Chapter 5

Wave equation

In this chapter, we discuss the *wave equation*

$$u_{tt} - a^2 \Delta u = f, \tag{5.1}$$

where $a > 0$ is a constant. We will discover that solutions of the wave equation behave in a different way comparing with the solutions of Laplace's equation or the heat equation.

5.1 Physical derivation

In physics or mechanics, the wave equation serves as a simplified model for the oscillations on a vibrating string ($n = 1$), membrane ($n = 2$), or elastic solid ($n = 3$). In these models, $u(x,t)$ is the displacement of the pointed mass in certain directions of the point x at time $t \geq 0$. In (5.1), f models the external force.

For instance, we assume that an elastic solid occupied a region Ω in $\mathbb{R}^3$ without external force. For any subregion $G \subset \Omega$, $\vec{F}(x,t)$ is the contact force density acting on G through the boundary ∂G. Normalize the mass density to be unity. Let ν be the unit outer normal vector of ∂G. The acceleration within G is

$$\frac{d^2}{dt^2} \int_G u \, dx = \int_G u_{tt} \, dx,$$

while the net contact force is

$$- \int_{\partial G} \vec{F} \cdot \nu \, dS,$$

then, by Newton's law, one has

$$\int_G u_{tt} \, dx = - \int_{\partial G} \vec{F} \cdot \nu \, dS = \int_G \nabla \cdot \vec{F} \, dx$$

where we have applied the *divergence theorem*. Therefore, we conclude that

$$u_{tt} = -\nabla \cdot \vec{F}.$$

For elastic body, $\vec{F} = \vec{F}(\nabla u)$, so

$$u_{tt} + \nabla \cdot \vec{F}(\nabla u) = 0.$$

In the case of small oscillations, $|\nabla u|$ is very small, and so $\vec{F}(\nabla u) \approx -a^2 \nabla u$, therefore,

$$u_{tt} - a^2 \Delta u = 0.$$

We remark that, from the physical interpretation, it is mathematically appropriate to specify two initial conditions, on the displacement u and the velocity u_t at $t = 0$. This will appears often in the rest of this chapter.

5.2 Solution by Spherical means

Unlike the heat equation or Laplace equation, we will present an elegant method to solve (5.1) first for $n = 1$ directly and then for $n \geq 2$ by the method of spherical means.

5.2.1 d'Alembert's Formula, $n = 1$

We first start with the Cauchy problem for the wave equation in one space dimension. Consider

$$\begin{cases} u_{tt} - u_{xx} = 0, \ in \ \mathbb{R} \times (0, \infty) \\ u(x,0) = g(x), \ u_t(x,0) = h(x), \ x \in \mathbb{R}, \end{cases} \tag{5.2}$$

where g and h are given functions.

From the characteristic method we showed in Chapter 1, we know that the general solution of the 1-D wave equation takes the following form

$$u(x,t) = F(x+t) + G(x-t)$$

for any C^2 functions F and G. We now determine the F and G through the initial data. It turns out that

$$F(x) + G(x) = g(x), \ F'(x) - G'(x) = h(x), \tag{5.3}$$

which implies

$$F'(x) = \frac{1}{2}(g'(x) + h(x)), \ G'(x) = \frac{1}{2}(g'(x) - h(x)). \tag{5.4}$$

Therefore, one derives the following *d'Alembert's formula*:

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\ dy. \tag{5.5}$$

We thus proved the following Theorem.

Theorem 5.2.1 *Assume $g(x) \in C^2(\mathbb{R})$ and $h(x) \in C^1(\mathbb{R})$. Then $u(x,t)$ defined by d'Alembert's formula (5.5) is the unique solution of (5.2).*

In the following example, we apply the d'Alembert's formula to an initial boundary value problem using the *reflection method*.

Example 5.2.2 Consider

$$\begin{cases} u_{tt} - u_{xx} = 0, \ in \ \mathbb{R}_+ \times (0, \infty) \\ u(x,0) = g(x), \ u_t(x,0) = h(x), \ x \in \mathbb{R}_+, \\ u(0,t) = 0, \ t > 0 \end{cases} \tag{5.6}$$

where g and h are given functions such that $g(0) = h(0) = 0$.

We perform the following *odd reflection*.

$$\tilde{u}(x,t) = \begin{cases} u(x,t), \ x \geq 0, \ t \geq 0 \\ -u(-x,t), \ x \leq 0, \ t \geq 0, \end{cases}$$
$$\tilde{g}(x) = \begin{cases} g(x), \ x \geq 0, \\ -g(-x), \ x \leq 0, \end{cases} \tag{5.7}$$
$$\tilde{h}(x,t) = \begin{cases} h(x), \ x \geq 0 \\ -h(-x), \ x \leq 0. \end{cases}$$

It is clear that

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = 0, \ in \ \mathbb{R} \times (0, \infty) \\ \tilde{u}(x,0) = \tilde{g}(x), \ \tilde{u}_t(x,0) = \tilde{h}(x), \ x \in \mathbb{R}, \end{cases} \tag{5.8}$$

Hence, the d'Alembert's formula gives

$$\tilde{u}(x,t) = \frac{1}{2}[\tilde{g}(x+t) + \tilde{g}(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\ dy. \tag{5.9}$$

Finally, we transform this expression into the region for $x \geq 0$ and $t \geq 0$:

$$u(x,t) = \begin{cases} \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\ dy, \ if \ x \geq t \geq 0, \\ \frac{1}{2}[g(x+t) - g(t-x)] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\ dy, \ if \ 0 \leq x \leq t. \end{cases} \tag{5.10}$$

5.2.2 Spherical means

For $n \geq 2$, $u \in C^m(\mathbb{R}^n \times [0, \infty))$ $(m \geq 2)$ solves the initial problem

$$\begin{cases} u_{tt} - \Delta u = 0, \ in \ \mathbb{R}^n \times (0, \infty) \\ u(x,0) = g(x), \ u_t(x,0) = h(x), \ x \in \mathbb{R}^n, \end{cases} \tag{5.11}$$

From the physical experience on the wave propagation in $\mathbb{R}^3$, it appears that the wave propagates spherically. This motivates the approach of spherical means. We will first study the average of u over certain spheres. These averages, as functions of the time t and the radius r, solve the Euler-Poisson-Darboux equation, which could be transfered into the one-dimensional wave equation for which we know how to solve.

