Applied PDEs
Homework 4, Solutions

Problem 1. Use Fourier transform method to give a solution formula to the following Cauchy problem

$$\begin{cases} u_t - u_{xx} + xu = 0, \ x \in \mathbb{R}, t > 0, \\ u(x,0) = \phi(x). \end{cases}$$

Solution: Apply Fourier transform in x,

$$\hat{u}_t + i\hat{u}_\xi + \xi^2 \hat{u} = 0, \ \hat{u}(\xi, 0) = \hat{\phi}(\xi).$$

Define $w = e^{-i\frac{\xi^3}{3}}\hat{u}$, one finds

$$w_t + iw_\xi = 0, \ w(\xi, 0) = \hat{\phi}(\xi)e^{-i\frac{\xi^3}{3}},$$

which is a transport equation. Therefore,

$$\hat{u}(\xi, t) = \hat{\phi}(\xi - it)e^{-i\frac{(\xi - it)^3}{3}}e^{i\frac{\xi^3}{3}}.$$

We now apply the inverse Fourier transform to obtain

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}e^{\frac{t^3}{3}} \int_{-\infty}^{\infty} \phi(y) \exp\{-ty - \frac{(t^2 - y + x)^2}{4t}\} \ dy.$$

Problem 4(a). Solve the Cauchy problem of 1-d heat equation $u_t - a^2 u_{xx} = 0$, $(x \in \mathbb{R}, t > 0)$ with the following initial conditions: $u(x,0) = \sin(x)$.

Solution: We can assume $a > 0$, let $z = \frac{x}{a}$, one has

$$\begin{cases} u_t - u_{zz} = 0, \ z \in \mathbb{R}, \ t > 0 \\ u(z,0) = \sin(az). \end{cases}$$

Therefore,

$$\begin{aligned} u(z,t) &= E(z,t) * \sin(az) \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \sin(ay) e^{-\frac{(y-z)^2}{4t}} \ dy, \end{aligned}$$

and

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\sin(ay)e^{-\frac{(y-\frac{x}{a})^2}{4t}}\ dy\\
&= \frac{1}{a\sqrt{4\pi t}}\int_{-\infty}^{\infty}\sin(v)e^{-\frac{(x-v)^2}{4a^2t}}\ dv\\
&= \frac{1}{a\sqrt{4\pi t}}\int_{-\infty}^{\infty}\sin(x-v)e^{-\frac{v^2}{4a^2t}}\ dv\\
&= \frac{1}{a\sqrt{4\pi t}}\int_{-\infty}^{\infty}\frac{1}{2i}(e^{i(x-v)}-e^{-i(x-v)})e^{-\frac{v^2}{4a^2t}}\ dv.
\end{aligned}$$

Compute this integral as follows

$$\begin{aligned}
u(x,t) &= \frac{1}{2i}\frac{1}{a\sqrt{4\pi t}}[e^{ix}\int_{-\infty}^{\infty}e^{-iv-\frac{v^2}{4a^2t}}\ dv - e^{-ix}\int_{-\infty}^{\infty}e^{iv-\frac{v^2}{4a^2t}}\ dv]\\
&= \frac{1}{2i}\frac{1}{a\sqrt{4\pi t}}e^{-a^2t}[e^{ix}\int_{-\infty}^{\infty}e^{-\frac{(v+2ia^2t)^2}{4a^2t}}\ dv - e^{-ix}\int_{-\infty}^{\infty}e^{-\frac{(v-2ia^2t)^2}{4a^2t}}\ dv]\\
&= \frac{1}{2i}\frac{1}{a\sqrt{4\pi t}}\sqrt{4a^2t}e^{-a^2t}[e^{ix}\int_{-\infty}^{\infty}e^{-\eta^2}\ dv - e^{-ix}\int_{-\infty}^{\infty}e^{-\eta^2}\ d\eta]\\
&= \frac{1}{2i}e^{-a^2t}(e^{ix}-e^{-ix})\\
&= e^{-a^2t}\sin(x).
\end{aligned}$$

Problem 6. Use extension method to solve the following problem

$$\begin{cases}
u_t - u_{xx} = 0,\ \ x>0, t>0,\\
u(x,0) = \phi(x),\ \ x>0\\
u_x(0,t) = 0,\ \ t>0;\ \ \phi(0)=\phi'(0)=0.
\end{cases}$$

