varepsilon.$$

Integrating by parts, we find

$$\begin{aligned} I_\varepsilon &= \int_\varepsilon^t \int_{\mathbb{R}^n} [(-\partial_s - \Delta_y) E(y,s)] f(x-y,t-s)]\ dyds \\ &\quad + \int_{\mathbb{R}^n} E(y,\varepsilon) f(x-y,t-\varepsilon)\ dy - \int_{\mathbb{R}^n} E(y,s) f(x-y,0)\ dy. \\ &= \int_{\mathbb{R}^n} E(y,\varepsilon) f(x-y,t-\varepsilon)\ dy - K, \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} u_t - \Delta u &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} E(y,\varepsilon) f(x-y,t-\varepsilon)\ dy \\ &= f(x,t),\ x \in \mathbb{R}^n,\ t > 0. \end{aligned}$$

Finally, we note that

$$\|u(x,t)\|_{L^\infty} \le t\|f\|_{L^\infty},$$

which tends to zero as $t \to 0+$. This concludes the proof.

By linear superposition principle, we know that

$$u(x,t) = \int_{\mathbb{R}^n} E(x-y,t)\phi(y)\ dy + \int_0^t \int_{\mathbb{R}^n} E(x-y,t-s)f(s,y)\ dyds \tag{4.14}$$

solves

$$\begin{cases} u_t - \Delta u = f(x,t)\ x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = \phi(x), \end{cases} \tag{4.15}$$

under conditions on ϕ and f as above.

4.4 Maximum Principle and applications

4.4.1 Maximum Principle

Assume $\Omega \subset \mathbb{R}^n$ is open, bounded set. We first introduce the following concept.

Definition 4.4.1 *Fix a time $T > 0$, the parabolic cylinder*

$$\Omega_T = \Omega \times (0, T].$$

The parabolic boundary of Ω_T is

$$\Gamma_T = \bar{\Omega}_T \setminus \Omega_T.$$

We remark that the *parabolic interior* of $\bar{\Omega}_T$ contains the top $\Omega \times \{t = T\}$. The parabolic boundary Γ_T comprises the bottom, the vertical sides $\partial\Omega \times [0,T]$, but not the top.

We now state the maximum principle.

Theorem 4.4.2 *(Maximum Principle) Assume $u(x,t) \in C_1^2(\bar{\Omega}_T)$ solves the heat equation in Ω_T, then*

$$\max_{\bar{\Omega}_T} u = \max_{\bar{\Gamma}_T} u.$$

Proof. Assume that $u(x,t)$ does not attains it maximum at Γ_T but at a point $(x^*,t^*) \in \Omega_T$. Namely, there exists $m < M$ such that

$$\max_{\bar{\Omega}_T} u = u(x^*,t^*) = M > m = \max_{\bar{\Gamma}_T} u.$$

Define

$$v(x,t) = u(x,t) + \frac{M-m}{2nd^2}|x-x^*|^2,$$

where $d = 2\max dist\{x^*, \partial\Omega\}$. We see that $v(x^*,t^*) = M$ and

$$v(x,t)|_{\Gamma_T} < m + \frac{M-m}{2n} < M.$$

Therefore, $v(x,t)$ attains its maximum at some $(x_1,t_1) \in \Omega_T$. We observe that at this point (x_1,t_1),

$$\Delta v(x_1,t_1) \le 0, \;\; v_t(x_1,t_1) \ge 0,$$

and so

$$v_t - \Delta v \ge 0, \;\; at \; (x_1,t_1).$$

But on the other hand, direct computation gives

$$v_t - \Delta v = -\frac{M-m}{d^2} < 0,$$

a contradiction. Therefore,

$$\max_{\bar{\Omega}_T} u = \max_{\bar{\Gamma}_T} u.$$

If one replaces u with $-u$, one actually obtains that

$$\max_{\bar{\Omega}_T} |u| = \max_{\bar{\Gamma}_T} |u|.$$

The following *comparison principle* is a direct consequence of the above maximum principle.