Definition 5.2.3 *Let $x \in \mathbb{R}^n$, $t > 0$, $r > 0$. Define*

$$U(x; r, t) = (u(y,t))_{\partial B(x,r)} = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y,t)\ dS_y. \tag{5.12}$$

Similarly, define

$$G(x; r) = (g(x))_{\partial B(x,r)},\ \ H(x; r) = (h(x))_{\partial B(x,r)}.$$

Lemma 5.2.4 *Fix $x \in \mathbb{R}^n$, and let u solves (5.11), then $U \in C^m(\bar{\mathbb{R}}_+ \times [0, \infty))$ solves the initial value problem for Euler-Poisson-Darboux equation*

$$\begin{cases} U_{tt} - U_{rr} - \dfrac{n-1}{r} U_r = 0,\ \ in\ \mathbb{R}_+ \times (0\infty) \\ U(x; r, 0) = G(x; r),\ \ U_t(x; r, 0) = H(x; r). \end{cases},\ \ in\ \mathbb{R}_+. \tag{5.13}$$

Proof. For $r > 0$, one has

$$U_r(x; r, t) = \frac{r}{n} (\Delta u(y,t))_{B(x,r)}. \tag{5.14}$$

Therefore, one has $U_r(x; r, t) \to 0$ as $r \to 0^+$. Then we differentiate (5.14) to have

$$U_{rr} = (\Delta u)_{\partial B(x,r)} + (\frac{1}{n} - 1)(\Delta u)_{B(x,r)},$$

which implies that

$$\lim_{r \to 0^+} U_{rr} = \frac{1}{n} \Delta u.$$

Similar calculations show the regularity of U. Now, by the wave equation, one has

$$\begin{aligned} (r^{n-1} U_r)_r &= \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}(y,t)\ dS_y \\ &= r^{n-1} (u_{tt})_{\partial B(x,r)} = r^{n-1} U_{tt}. \end{aligned}$$

This completes the proof of the theorem.

5.2.3 Kirchhoff's formula, $n = 3$

We now take $n = 3$ and suppose $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ solves the initial value problem (5.11), and U, G, H defined in the Definition 5.2.3. The magic transform is

$$\tilde{U} = rU, \ \tilde{G} = rG, \ \tilde{H} = rH, \tag{5.15}$$

which solves the following problem

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0, \ in \ \mathbb{R}_+ \times (0, \infty) \\ \tilde{U}(x; r, 0) = \tilde{G}(x; r), \ \tilde{U}_t(x; r, 0) = \tilde{H}(x; r), \ r \in \mathbb{R}_+, \\ \tilde{U}(x; 0, t) = 0, \ t > 0. \end{cases} \tag{5.16}$$

From the Example 5.2.2, we know that for $0 \le r \le t$,

$$\tilde{U}(x; r, r) = \frac{1}{2}[\tilde{G}(x; r + t) - \tilde{G}(x; t - r)] + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y) \ dy, \tag{5.17}$$

We note that for continuous function u, one has

$$\lim_{r \to 0^+} U(x; r, t) = u(x, t).$$

Therefore,

$$\begin{aligned} u(x, t) &= \lim_{r \to 0^+} \frac{\tilde{U}(x; r, t)}{r} \\ &= \lim_{r \to 0^+} [\frac{\tilde{G}(x; r + t) - \tilde{G}(x; t - r)}{2r} + \frac{1}{2r}\int_{-r+t}^{r+t} \tilde{H}(y) \ dy \\ &= \tilde{G}'(x; t) + \tilde{H}(x; t). \end{aligned}$$

Hence one reaches the Kirchhoff's formula

$$u(x, t) = \frac{\partial}{\partial t}(t(g)_{\partial B(x,t)}) + t(h)_{\partial B(x,t)}. \tag{5.18}$$

A further calculation gives the more explicit form

$$u(x, t) = \frac{1}{|\partial B(x, t)|}\int_{\partial B(x,t)} [th(y) + g(y) + Dg(y) \cdot (y - x)] \ dS_y. \tag{5.19}$$

5.2.4 Poisson's formula, $n = 2$

Unlike the 3D case, when $n = 2$, the Euler-Poisson-Darboux equation cannot be transferred into the one-dimensional wave equation. We will apply the *method of descent* introduced by Hadamard. The idea is to take the initial value problem (5.11) for $n = 2$ and regard it as a problem for $n = 3$ where the third spatial variable x_3 does not appear.

To this purpose, we assume $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ solves (5.11) for $n = 2$, and we write

$$\bar{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t). \tag{5.20}$$

Therefore, for

$$\bar{g}(x_1, x_2, x_3) = g(x_1, x_2),\ \bar{h}(x_1, x_2, x_3) = h(x_1, x_2),$$

we have

$$\begin{cases} \bar{u}_{tt} - \Delta \bar{u} = 0,\ \textit{in } \mathbb{R}^3 \times (0,\infty), \\ \bar{u} = \bar{g},\ \bar{u}_t = \bar{h},\ \textit{on } \mathbb{R}^3 \times \{t = 0\}. \end{cases} \tag{5.21}$$

Denote by $x = (x_1, x_2) \in \mathbb{R}^2$, and $\bar{x} = (x_1, x_2, 0) \in \mathbb{R}^3$, we know from the Kirchhoff's formula (5.18) that

$$u(x,t) = \bar{u}(\bar{x},t) = \frac{\partial}{\partial t}(t(\bar{g})_{\partial \bar{B}(\bar{x},t)}) + t(\bar{h})_{\partial \bar{B}(\bar{x},t)}, \tag{5.22}$$

where $\bar{B}(\bar{x},t)$ is the ball in $\mathbb{R}^3$ centered at $\bar{x}$ with radius $t > 0$. For $y \in B(x,t)$ and $\gamma(y) = (t^2 - |y-x|^2)^{\frac{1}{2}}$, we note that

$$\begin{aligned} (\bar{g})_{\partial \bar{B}(\bar{x},t)} &= \frac{1}{4\pi t^2} \int_{\partial \bar{B}(\bar{x},t)} \bar{g}\ d\bar{S} \\ &= \frac{2}{4\pi t^2} \int_{B(x,t)} g(y)(1 + |D\gamma(y)|^2)^{\frac{1}{2}}\ dy. \end{aligned}$$

Since $(1 + |D\gamma(y)|^2)^{\frac{1}{2}} = t(t^2 - |y-x|^2)^{\frac{1}{2}}$, we thus have

$$\begin{aligned} (\bar{g})_{\partial \bar{B}(\bar{x},t)} &= \frac{2}{2\pi t} \int_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{\frac{1}{2}}}\ dy \\ &= \frac{t}{2} \left(\frac{g(y)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} \right)_{B(x,t)}. \end{aligned}$$

Further simplification gives the following *Poisson's formula*:

$$u(x,t) = \frac{1}{2} \left(\frac{tg(y) + t^2 h(y) + tDg(y) \cdot (y-x)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} \right)_{B(x,t)} \tag{5.23}$$

for $x \in \mathbb{R}^2$, $t > 0$.

5.2.5 Further generalization

We now generalized the previous results to any dimensions for $n \geq 3$. The following identities can be proved by induction.

Lemma 5.2.5 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be C^{k+1}. Then for $k = 1, 2, \cdots$, the following identities hold*

- $(\frac{d^2}{dr^2})(\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}\phi(r)) = (\frac{1}{r}\frac{d}{dr})^k(r^{2k}\frac{d\phi(r)}{dr})$.
- $(\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}\phi(r)) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^j\phi(r)}{dr^j}$, *where β_j^k are constants independent of ϕ.*
- $\beta_0^k = 1\cdot 3\cdot 5\cdots(2k-1)$.

Now, we assume that $n \geq 3$ is an odd integer and set $n = 2k+1$. Then the following transformation

$$\begin{cases} \tilde{U}(r,t) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U(x;r,t)) \\ \tilde{G}(r) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}G(x;r)) \\ \tilde{H}(r) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}H(x;r)), \end{cases} \tag{5.24}$$

converts the Euler-Poisson-Darboux equation into the one-dimensional wave equation:

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0, \ in\ \mathbb{R}_+ \times (0,\infty) \\ \tilde{U}(x;r,0) = \tilde{G}(x;r), \ \tilde{U}_t(x;r,0) = \tilde{H}(x;r), \ r \in \mathbb{R}_+, \\ \tilde{U}(x;0,t) = 0, \ t > 0. \end{cases} \tag{5.25}$$

Solution for odd n. Follow the similar steps in the case of $n = 3$, we have for $n = 2k+1$ and $\gamma_n = 1\cdot 3\cdot 5\cdots(n-2)$ the following representation formula:

$$u(x,t) = \frac{1}{\gamma_n}\left[(\frac{\partial}{\partial t})(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}}(t^{n-2}(g)_{\partial B(x,t)}) + (\frac{1}{t}\frac{d}{dt})^{\frac{n-3}{2}}(t^{n-2}(h)_{\partial B(x,t)})\right]. \tag{5.26}$$

Solution for even n. Then, the *method of descent* will give the results for even n. For even n, and $\gamma_n = 2\cdot 4\cdot 5\cdots(n-2)\cdot n$, we have the following solution formula

$$\begin{aligned} &u(x,t) = \\ &\frac{1}{\gamma_n}\left[(\frac{\partial}{\partial t})(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}}(t^n(\frac{g(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}})_{B(x,t)})\right] \\ &+\frac{1}{\gamma_n}\left[(\frac{1}{t}\frac{d}{dt})^{\frac{n-2}{2}}(t^n(\frac{g(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}})_{B(x,t)})\right]. \end{aligned} \tag{5.27}$$

5.3 Nonhomogeneous problem

We now study the initial value problem for the nonhomogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = f, \ \textit{in} \ \mathbb{R}^n \times (0, \infty) \\ u(x,0) = 0, \ u_t(x,0) = 0, \ x \in \mathbb{R}^n. \end{cases} \tag{5.28}$$

By Duhamel's principle, let $u(x,t;s)$ be the solutions of

$$\begin{cases} u_{tt}(\cdot;s) - \Delta u(\cdot;s) = 0, \ \textit{in} \ \mathbb{R}^n \times (0, \infty) \\ u(x,0) = 0, \ u_t(x,0) = f(\cdot, s), \ \textit{on} \ \mathbb{R}^n \times \{t = s\}, \end{cases} \tag{5.29}$$

we set

$$u(x,t) = \int_0^t u(x,t;s)\ ds, \ (x \in \mathbb{R}^n, \ t \geq 0). \tag{5.30}$$

The following theorem asserts this gives the solution of (5.28).

Theorem 5.3.1 *For $n \geq 2$ and $f \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \times [0,\infty))$, $u(x,t)$ defined in (5.30) is a solution of (5.28).*

Proof. Through the solutions constructed in the last section, $u(\cdot,\cdot;s) \in C^2(\mathbb{R}^n \times [0,\infty)$. Then, we compute

$$u_t(x,t) = u(x,t;t) + \int_0^t u_t(x,t;s)\ ds = \int_0^t u_t(x,t;s)\ ds,$$

$$\begin{aligned} u_{tt}(x,t) &= u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)\ ds \\ &= f(x,t) + \int_0^t u + tt(x,t;s)\ ds. \end{aligned}$$

Furthermore

$$\Delta u(x,t) = \int_0^t \Delta u(x,t;s)\ ds = \int_0^t u_{tt}(x,t;s)\ ds.$$

Therefore,

$$u_{tt}(x,t) = \Delta u(x,t) = f(x,t), \ (x \in \mathbb{R}^n, \ t > 0).$$

Clearly, $u(x,0) = u_t(x,0) = 0$ for any $x \in \mathbb{R}^n$.

Example 5.3.2 When $n = 3$, Kirchhoff's formula implies

$$u(x,t,;s) = (t-s)(f(y,s)_{\partial B(x,t-s)}.$$

Therefore,

$$\begin{aligned} u(x,t) &= \int_0^t (t-s)(f(y,s)_{\partial B(x,t-s)}\, ds \\ &= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{t-s}\, dSds \\ &= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\, dSdr. \end{aligned}$$

Hence, one obtains the solution of (5.28) for $n=3$

$$u(x,t) = \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\, dy, \ \ (x \in \mathbb{R}^3, \ t \geq 0) \tag{5.31}$$

where

$$\frac{f(y,t-|y-x|)}{4\pi|y-x|}$$

is called *retarded potential*.

5.4 Energy method

In this section, we introduce the *energy method*, which is a powerful tool in the theory of partial differential equations. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with a smooth boundary $\partial\Omega$. Define $\Omega_T = \Omega \times (0,T]$, $\Gamma_T = \bar{\Omega}_T \setminus \Omega_T$, where $T > 0$.

5.4.1 Wave equation

We first consider the following initial boundary value problem

$$\begin{cases} u_{tt} - \Delta u = f, \ \ in \ \Omega_T, \\ u = g, \ \ on \ \Gamma_T, \\ u_t = h, \ \ on \ \Omega \times \{t=0\}. \end{cases} \tag{5.32}$$

We now prove the uniqueness for the above problem using *energy method*.

Theorem 5.4.1 *There exists at most one solution $u \in C^2(\bar{\Omega}_T)$ for (5.32).*

Proof: Let u and $\bar{u}$ be two solutions of (5.32), then $w = u - \bar{u}$ soloves

$$\begin{cases} w_{tt} - \Delta w = 0, \ \ in \ \Omega_T, \\ w = 0, \ \ on \ \Gamma_T, \\ w_t = 0, \ \ on \ \Omega \times \{t=0\}. \end{cases}$$

Define the energy

$$e(t) = \frac{1}{2}\int_{\Omega}(w_t^2 + |Dw|^2)(x,t)\ dx,\ (0 \le t \le T).$$

It is clear that

$$\begin{aligned} e'(t) &= \int_{\Omega} w_t w_{tt} + Dw \cdot Dw_t\ dx \\ &= \int_{\Omega} w_t(w_{tt} - \Delta w)\ dx = 0. \end{aligned}$$

Therefore, $e(t) = e(0) = 0$ for all $t \in [0,T]$, and so $w_t \equiv 0$ and $Dw \equiv 0$ in Ω_T. By the initial boundary conditions, we know that $w \equiv 0$ and thus $u = \bar{u}$ in Ω_T.