Theorem 4.4.3 *(Comparison Principle) Let $u_1, u_2 \in C_1^2(\bar{\Omega}_T)$ be two solutions of the heat equation. If*

$$u_1 \le u_2, \;\; on \; \Gamma_T,$$

then

$$u_1 \le u_2, \;\; on \; \bar{\Omega}_T.$$

In order to restore the strong maximum principle, we introduce the parabolic mean-value formula. we introduce the following concept.

Definition 4.4.4 *For fixed $x \in \mathbb{R}^n$, $t > 0$ and $r > 0$, we define*

$$V(x,t;r) = \left\{ (y,s) \in \mathbb{R}_+^{n+1} | s \le t, E(x-y), t-s) \ge \frac{1}{r^n} \right\}.$$

We remark that the boundary of $V(x,t;r)$ is a level set of $E(x-y,t-s)$, and the point (x,t) is at the center of the top. With the help of this notion, we have

Theorem 4.4.5 *(Mean-value property for heat equation). Let $u \in C_1^2(\Omega_T)$ solve the heat equation. Then*

$$u(x,t) = \frac{1}{4r^2} \iint_{V(x,t;s)} u(y,s) \frac{|x-y|^2}{(t-s)^2} \, dyds, \tag{4.16}$$

for each $V(x,t;r) \subset \Omega_T$.

With the help of this theorem, a similar argument as for Laplace equation, one can prove

Theorem 4.4.6 *(Strong maximum principle). Let $u \in C_1^2(\Omega_T)$ solve the heat equation. If Ω is connected and there exists a point $(x_0, t_0) \in \Omega_T$ such that*

$$u(x_0, t_0) = \max_{\bar{\Omega}_T} u,$$

then u is constant in $\bar{\Omega}_{t_0}$.

4.4.2 Uniqueness and Stability

The first application is to the following initial boundary value problem:

$$\begin{cases} u_t - \Delta u = f(x,t), \ (x,t) \in \Omega_T, \\ u|_{\Gamma_T} = \phi(x,t). \end{cases} \tag{4.17}$$

Theorem 4.4.7 *The solution of (4.17) is unique and continuously depends on the initial boundary data.*

Proof. If $u^i (i = 1, 2)$ are solutions of

$$\begin{cases} u_t^i - \Delta u^i = f(x,t), \ (x,t) \in \Omega_T, \\ u^i|_{\Gamma_T} = \phi_i(x,t), \end{cases}$$

then $w = u^1 - u^2$ solves

$$\begin{cases} w_t - \Delta w = f(x,t), \ (x,t) \in \Omega_T, \\ w|_{\Gamma_T} = \phi_1 - \phi_2. \end{cases}$$

By maximum principle, one has

$$\max_{\bar{\Omega}_T} |w| = \max_{\bar{\Omega}_T} |\phi^1 - \Phi^2|.$$

This shows the continuous dependence. In particular, if $\phi^1 = \phi^2$, $w \equiv 0$, one finds the uniqueness.

We now turn to the initial value problem. For this purpose we first establish the maximum principle for the Cauchy problem (4.7). From Theorem 4.3.5, we know that if $\phi(x) \in C(\mathbb{R}^n)$ satisfies the growth condition

$$|\phi(x)| \le M e^{A|x|^2},$$

then the formula (4.9) $u(x,t) = E(x,t) * \phi(x)$ solves (4.7) for any $x \in \mathbb{R}^n$, $0 \le t \le T$ $(T < \frac{1}{4T})$. Furthermore, from the proof of Theorem 4.3.5, we know that there exist $M_1 > 0$ and $A_1 \ge 0$, such that

$$|u(x,t)| \le M_1 e^{A_1|x|^2}, \ (x,t) \in \mathbb{R}^n \times [0,T].$$

This motivates the following maximum principle.