5.4.2 Domain of dependence

Energy method is also useful to study some local behavior of solutions. We will illustrate the domain of dependence of solutions to the wave equation and the *finite propagation speed* properties.

Now, suppose $u \in C^2$ solves

$$u_{tt} - \Delta u = 0,\ in\ \mathbb{R}^n \times (0,\infty).$$

Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$. We define the cone

$$\mathcal{C} = \{(x,t)|0 \le t \le t_0, |x - x_0| \le t_0 - t\}.$$

We first prove the following *finite propagation speed* property, which says that any disturbance originating outside $B(x_0,t_0)$ has no effect on the solution within $\mathcal{C}$.

Theorem 5.4.2 *If $u \equiv u_t \equiv 0$ on $B(x_0,t_0)$, then $u \equiv 0$ within the cone $\mathcal{C}$.*

Proof: Define the local energy

$$e(t) = \frac{1}{2}\int_{B(x_0,t_0-t)}(u_t^2 + |Du|^2)(x,t)\ dx,\ t \in [0,t_0].$$

We now compute

$$\begin{aligned} e'(t) &= \int_{B(x_0,t=0-t)} u_t u_{tt} + Du \cdot Du_t\ dx - \frac{1}{2}\int_{\partial B(x_0,t_0-t)}(u_t^2 + |Du|^2)\ dS \\ &= \int_{B(x_0,t_0-t)} u_t(u_{tt} - \Delta u)\ dx - \frac{1}{2}\int_{\partial B(x_0,t_0-t)}(u_t^2 + |Du|^2 - 2u_t\frac{\partial u}{\partial \nu})\ dS \\ &= -\frac{1}{2}\int_{\partial B(x_0,t_0-t)}(u_t^2 + |Du|^2 - 2u_t\frac{\partial u}{\partial \nu})\ dS \end{aligned}$$

We now observe that, by Cauchy-Schwartz inequality,

$$|u_t \frac{\partial u}{\partial \nu}| \leq \frac{1}{2} u_t^2 + \frac{1}{2} |Du|^2.$$

Therefore, we have

$$e'(t) \leq 0$$

which means $e(t) \leq e(0) = 0$ for all $t \in [0, t_0]$. Thus $u_t \equiv 0$ and $Du \equiv 0$ and so $u \equiv 0$ within the cone $\mathcal{C}$.

The bottom of the cone $\mathcal{C}$ is $B(x_0, t_0)$ on the initial hyper-plane, is called the domain of dependence for the point (x_0, t_0).

It is now clear that Theorem 5.4.2 also implies the uniqueness for the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = f, \ in \ \mathbb{R}^n, \\ u = g, \ u_t = h, \ on \ \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{5.33}$$

5.4.3 Energy method for Heat equation

We now study the large time asymptotic behavior of the solutions to the following initial boundary value problem for heat equation.

$$\begin{cases} u_t - \Delta u = 0, \ in \ \Omega \times (0, \infty), \\ u = 0, \ on \ \partial\Omega, \\ u = g, \ on \ \Omega \times \{t = 0\}. \end{cases} \tag{5.34}$$

Theorem 5.4.3 *The solution of (5.34) converges to zero exponentially fast in time.*

Proof: Let u be the solution of the problem (5.34), we define the energy

$$e(t) = \int_\Omega u^2 \ dx.$$

Now we compute

$$\begin{aligned} e'(t) &= 2 \int_\Omega u u_t \ dx \\ &= 2 \int_\Omega u \Delta u \ dx \\ &= -2 \int_\Omega |Du|^2 \ dx. \end{aligned}$$

However, by Poincaré inequality, one knows that there exists a positive constant $C = C(\Omega)$ such that

$$C \int_\Omega u^2 \ dx \leq \int_\Omega |Du|^2 \ dx,$$

for any $u \in H_0^1(\Omega)$. Therefore, we have

$$e'(t) + 2Ce(t) \leq 0,$$

which implies

$$e(t) \leq e(0)e^{-2Ct}, \ \forall t > 0.$$

Therefore, one sees that $e(t) \to 0$ as $t \to \infty$ and thus u converges to zero in energy norm exponentially fast in time.

5.5 Initial boundary value problem: $n = 1$

For wave equation in one spatial dimension, there are several approaches to solve the initial boundary value problem, including the characteristic method, reflection method and the Fourier method (separation of variables). We will focus on the method of separation of variables.

5.5.1 Homogeneous equation

The Fourier method is motivated from the physical fact: vibrations can be decomposed into simple oscillations according to its frequency. Therefore, one can try to look for the solution of the form

$$u_n(x,t) = X_n(x)T_n(t), \ n = 1, 2, \cdots,$$

then, determine the constants in the following formula

$$u(x,t) = \sum_{n=1}^{\infty} C_n X_n(x) T_n(t),$$

to solve the problem.

We now show this idea using the following example.

Example 5.5.1 Solve the following problem

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \ 0 < x < l, \ t > 0, \\ u(0,t) = 0, \ u(l,t) = 0, \ t \geq 0, \\ u(x,0) = f(x), \ u_t(x,0) = g(x), \ 0 \leq x \leq l, \end{cases} \tag{5.35}$$

where f and g are C^1 satisfying the compatibility condition

$$f(0) = f(l) = 0, \ g(0) = g(l) = 0.$$

Solution: The Fourier method consists the following steps.

(i) Separation of variables. We first look for the solution of the form

$$u(x,t) = X(x)T(t) \neq 0$$

satisfying the boundary conditions. This leads to

$$\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda,\ X(x)T(t) \neq 0.$$

It is clear that λ is a constant. We thus have

$$\begin{cases} T''(t) + \lambda a^2 T(t) = 0,\ t > 0 \\ X''(x) + \lambda X(x) = 0,\ 0 < x < l. \end{cases} \tag{5.36}$$

For $u(x,t)$ to satisfy the boundary condition, one requires $X(0) = X(l) = 0$.

(ii) Solve the eigenvalue problem. We now solve the following eigenvalue problem

$$\begin{cases} X''(x) + \lambda X(x) = 0,\ 0 < x < l \\ X(0) = X(l) = 0. \end{cases} \tag{5.37}$$

where the λ is called the eigenvalue if it gives a non-zero solution $X_\lambda(x)$ which is called the associated eigenfunction. There are three cases concerning with the sign of λ.