Theorem 4.4.8 *(Maximum principle of Cauchy problem). Let $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves (4.7), and satisfies the growth estimate*

$$|u(x,t)| \leq Me^{A|x|^2}, \ (x,t) \in \mathbb{R}^n \times [0,T], \tag{4.18}$$

for constants $M > 0$, $A \geq 0$. Then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} \phi.$$

Proof. We first assume

$$4AT < 1 \tag{4.19}$$

and so there is $\varepsilon > 0$ that

$$4A(T+\varepsilon) < 1.$$

Fix $y \in \mathbb{R}^n$, $\mu > 0$, we define

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}. \tag{4.20}$$

Clearly, it holds

$$v_t - \Delta v = 0, \ in \ \mathbb{R}^n \times (0,T].$$

Fix $r > 0$, define $\Omega_T = B(y,r) \times (0,T]$. By maximum principle,

$$\max_{\Omega_T} v = \max_{\Gamma_T} v.$$

We not check the value of v at Γ_T. First of all, if $x \in \mathbb{R}^n$,

$$\begin{aligned} v(x,0) &= u(x,0) - \frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\varepsilon)}} \\ &\leq u(x,0) = \phi(x). \end{aligned} \tag{4.21}$$

On the sides where $|x-y| = r$, $t \in [0,T]$, we have

$$\begin{aligned} v(x,0) &= u(x,0) - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \\ &\leq Me^{A(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\varepsilon)}}. \end{aligned} \tag{4.22}$$

We now know $\frac{1}{4(T+\varepsilon)} = A + \delta$ for some small $\delta > 0$. Thus we have

$$v(x,t) \leq Me^{A(|y|+r)^2} - \mu(4(A+\delta)^{\frac{n}{2}} e^{(A+\delta)r^2} \leq \sup_{\mathbb{R}^n} \phi,$$

for r sufficiently large. We thus conclude that

$$v(x,t) \leq \sup_{\mathbb{R}^n} \phi,$$

for all $x \in \mathbb{R}^n$ and $t \leq T$ as long as $4AT < 1$. Let $\mu \to 0$, we showed that

$$u(x,t) \leq \sup_{\mathbb{R}^n} \phi, \; if \; 4AT < 1.$$

For general case where (4.19) fails, we can repeat the above procedure on $[0, \frac{1}{8A}]$, $[\frac{1}{8A}, \frac{2}{8A}]$, $\cdots$. Thus the proof of the theorem is complete.

Using this maximum principle, one easily prove the following uniqueness result.

Theorem 4.4.9 *(Uniqueness for Cauchy problem). Let $\phi(x) \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0,T])$. Then there exists at most one solution $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ of (4.15) , satisfying the growth estimate*

$$|u(x,t)| \leq M e^{A|x|^2}, \; (x,t) \in \mathbb{R}^n \times [0,T], \tag{4.23}$$

for constants $M > 0$, $A \geq 0$.

Now, a natural question arises: Is there any other solutions to (4.15) if we don't require the growth restriction (4.23). The answer is YES. This will be explained in the examples.

4.5 Examples

The first example of A. N. Tychonov is to answer the uniqueness problem for the Cauchy problem without growth restriction (4.23).

Example 4.5.1 There are infinitely many solutions to the initial value problem

$$\begin{cases} u_t - u_{xx} = 0, \; x \in \mathbb{R}, \; t > 0, \\ u(x,0) = 0, \end{cases}$$

without the growth restriction (4.23).