(a) If $\lambda < 0$, the general solution is

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x},$$

where A and B will be determined through the boundary conditions. In deed,

$$\begin{aligned} X(0) &= A + B = 0 \\ X(l) &= Ae^{\sqrt{-\lambda}l} + Be^{-\sqrt{-\lambda}l} = 0. \end{aligned}$$

Therefore, $A = B = 0$ and $\lambda < 0$ is not the eigenvalue.

(b) If $\lambda = 0$, then $X(x) = A + Bx$ which is identically zero with the boundary conditions $X(0) = 0 = X(l)$. So, $\lambda = 0$ is not an eigenvalue either.

(c) We now consider $\lambda = k^2 > 0$. The general solution takes the form

$$X(x) = A\cos(kx) + B\sin(kx).$$

Using the boundary condition $X(0) = 0 = X(l)$, one has $A = 0$ and $B\sin(kl) = 0$. Since we need the non-zero solution, $B \neq 0$ and

$$k = \frac{n\pi}{l},\ \lambda_n = (\frac{n\pi}{l})^2,\ n = 1, 2, \cdots \tag{5.38}$$

This gives the eigenvalues for problem (5.37) and the corresponding eigenfunctions are

$$X_n(x) = \sin(\frac{n\pi}{l}x), \ n = 1, 2, \cdots \tag{5.39}$$

We now substitute λ_n into the first equation in (5.36),

$$T_n(t) = c_n \cos(\frac{an\pi}{l}t) + d_n \sin(\frac{an\pi}{l}t), \ n = 1, 2, \cdots$$

Therefore, we found

$$u_n(x,t) = X_n(x)T_n(t) = [c_n \cos(\frac{an\pi}{l}t) + d_n \sin(\frac{an\pi}{l}t)] \sin(\frac{n\pi}{l}x), \ n = 1, 2, \cdots \tag{5.40}$$

where c_n and d_n are arbitrary constants to be determined later.

(iii) In this step, we hope to find the solution of the initial boundary value problem (5.35) through linear combination of $u_n(x,t)$. Let

$$u(x,t) = \sum_{n=1}^{\infty} [c_n \cos(\frac{an\pi}{l}t) + d_n \sin(\frac{an\pi}{l}t)] \sin(\frac{n\pi}{l}x). \tag{5.41}$$

Using the initial conditions, we require

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{l}x) = f(x), \\ u_t(x,0) &= \sum_{n=1}^{\infty} d_n \frac{an\pi}{l} \sin(\frac{n\pi}{l}x) = g(x). \end{aligned} \tag{5.42}$$

From the conditions of f and g and we require they are C^1, then

$$\begin{aligned} c_n &= \frac{2}{l} \int_0^l f(x) \sin(\frac{n\pi}{l}x) \ dx, \\ d_n &= \frac{2}{an\pi} \int_0^l g(x) \sin(\frac{n\pi}{l}x) \ dx. \end{aligned} \tag{5.43}$$

We therefore obtained the formal solution of the problem (5.35). One could further verify that the series solution has good convergent properties to be a classical solution if f and g are $C^4([0,l])$ and $f(0) = f''(0) = g(0) = f(l) = f''(l) = g(l) = 0$.

We see from this example that as long as the corresponding eigenvalue problem is well solved, we are able to find at least the formal solution of the initial value problem for homogeneous wave equation in one space dimension. The eigenvalue problem will be discussed in great details at the end of this chapter. For the formal solution to be classical solution, it often requires very restrictive properties on the data, this however could be resolved by the notion of *weak solution*.

5.5.2 Non-homogeneous equation

In this section, we further show that the eigenfunctions have further applications to solve the initial boundary value problem for the non-homogeneous equation. This is so-called *expansion about eigenfunctions.*

Again, we will illustrate our ideas using specific example.

Example 5.5.2 Solve the following problem

$$\begin{cases} u_{tt} - a^2 u_{xx} = f(x,t), \ 0 < x < l, \ t > 0, \\ u(0,t) = 0, \ u(l,t) = 0, \ t \geq 0, \\ u(x,0) = 0, \ u_t(x,0) = 0, \ 0 \leq x \leq l, \end{cases} \tag{5.44}$$

Solution: Unless $f(x,t)$ has certain specific structure, $u(x,t) = X(x)T(t)$ is not a solution of the non-homogeneous wave equation. However, motivated by the constant variation method of ODEs, we seek the solution of the following form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) \tag{5.45}$$

where $X_n(x)$ are the eigenfunctions for the corresponding homogeneous wave equation under the same boundary conditions. In this case,

$$X_n(x) = \sin(\frac{n\pi}{l}x), \ n = 1, 2, \cdots .$$

We hope that the solution could be expanded into the series of $X_n(x)$ with variant coefficients $T_n(t)$. Clearly, (5.45) satisfies the boundary conditions. We will determine $T_n(t)$ through the equation and the initial conditions. Therefore, $T_n(t)$ satisfies

$$\begin{aligned} &\sum_{n=1} [T''(t) + (\frac{an\pi}{l})^2 T_n(t)] \sin(\frac{n\pi}{l}x) = f(x,t), \\ &u(x,0) = \sum_{n=1}^{\infty} T_n(0) \sin(\frac{n\pi}{l}x) = 0 \\ &u_t(x,0) = \sum_{n=1}^{\infty} T_n'(0) \sin(\frac{n\pi}{l}x) = 0. \end{aligned} \tag{5.46}$$

Setting

$$f_n(t) = \frac{l}{2} \int_0^l f(x,t) \sin(\frac{n\pi}{l}x) \ dx, \ n = 1, 2, \cdots ,$$

one has

$$\begin{cases} T_n''(t) + (\frac{an\pi}{l})^2 T_n(t) = f_n(t) \\ T_n(0) = T_n'(0) = 0, \ n = 1, 2, \cdots . \end{cases}$$

Hence, we found

$$T_n(t) = \frac{l}{an\pi} \int_0^t f_n(\tau) \sin(\frac{an\pi}{l}(t-\tau))\ d\tau,\ n = 1, 2, \cdots .$$

We finally obtain the formal solution of the problem (5.44)

$$u(x,t) = \sum_{n=1}^{\infty} [\frac{l}{an\pi} \int_0^t f_n(\tau) \sin(\frac{an\pi}{l}(t-\tau))\ d\tau] \sin(\frac{n\pi}{l}x). \tag{5.47}$$

For general initial boundary value problem, one can apply the linear superposition principle and some basic techniques to transfer the non-homogeneous boundary conditions to homogeneous ones. We omit the details.

5.5.3 Sturm-Liouville Theory

In this subsection, we briefly discuss the theory of Sturm-Liouville for eigenvalue problem. Consider the initial boundary value problem

$$\begin{cases} Lu = 0,\ x \in (a,b), t > 0, \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x),\ x \in [a,b] \\ \alpha_1 u(a,t) + \alpha_2 u_x(a,t) = 0, \beta_1 u(b,t) + \beta_2 u_x(b,t) = 0,\ t \geq 0, \end{cases} \tag{5.48}$$

where

$$Lu = A(t)u_{tt} + C(x)u_{xx} + D(t)u_t + E(x)u_x + (F_1(t) + F_2(x))u.$$

We assume that α_i and β_i are constants for $i = 1, 2$ such that

$$\alpha_1^2 + \alpha_2^2 \neq 0,\ \beta_1^2 + \beta_2^2 \neq 0.$$

Suppose $A(t) \geq A_0 > 0$, $C(x) \leq C_0 < 0$, A_0 and C_0 are constants. All other coefficients are continuous. We also assume $F_2(x) > 0$.

If we want to perform the method of separation of variables, $u(x,t) = X(x)T(t)$, for λ the possible eigenvalue, we need to solve the equation of $T(t)$

$$AT''(t) + DT'(t) + F_1 T + \lambda T = 0 \tag{5.49}$$

and the eigenvalue problem

$$\begin{cases} CX'' + EX' + F_2 X - \lambda X = 0, \\ \alpha_1 X(a) + \alpha_2 X'(a) = 0, \\ \beta_1 X(b) + \beta_2 X'(b) = 0. \end{cases} \tag{5.50}$$

In order to solve this eigenvalue problem, we first convert it into the self-adjoint form. Multiplying the equation by

$$S = -\frac{1}{C}\exp\{\int_0^x \frac{E}{C}\,dx\},$$

the equation became

$$[p(x)X']' - q(x)X + \lambda SX = 0, \tag{5.51}$$

where

$$p(x) = -SC \geq \exp\{\int_0^x \frac{E}{C_0}\,dx\} = p_0 > 0,$$
$$q(x) = SF_2 > 0,\ S(x) > 0.$$

Therefore, (5.50) has been changed into the standard form of Sturm-Liouville eigenvalue problem

$$\begin{cases} [p(x)X']' - q(x)X + \lambda SX = 0 \\ \alpha_1 X(a) + \alpha_2 X'(a) = 0, \\ \beta_1 X(b) + \beta_2 X'(b) = 0. \end{cases} \tag{5.52}$$

We first list some properties of the eigenfunctions.

Theorem 5.5.3 *Let X_1 and X_2 are eigenfunctions corresponding to the same eigenvalue λ, then $X_1 = CX_2$ for certain nonzero constant C.*

Theorem 5.5.4 *If X_1 and X_2 are eigenfunctions corresponding to eigenvalues λ_1 and λ_2 respectively, then*

$$\int_a^b SX_1X_2\,dx = 0.$$

We now introduce the working space for the Sturm-Liouville theory. For $0 < S(x) \in C([a,b])$, we first define

$$L_S^2 \equiv \{y(x) \in L_{loc}^1([a,b]) : \int_a^b S(x)y^2(x)\,dx < \infty\},$$

equipped with the inner product

$$(y_1, y_2)_S \equiv \int_a^b S(x)y_1(x)y_2(x)\,dx,\ \forall y_1, y_2 \in L_S^2.$$

For the functions from $C_0^1([a,b])$ (C^1 functions vanishing at endpoints), we introduce an inner product

$$(y_1, y_2)_H \equiv \int_a^b [p(x)y_1'y_2' + q(x)y_1y_2]\,dx,\ \forall y_1, y_2 \in C_0^1([a,b]), \tag{5.53}$$

where $p(x)$ and $q(x)$ are continuous functions such that there is a constant $p_0 > 0$ and $p(x) > p_0$, $q(x) > 0$ for any $x \in [a, b]$.

(5.53) defines a norm $\|\cdot\|_H$, we denote $H^{0,1}_{p,q}$ for the closure space of $C^1_0([a, b])$ under the norm $\|\cdot\|_H$. This is a Hilbert space.

We now use the Dirichlet boundary condition as the example to illustrate the procedure. Consider now

$$\begin{cases} [p(x)X']' - q(x)X + \lambda SX = 0 \\ X(a) = X(b) = 0. \end{cases} \tag{5.54}$$

Definition 5.5.5 *$X \in H^{0,1}_{p,q}$ is said to be a weak solution of (5.54) if it satisfies*

$$(X, y)_H = \lambda(X, y)_S$$

for any $y \in C^1_0[a, b]$.

Consider the functional

$$J(X) = \frac{(X, X)_H}{(X, X)_S}, \ \forall X \in H^{0,1}_{p,q}. \tag{5.55}$$

Theorem 5.5.6 *There exits $0 \neq X \in H^{0,1}_{p,q}$ such that*

$$J(X) = \inf_{z \in H^{0,1}_{p,q}} J(z).$$

Now, we set $K_1 = H^{0,1}_{p,q}$, we call

$$\lambda_1 = \inf_{0 \neq X \in K_1} J(X)$$

the first eigenvalue of (5.54), and the function $0 \neq X_1 \in K_1$ such that

$$\frac{(X_1, X_1)_H}{(X_1, X_1)_S} = \lambda_1$$

the eigenfunction corresponding to λ_1.

Now, we set

$$K_2 = \{X \in K_1 | (X, X_1)_S = 0\}.$$

Similarly, we define

$$\lambda_2 = \inf_{X \in K_2} J(X)$$

as the second eigenvalue and $0 \neq X_2 \in K_2$ such that

$$\frac{(X_2, X_2)_H}{(X_2, X_2)_S} = \lambda_2$$

the eigenfunction corresponding to λ_2. Inductively, we set

$$K_n = \{X \in K_1 | (X, X_1)_S = (X, X_2)_S = \cdots = (X, X_{n-1})_S = 0\},$$

and the n-th eigenvalue

$$\lambda_n = \inf_{X \in K_n} J(X)$$

and $0 \neq X_n \in K_n$ such that

$$\frac{(X_n, X_n)_H}{(X_n, X_n)_S} = \lambda_n$$

the eigenfunction corresponding to λ_n.

Theorem 5.5.7 *Let λ_n be eigenvalues of (5.54) and X_n the corresponding eigenfunctions of λ_n. Then*

- $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$
- $\lambda_n \to \infty$ *as* $n \to \infty$.
- $\{X_n\}_{n=1}^{\infty}$ *form an orthogonal basis of* L^2_S.

We therefore solved the Sturm-Liouville problem (5.54). The general case can be treated in a similar way.