Solution. For some $g(t) \in C^\infty(\mathbb{R})$, we define

$$u(x,t) = \sum_{k=0}^{\infty} \frac{d^k g(t)}{dt^k} \frac{x^{2k}}{(2k)!}, \; x \in \mathbb{R}, \; t \in \mathbb{R}. \tag{4.24}$$

If the series converge in a nice way, we have

$$u_t = \sum_{k=0}^{\infty} \frac{d^{k+1} g(t)}{dt^{k+1}} \frac{x^{2k}}{2k!},$$

and

$$u_{xx} = \sum_{k=1}^{\infty} \frac{d^k g(t)}{dt^k} \frac{x^{2k-2}}{(2k-2)!} = \sum_{k=0}^{\infty} \frac{d^{k+1} g(t)}{dt^{k+1}} \frac{x^{2k}}{2k!}.$$

Therefore, we see $u(x,t)$ solves the heat equation. Now, we choose

$$g(t) = \begin{cases} \exp\{-t^{-\alpha}\}, \ \alpha > 1, \ t > 0 \\ 0, \quad t \leq 0. \end{cases}$$

It is clear that $g(t)$ is analytic except for $t = 0$. It remains to verify that $u(x,t)$ attains the initial data when $t \to 0+$. For this purpose, we need to compute the derivatives of $g(t)$. Due to complex analysis, the Cauchy integral formula, we have

$$\frac{d^k g(t)}{dt^k} = \frac{k!}{2\pi i} \int_\Gamma \frac{e^{-z^{-\alpha}}}{|z-t|^{k+1}} \, dz,$$

where the path Γ is chosen as the circle: $|z - t| = \theta t$ for $\theta \in (0,1)$. For $Re(z) > 0$, z^α is defined as its principal value. Now, for some $\lambda \in \mathbb{R}$, the point on Γ is described as

$$z = t + \theta t e^{i\lambda} = t(1 + \theta e^{i\lambda}), \ Re(-z^{-\alpha}) = -t^\alpha Re(1 + e^{i\lambda})^{-\alpha}.$$

For small θ, we have

$$Re(1 + e^{i\lambda})^{-\alpha} > \frac{1}{2}$$

, and so

$$Re(-z^{-\alpha}) < -\frac{1}{2} t^{-\alpha}, \ |\frac{d^k g(t)}{dt^k}| \leq \frac{k!}{(\theta t)^k} e^{-\frac{1}{2t^\alpha}}.$$

Note that $\frac{k!}{(2k)!} < \frac{1}{k!}$, we have

$$|u(x,t)| \leq \sum_{k=0}^{\infty} \frac{|x|^{2k}}{k!(\theta t)^k} e^{-\frac{1}{2t^\alpha}} = \exp\{\frac{1}{t}(\frac{|x|^2}{\theta} - \frac{1}{2} t^{1-\alpha})\}.$$

Now, it is clear that on each interval $[x_1, x_2]$, when $t \to 0+$, $u(x,t) \to 0$ uniformly. This shows that (4.24) determines a solution (called Tychonov's solution) for each $\alpha > 1$.

The second example is due to E Rothe showing that the backward heat equation is ill-posed.

Example 4.5.2 Consider the initial value problem

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = \phi(x), \ x \in \mathbb{R}. \end{cases}$$

If $\phi(x) = 0$, we see $u(x,t) = 0$ is a solution. Now, if

$$\phi(x) = \lambda \sin(\frac{x}{\lambda}), \ \lambda > 0,$$

then

$$u(x,t) = \lambda e^{-\frac{t}{\lambda^2}} \sin(\frac{x}{\lambda})$$

is a solution. We see when λ is sufficiently small, then $\phi(x)$ is arbitrarily close to 0. This is true when $t > 0$; but for $t < 0$, it is not. So the solution is not stable about the initial data if $t < 0$.

In what follows, we show some tricks in solving initial boundary value problem of heat equation in one space dimension.

Example 4.5.3 Consider the heat conduction problem on a finite bar

$$\begin{cases} u_t - u_{xx} = 0, \ t > 0, x \in (0,l), \\ u(x,0) = \phi(x), \ x \in (0,l), \\ u(0,t) = u(l,t) = 0, \ t \geq 0, \end{cases} \tag{4.25}$$

where $\phi(x) \in C[0,l]$ such that $\phi(0) = \phi(l) = 0$.

Solution. If $u(x,t)$ is a solution, since $u(0,t) = 0$, we could extend $u(x,t)$ as an odd function in x into $(-l,0)$. Then we further extend $u(x,t)$ as a periodic function in x with period $2l$ to whole real line. This will transfer the problem into a Cauchy problem

$$\begin{cases} u_t - u_{xx} = 0, \ t > 0, x \in \mathbb{R}, \\ u(x,0) = \Phi(x), \ x \in \mathbb{R}, \end{cases}$$

where

$$\begin{aligned} &\Phi(x) = \phi(x), \ x \in (0,l) \\ &\Phi(x) = -\phi(-x), \ x \in (-l,0) \\ &\Phi(x+2l) = \Phi(x), \ x \in \mathbb{R}. \end{aligned}$$

Since $\phi(x) \in C[0,l]$, $\Phi(x)$ satisfies all conditions for existence and uniqueness, and therefore

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(y)E(x-y,t)\ dy \tag{4.26}$$

We know that $\Phi(x)$ is odd in x, so

$$u(0,t) = \int_{-\infty}^{\infty} \Phi(y)E(y,t)\ dy = 0.$$

Also, $\Phi(l-x)$ is odd in x, so

$$u(l,t) = \int_{-\infty}^{\infty} \Phi(l-y)E(y,t)\ dy = 0.$$

Therefore, $u(x,t)$ determined by the formula (4.26) is a solution of (4.25). We could rewrite (4.26) as

$$u(x,t) = \int_0^l \phi(y)G(x,y,t)\ dy$$

where

$$G(x,y,t) = \frac{1}{2\sqrt{\pi t}}\sum_{n=-\infty}^{\infty}\left(\exp\{-\frac{x-y-2nl)^2}{4t}\} - \exp\{-\frac{x+y-2nl)^2}{4t}\}\right).$$

Example 4.5.4 We solve the above problem (4.25) again by the method of separation of variables, called Fourier method.

Solution. Assuming

$$u(x,t) = X(x)T(t),$$

one gets

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

for λ the constant ratio as the first term depends only on t while the second depends only on x. Therefore, we obtained two equations

$$T' + \lambda T = 0,\ X'' + \lambda X = 0. \tag{4.27}$$

For X, it satisfies the following two points boundary value problem

$$\begin{cases} X'' + \lambda X = 0,\ 0 < x < l, \\ X(0) = X(l) = 0. \end{cases} \tag{4.28}$$

This is a typical Sturm-Liouville type eigenvalue problem. We now derive the general method for such type of problems.

First of all, if $\lambda < 0$, we have the general solution of X as

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x},$$

which gives zero solution by the boundary condition.

If $\lambda = 0$, $X'' = 0$, which with the boundary condition, give zero solution.

Now, if $\lambda = k^2 > 0$ for $k > 0$, we have the general solution of X is

$$X(x) = A\cos(kx) + B\sin(kx),$$

where $A = 0$ as $X(0) = 0$, and $B\sin(kl) = 0$ as $X(l) = 0$. If $B = 0$, we get zero solution again. If $B \neq 0$, we have

$$k = \frac{n\pi}{l},\ or,\ \lambda_n = (\frac{n\pi}{l})^2,\ n = 1, 2, \cdots \tag{4.29}$$

We remark that $n < 0$ does not give new values of λ. We thus obtained the non-zero solutions

$$X_n(x) = B\sin(\frac{n\pi}{l}x),\ n = 1, 2, \cdots \tag{4.30}$$

We call λ_n the eigenvalue and X_n the corresponding eigenfunction. We now substitute the λ_n into the equation of T to obtain

$$T_n = a_n e^{-\lambda_n t}. \tag{4.31}$$

We thus obtained the formal solution of the heat equation with the boundary data

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(\frac{n\pi}{l})^2 t}\sin(\frac{n\pi}{l}x). \tag{4.32}$$