5.6 Problems

Problem 1. Let $u \in C^2(\mathbb{R} \times [0, \infty))$ be the solution of

$$\begin{cases} u_{tt} - u_{xx} = 0, \ x \in \mathbb{R}, t > 0, \\ u(x, 0) = \phi(x), \ u_t(x, 0) = \psi(x), \end{cases}$$

where $\phi(x)$ and $\psi(x)$ have compact support. Define

$$K(t) = \frac{1}{2} \int_{-\infty}^{+\infty} u_t^2(x, t) \ dx$$
$$P(t) = \frac{1}{2} \int_{-\infty}^{+\infty} u_x^2(x, t) \ dx.$$

Prove the following

- (a). $K(t) + P(t)$=constant;
- (b). When t is sufficiently large, $K(t) = P(t)$.

Problem 2. Solve the following initial value problem

$$\begin{cases} u_{tt} - u_{xx} = 0, \ x \in \mathbb{R}, t > ax, \\ u|_{t=ax} = \phi(x), \ x \in \mathbb{R}, \\ u_t|_{t=ax} = \psi(x), \ x \in \mathbb{R}, \end{cases}$$

where $a \neq \pm 1$. If the initial data are given on $x \in [b_1, b_2]$, then on what region can you determine the solution?

Problem 3. Prove the initial value problem

$$\begin{cases} u_{tt} - u_{xx} = 6(x+t), \ x \in \mathbb{R}, t > x, \\ u|_{t=x} = 0, \ u_t|_{t=x} = \psi(x), \ x \in \mathbb{R}, \end{cases}$$

has solutions if and only if $\psi(x) - 3x^2 = constat$. When the solutions exist, it is not unique. Why does this problem behave differently from problem 2?

Problem 4. Solve the following problems

- (a) $\begin{cases} u_{tt} - a^2 u_{xx} = 0, \ x \in \mathbb{R}, t > 0, \\ u|_{x-at=0} = \phi(x), \ u|_{x+at=0} = \psi(x), \ \phi(0) = \psi(0); \end{cases}$

- (b) $\begin{cases} u_{tt} - a^2 u_{xx} = 0, \ x \in \mathbb{R}, t > 0, \\ u|_{t=0} = \phi(x), \ u|_{x-at=0} = \psi(x), \ \phi(0) = \psi(0); \end{cases}$

- (c) $\begin{cases} u_{xx} + 2\cos(x) u_{xy} - \sin^2(x) u_{yy} - \sin(x) u_y = 0, \\ u|_{y=\sin(x)} = \phi(x), \ u_y|_{y=\sin(x)} = \psi(x), \ x, y \in \mathbb{R}; \end{cases}$

- (d) $\begin{cases} u_{xx} + y u_{xy} + \frac{1}{2} u_y = 0, \ x \in \mathbb{R}, \ y < 0 \\ u|_{y=0} = \phi(x), \ u_y|_{y=0} < \infty; \end{cases}$

- (e) $\begin{cases} y^2 u_{yy} - x^2 u_{xx} = 0, \ x \in \mathbb{R}, \ y > 1, \\ u|_{y=1} = f(x), \ u_y|_{y=1} = g(x). \end{cases}$

Problem 5. Let h be a nonzero constant, F and G be C^2 functions. Prove that

$$u(x,t) = \frac{1}{h-x}(F(x-at) + G(x+at))$$

is the general solution of

$$[(1 - \frac{x}{h})^2 u_x]_x = \frac{1}{a^2}(1 - \frac{x}{h})^2 u_{tt}.$$

Find the solution of this equation with the following initial conditions

$$u(x,0) = \phi(x),\ u_t(x,0) = \psi(x).$$

Problem 6. If $w(x,t;\tau)$ is the solution of

$$(A)\begin{cases} w_{tt} - a^2 w_{xx} = 0,\ x \in \mathbb{R}, t > \tau, \\ w(x,\tau;\tau) = 0,\ w_t(x,\tau;\tau) = f(x,\tau), \end{cases}$$

prove that the function

$$u(x,t) = \int_0^t w(x,t;\tau)\ d\tau$$

is the solution of

$$(B)\begin{cases} u_{tt} - a^2 u_{xx} = f(x,t),\ x \in \mathbb{R}, t > 0, \\ u(x,0) = 0,\ u_t(x,0) = 0, \end{cases}$$

This is the so-called Duhamel principle.

Problem 7. Use the separation of variable method to solve the following problems

- (a) $\begin{cases} u_{tt} - a^2 u_{xx} = 0,\ 0 < x < l,\ t > 0, \\ u(x,0) = x^2 - 2lx,\ u_t(x,0) = 0, \\ u(0,t) = u_x(l,t) = 0,\ t \geq 0; \end{cases}$

- (b) $\begin{cases} u_{tt} - a^2 u_{xx} = 0,\ 0 < x < l,\ t > 0, \\ u(x,0) = \begin{cases} \frac{hx}{c},\ x \in [0,c] \\ h\frac{l-x}{l-c},\ x \in (c,l]. \end{cases} \\ u_t(x,0) = 0, \\ u(0,t) = u(l,t) = 0,\ t \geq 0; \end{cases}$

Problem 8. Solve the following initial boundary value problems.

- (a) $\begin{cases} u_{tt} - a^2 u_{xx} = Ax,\ 0 < x < l,\ t > 0, \\ u(x,0) = u_t(x,0) = 0,\ 0 \leq x \leq l, \\ u(0,t) = u(l,t) = 0,\ t \geq 0; \end{cases}$

- (b) $\begin{cases} u_{tt} - a^2 u_{xx} = 0,\ 0 < x < l,\ t > 0, \\ u(x,0) = u_t(x,0) = 0,\ 0 \leq x \leq l, \\ u(0,t) = 0,\ u(l,t) = A(\sin(\omega t) - \omega t),\ t \geq 0; \end{cases}$

- (c) $\begin{cases} u_{tt} - a^2 u_{xx} = bx, \ 0 < x < l, \ t > 0, \\ u(x,0) = u_t(x,0) = 0, \ 0 \le x \le l, \\ u(0,t) = 0, \ u(l,t) = Bt, \ t \ge 0. \end{cases}$

Problem 9. Let $x = r\cos(\phi)$, $y = r\sin(\phi)$ be the coordinates in $\mathbb{R}^2$. Solve the following problem

$$\begin{cases} u_{xx} + u_{yy} = 0, \ 0 < r < R, \ 0 < \phi < \alpha, \\ u|_{\phi=0} = u|_{\phi=\alpha} = 0, \ u|_{r=R} = f(x,y). \end{cases}$$

Problem 10. Solve the initial boundary value problem

$$\begin{cases} u_{tt} + a^2 u_{xxxx} = 0, \ 0 < x < l, t > 0, \\ u(x,0) = x(x-l), \ u_t(x,0) = 0, \ 0 \le x \le l, \\ u(0,t) = u(l,t) = u_{xx}(0,t) = u_{xx}(l,t) = 0, \ t \ge 0. \end{cases}$$