The constant B is absorbed by a_n, which will be determined by the initial condition,

$$\phi(x) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi}{l}x),$$

and therefore,

$$a_n = \frac{2}{l}\int_0^l \phi(x)\sin(\frac{n\pi}{l}x)\ dx.$$

The equation (4.32) gives a formal solution, we need to prove the convergent properties for this series. As $\phi(x) \in C([0,l])$, there exists a constant $K > 0$ such that $|a_n| \leq K$ for all n. Now, for any $t_1 > t_0 > 0$, on the domain $[0,l] \times [t_0, t_1]$, we have

$$|a_n e^{-(\frac{n\pi}{l})^2 t}\sin(\frac{n\pi}{l}x)| \leq K e^{-(\frac{n\pi}{l})^2 t}.$$

Therefore, the series (4.32) converges uniformly and absolutely. Therefore, $u(x,t)$ is continuous in this domain and thus is continuous for $x \in [0,l]$, $t > 0$. Furthermore, it verifies the boundary conditions. Similarly, one show the continuous differentiability in t and in x. Therefore, (4.32) defines the solution to problem (4.25).

Example 4.5.5 Assume there is a infinitely long, heat conductive, thin rod ($x \in \mathbb{R}$). Assume the rod is made of two different materials on the part of $x < 0$ and on the part of $x > 0$ respectively. At the transition point $x = 0$, the heat flux must be continuous on both side. We denote the left temperature by $u(x,t)$ and the right temperature by $v(x,t)$. Therefore, we need to solve

$$\begin{cases} u_t - \mu u_{xx} = 0,\ x < 0,\ t > 0, \\ v_t - \nu v_{xx} = 0,\ x > 0,\ t > 0, \\ u(x,0) = \phi(x),\ x \leq 0, \\ v(x,0) = \psi(x),\ x \geq 0, \\ u(0,t) = v(0,t),\ t \geq 0, \\ \omega u_x(0,t) = \Omega v_x(0,t),\ t \geq 0, \end{cases} \tag{4.33}$$

where, $\phi(0) = \psi(0)$, μ and ν are head conduction constants, and ω and Ω are the heat transfer constants.

Solution. We assume this problem has solution $(u(x,t), v(x,t))$. We define

$$w(x,t) = au(-x,t) + bv(\sqrt{\frac{\nu}{\mu}}x, t), \ x \geq 0 \tag{4.34}$$

where a and b are constants to be determined. We easily verify that w satisfies the equation

$$w_t - \mu w_{xx} = 0, \ x > 0, \ t > 0.$$

We now introduce a function

$$u^*(x,t) = \begin{cases} u(x,t), \ x < 0, \\ w(x,t), \ x > 0. \end{cases}$$

It is clear that u^* satisfies $u^*_t - \mu u^*_{xx} = 0$ on $\mathbb{R} \setminus \{0\}$. Now, we require that $u^*(x,t)$ satisfies the following compatibility condition at $x = 0$,

$$\begin{cases} u(0,t) = w(0,t) = au(0,t) + bv(0,t), \\ u_x(0,t) = w_x(0,t) = -au_x(0,t) + b\sqrt{\frac{\nu}{\mu}}v_x(0,t). \end{cases} \tag{4.35}$$

By the conditions of (4.33), the above conditions implies that

$$\begin{cases} u(0,t) = au(0,t) + bu(0,t), \\ \Omega u_x(0,t) = -a\Omega u_x(0,t) + b\omega\sqrt{\frac{\nu}{\mu}}u_x(0,t), \end{cases} \tag{4.36}$$

which gives

$$a + b = 1, \ -a\Omega + b\omega\sqrt{\frac{\nu}{\mu}} = \Omega.$$

Thus,

$$a = 1 - b, \ b = \frac{2\Omega}{\Omega + \omega\sqrt{\frac{\nu}{\mu}}}.$$

We now substitute a and b into (4.34) to recover the initial data for u^* on $x \geq 0$:

$$u^*(x,0) = a\phi(-x) + b\psi(\sqrt{\frac{\nu}{\mu}}x), \ x > 0.$$

We now define

$$\phi^*(x) = \begin{cases} \phi(x), \ x < 0, \\ a\phi(-x) + b\psi(\sqrt{\frac{\nu}{\mu}}x), \ x \geq 0. \end{cases}$$

We thus obtained a Cauchy problem for u^*

$$\begin{cases} u_t^* - \mu u_{xx}^* = 0, \ x \in \mathbb{R}, t > 0 \\ u^*(x,0) = \phi^*(x), \ x \in \mathbb{R}. \end{cases} \tag{4.37}$$

Now, set $y = \frac{x}{\sqrt{\mu}}$, and then the solution of (4.37) is given by

$$u^*(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \phi^*(\sqrt{\mu} z) e^{-\frac{(\frac{x}{\sqrt{\mu}} - z)^2}{4t}} \, dz, \ x \in \mathbb{R}.$$

4.6 Problems

Problem 1. Use Fourier transform method to give a solution formula to the following Cauchy problem

$$\begin{cases} u_t - u_{xx} + xu = 0, \ x \in \mathbb{R}, t > 0, \\ u(x,0) = \phi(x). \end{cases}$$

Problem 2. Use Fourier transform method to solve

$$\begin{cases} u_{xx} + u_{yy} = 0, \ x \in \mathbb{R}, y > 0, \\ u(x,0) = \phi(x), \ \lim\limits_{\sqrt{x^2+y^2} \to \infty} u = 0. \end{cases} \tag{4.38}$$

Problem 3. Assume a static temperature distribution $u(x,y)$ on upper half plane satisfies the constraints

$$u(x,0) = \begin{cases} 1, \ if \ |x| \le a, \\ 0, \ if \ |x| > a. \end{cases}$$

Prove that

$$u(x,y) = \frac{1}{\pi}(\arctan(\frac{a+x}{y}) + \arctan(\frac{a-x}{y})).$$

Problem 4. Solve the Cauchy problem of 1-d heat equation $u_t - a^2 u_{xx} = 0$, $(x \in \mathbb{R}, t > 0)$ with the following initial conditions:

- (a) $u(x,0) = \sin(x)$;
- (b) $u(x,0) = x^2 + 1$.

Problem 5. Use extension method to solve the following problem

$$\begin{cases} u_t - a^2 u_{xx} = 0, \ x > 0, t > 0, \\ u(x,0) = \phi(x), \ x > 0 \\ u(0,t) = 0, \ t \geq 0; \ \phi(0) = 0. \end{cases}$$

Problem 6. Use extension method to solve the following problem

$$\begin{cases} u_t - u_{xx} = 0, \ x > 0, t > 0, \\ u(x,0) = \phi(x), \ x > 0 \\ u_x(0,t) = 0, \ t > 0; \ \phi(0) = \phi'(0) = 0. \end{cases}$$

Problem 7. For constant d, solve the following problem

$$\begin{cases} u_t - u_{xx} = 0, \ x > 0, t > 0, \\ u(x,0) = \phi(x), \ x > 0 \\ u_x(0,t) + du(0,t) = 0, \ t > 0. \end{cases}$$

Problem 8. For constant $\sigma > 0$, use separation of variables method to solve

$$\begin{cases} u_t - u_{xx} = 0, \ 0 < x < l, \ t > 0, \\ u(x,0) = \phi(x), \ 0 \leq x \leq l \\ u(0,t) = 0, \ u_x(l,t) + \sigma u(l,t) = 0, \ t \geq 0. \end{cases}$$

Problem 9. Prove the solution of

$$\begin{cases} u_t - u_{xx} = 0, \ x \in \mathbb{R}, \ t > 0, \\ u(x,0) = 1, \ x > 0, \\ u(x,0) = -1, \ x < 0, \end{cases}$$

is

$$u(x,t) = h(\frac{x}{2\sqrt{t}}),$$

where, h is the error function

$$h(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \ dt.$$

Problem 10. If $u_1(x,t)$ and $u_2(y,t)$ are respectively the solutions of the problems

$$\begin{cases} u_{1t} - u_{1xx} = 0, \ x \in \mathbb{R}, \ t > 0, \\ u_1(x,0) = \phi_1(x), \end{cases}$$

and

$$\begin{cases} u_{2t} - u_{2yy} = 0, \; y \in \mathbb{R}, \; t > 0, \\ u_2(y, 0) = \phi_2(y). \end{cases}$$