Problem 11. Solve the initial boundary value problem

$$\begin{cases} u_t = a^2 u_{xx} - b^2 u, \ 0 < x < l, t > 0, \\ u(x,0) = u_0, \ u(0,t) = 0, \ u_x(l,t) + hu(l,t) = 0. \end{cases}$$

Problem 12. Solve the initial boundary value problem

$$\begin{cases} u_t = c^2(u_{xx} + u_{yy}), \ 0 < x < a, \ 0 < y < b, t > 0, \\ u(x,y,0) = A, \\ u(0,y,t) = 0 = u_x(a,y,t), \\ u_y(x,0,t) = u(x,b,t) = 0. \end{cases}$$

Problem 13. Assume the eigenvalue problem

$$\begin{cases} [p(x)X'(x)]' - q(x)X(x) + \lambda\rho(x)X(x) = 0, \\ p \ge p_0 > 0, \rho > \rho_0 > 0, q \ge 0, x \in (0,l), \\ A_0X(0) + B_0X'(0) = 0, \\ A_1X(l) + B_1X'(l) = 0, \\ A_0^2 + B_0^2 \ne 0, \ A_1^2 + B_1^2 \ne 0, \end{cases}$$

has eigenvalue. Where A_i and B_i $(i = 0,1)$ are constants, $B_0 \le 0$, $A_0 \ge 0$, $A_1 \ge 0$, and $B_1 \ge 0$. Prove that the eigenvalue is positive. Furthermore, if X_1 and X_2 correspond to different eigenvalues λ_1 and λ_2, then

$$\int_0^l \rho(x)X_1(x)X_2(x) \ dx = 0.$$

Problem 14. Using the energy function

$$E(t) = \frac{1}{2}\int_0^l (ku_x^2 + \rho u_t^2 + qu^2)\ dx$$

to prove that the problem

$$\begin{cases} \rho(x)u_{tt} = (k(x)u_x)_x - q(x)u,\ 0 < x < l,\ t > 0, \\ u(x,0) = u_t(x,0) = 0, \\ u(0,t) = u(l,t) = 0, t > 0, \end{cases}$$

has the only solution $u \equiv 0$. Here $k(x) \geq k_0 > 0$, $q(x) \geq 0$, $\rho(x) \geq \rho_0 > 0$, k_0 and ρ_0 are two constants.

Problem 15. If $w(x, t; \tau)$ is the solution of

$$(C) \begin{cases} w_{tt} - a^2 w_{xx} = 0,\ 0 < x < l, t > \tau, \\ w|_{x=0} = w|_{x=l} = 0,\ t > \tau \\ w(x,\tau;\tau) = 0,\ w_t(x,\tau;\tau) = f(x,\tau),\ 0 \leq x \leq l, \end{cases}$$

prove that the function

$$u(x,t) = \int_0^t w(x,t;\tau)\ d\tau$$

is the solution of

$$(D) \begin{cases} u_{tt} - a^2 u_{xx} = f(x,t),\ 0 < x < l, t > 0, \\ u(0,t) = u(l,t) = 0,\ t \geq 0, \\ u(x,0) = 0,\ u_t(x,0) = 0,\ 0 < x < l, \end{cases}$$

This is the Duhamel principle.

Problem 16. Assume $\vec{E} = (E_1, E_2, E_3)$ and $\vec{B} = (B_1, B_2, B_3)$ are solution of the Maxwell system

$$\begin{cases} \vec{E}_t = \nabla \times \vec{B} \\ \vec{B}_t = -\nabla \times \vec{E} \\ \nabla \cdot \vec{B} = \nabla \cdot \vec{E} = 0. \end{cases}$$

Prove that if $u = E_i$ or $u = B_i$ for $i = 1, 2, 3$, the $u_{tt} - \Delta u = 0$.

Problem 17. Use the Kirchhoff formula to solve

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}),\ (x,y,z) \in \mathbb{R}^3, t > 0 \\ u|_{t=0} = x^3 + y^2 z,\ u_t|_{t=0} = 0. \end{cases}$$

Problem 18. Assume $u(x,t)$ is the solution for the following problem

$$\begin{cases} u_{tt} - a^2 \Delta u = 0, \ x \in \mathbb{R}^3, t > 0 \\ u(x,0) = \phi(x), \ u_t(x,0) = \psi(x). \end{cases}$$

If $\phi(x), \ \psi(x) \in C_c^\infty(\mathbb{R}^3)$, prove there is a constant C such that

$$|u(x,t)| \le \frac{C}{t}, \ x \in \mathbb{R}^3, \ t > 0.$$

Problem 19. Use the Poisson's formula to solve

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy}), \ (x,y) \in \mathbb{R}^2, t > 0 \\ u|_{t=0} = x^2(x+y), \ u_t|_{t=0} = 0. \end{cases}$$

Problem 20. Solve the following initial value problem

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy}) + c^2 u, \ (x,y) \in \mathbb{R}^2, t > 0 \\ u|_{t=0} = \phi(x,y), \ u_t|_{t=0} = \psi(x,y). \end{cases}$$

(Hint: choose $v(x,y,z) = e^{\frac{cz}{a}} u(x,y)$, apply Kirchhoff formula to v.)

Problem 21. Let $u(x,y,t)$ be the solution of

$$\begin{cases} u_{tt} - 4(u_{xx} + u_{yy}) = 0u, \ (x,y) \in \mathbb{R}^2, t > 0 \\ u|_{t=0} = \phi(x,y), \ u_t|_{t=0} = \psi(x,y). \end{cases}$$

Here

$$\phi(x,y) = \psi(x) = \begin{cases} 0, \ (x,y) \in \Omega, \\ 1, \ (x,y) \in \mathbb{R}^2 \setminus \Omega, \end{cases}$$

where $\Omega = \{(x,y) | |x| \le 1, |y| \le 1\}$. Determine the region where $u(x,y,t) \equiv 0$ for $t > 0$.

Problem 22. Solve the following problem

$$\begin{cases} u_{tt} - \Delta u = 2(y-t), \ (x,y,z) \in \mathbb{R}^3, t > 0 \\ u|_{t=0} = 0, \ u_t|_{t=0} = x^2 + yz. \end{cases}$$

(Hint: Decompose the problem into two using the linear superposition principle.)

Problem 23. A vibrating string under frictional force satisfies the following equation

$$u_{tt} - a^2 u_{xx} - c u_t = 0, \ c > 0.$$

Prove the energy of this equation decreasing in time. With this fact, prove the uniqueness of the following problem

$$\begin{cases} u_{tt} - a^2 u_{xx} - c u_t = f(x,t), \ 0 < x < l, \ t > 0, \\ u(0,t) = u(l,t) = 0, \ t \ge 0, \\ u(x,0) = \phi(x), \ u_t(x,0) = \psi(x), \ 0 \le x \le l. \end{cases}$$