Prove the function $u(x, y, t) = u_1 u_2$ is the solution of

$$\begin{cases} u_t - (u_{xx} + u_{yy}) = 0, \; (x, y) \in \mathbb{R}^2, \; t > 0, \\ u(x, y, 0) = \phi_1(x)\phi_2(y). \end{cases}$$

Problem 11. Derive the solution formula for the following problem

$$\begin{cases} u_t - (u_{xx} + u_{yy}) = 0, \; (x, y) \in \mathbb{R}^2, \; t > 0, \\ u(x, y, 0) = \sum_{i=1}^{n} \alpha_i(x)\beta_i(y). \end{cases}$$

Problem 12. Let $v(x, t)$ be the solution of

$$\begin{cases} v_t - a^2 v_{xx} = 0, \; x > 0, \; t > 0, \\ v(x, 0) = 0, \; v(0, t) = 1. \end{cases}$$

Prove the Duhamel integral

$$u(x, t) = \frac{\partial}{\partial t} \int_0^t v(x, t - \tau) g(\tau) \, d\tau$$

solves

$$\begin{cases} u_t - a^2 u_{xx} = 0, \; x > 0, \; t > 0, \\ u(x, 0) = 0, \; u(0, t) = g(t). \end{cases}$$

Problem 13. Prove the weak maximum principle: Let Ω be a bounded open set in $\mathbb{R}^n$, and Ω_T is the parabolic cylinder. If $u(x, t) \in C_1^2(\bar{\Omega}_T)$ satisfies

$$u_t - \Delta u \leq 0,$$

then

$$\max_{\Omega_T} u = \max_{\Gamma_T} u$$

where Γ_T is the parabolic boundary. If $\leq$ is replaced by $\geq$, and the *max* is repalced by *min*, then the statement is also valid.

Problem 14. Let $u(x, t) \in C_1^2(\bar{\Omega}_T)$ be the solution of

$$\begin{cases} u_t - \Delta u = f(x, t), \; (x, t) \in \Omega_T, \\ u(x, t)|_{\Gamma_T} = \phi(x, t). \end{cases}$$

Define

$$F = \sup_{\bar{Q}_T}|f|, \;\; B = \sup_{\Gamma_T}|\phi|.$$

Prove that

$$\max_{\bar{\Omega}_T} u \le FT + B.$$

Problem 15. Assume u solves $u_t - \Delta u = 0$, prove the following statements

- (a) If $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth convex function, then $v = \phi(u)$ satisfies

$$v_t - \Delta v \le 0.$$

- (b) Prove $v = |Du|^2 + u_t^2$ also satisfies the above inequality.

Problem 16. Define the following parabolic differential operator

$$Lu = u_t - a^2 u_{xx} + b(x,t)u_x + c(x,t)u.$$

Assume $c(x,t) > 0$, if $u(x,t) \in C_1^2(\bar{\Omega}_T)$ satisfies

$$Lu \le 0, \;\; in \;\Omega_T,$$

then

$$\max_{\Omega_T} u \le \max_{\Gamma_T} u^+,$$

where $u^+(x,t) = \max\{u(x,t), 0)\}$.

Problem 17. Consider the same operator L as in Problem 16. Assume for some constant $c_0 > 0$, $c(x,t) > -c_0$, and $u(x,t) \in C_1^2(\bar{\Omega}_T)$ satisfies

$$Lu \le 0, \;\; in \;\Omega_T,$$

if $\max_{\Gamma_T} u \le 0$, then

$$\max_{\Omega_T} u \le 0.$$