Solution: In view of the boundary condition, we use even extension:

$$\begin{cases}
u_t - u_{xx} = 0,\ \ x\in\mathbb{R},\ \ t>0,\\
u(x,0) = \bar{\phi}(x),\ \ x\in\mathbb{R},
\end{cases}$$

where

$$\bar{\phi}(x) = \begin{cases}
\phi(x),\ \ x>0\\
\phi(-x),\ \ x<0.
\end{cases}$$

We thus have the solution

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \bar{\phi}(y) e^{-\frac{(y-x)^2}{4t}}\ dy \\
&= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{0} \phi(-y) e^{-\frac{(y-x)^2}{4t}}\ dy + \frac{1}{\sqrt{4\pi t}} \int_{0}^{\infty} \phi(y) e^{-\frac{(y-x)^2}{4t}}\ dy \\
&= \frac{1}{\sqrt{4\pi t}} \int_{0}^{\infty} \phi(y) [e^{-\frac{(y-x)^2}{4t}} + e^{-\frac{(y+x)^2}{4t}}]\ dy.
\end{aligned}$$

One easily verifies

$$u_x(0,t) = \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \phi(y) e^{-\frac{y^2}{4t}} (\frac{2y}{4t} - \frac{2y}{4t})\ dy = 0.$$

Problem 8. For constant $\sigma > 0$, use separation of variables method to solve

$$\begin{cases}
u_t - u_{xx} = 0,\ 0 < x < l,\ t > 0, \\
u(x,0) = \phi(x),\ 0 \le x \le l \\
u(0,t) = 0,\ u_x(l,t) + \sigma u(l,t) = 0,\ t \ge 0.
\end{cases}$$

Solution: Assume $u(x,t) = X(x)T(t)$ which gives

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

Therefore it gives two equations

$$T' = -\lambda T,\ X'' + \lambda X = 0.$$

We solve for eigenvalue problem for X with the boundary conditions

$$X(0) = 0,\ X'(l) + \sigma X(l) = 0.$$

It is easy to check that $X(x) \equiv 0$ if $\lambda \le 0$, therefore, we come up with $\lambda = k^2 > 0$ with $k > 0$, if we want non-trivial solution for $X(x)$. In the latter case, one has

$$X(x) = A\cos(kx) + B\sin(kx).$$

Since $X(0) = 0$, $A = 0$. For non-zero B, we use the other boundary condition to find

$$k\cos(kl) + \sigma\sin(kl) = 0.$$

Therefore, k is the solution of

$$k = -\sigma\tan(kl),$$

which has infinitely many solutions k_n $(n = 1, 2, \cdots)$, and $\lambda_n = k_n^2$. For each λ_n we find

$$T_n(t) = a_n e^{-\lambda_n t}.$$

Therefore, we expect the solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k_n^2 t} \sin(k_n x).$$

where a_n is determined as the coefficients of the expansion for $\phi(x)$ with respect to $\{\sin(k_n x)\}_{n=1}^{\infty}$. One can show (with some estimates on k_n), that

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k_n^2 t} \sin(k_n x),$$

with

$$a_n = \frac{2}{l} \int_0^l \phi(x) \sin(k_n x)\ dx,$$

gives the solution to the problem.

Problem 14. Let $u(x,t) \in C_1^2(\bar{\Omega}_T)$ be the solution of

$$\begin{cases} u_t - \Delta u = f(x,t),\ (x,t) \in \Omega_T, \\ u(x,t)|_{\Gamma_T} = \phi(x,t). \end{cases}$$

Define

$$F = \sup_{\bar{Q}_T} |f|,\ B = \sup_{\Gamma_T} |\phi|.$$

Prove that

$$\max_{\bar{\Omega}_T} |u| \leq FT + B.$$

Proof. We first define

$$w_1(x,t) = -(Ft + B) + u(x,t).$$

Clearly, $w_1(x,t)$ satisfies

$$\begin{cases} w_{1t} - \Delta w_1 = f(x,t) - F \leq 0, \\ w_1|_{\Gamma} = \phi(x,t) - (Ft + B) \leq 0. \end{cases}$$

We thus apply the weak maximum principle to obtain

$$\max_{\bar{\Omega}_T} w_1 = \max_{\Gamma_T} w_1 \leq 0,$$

therefore

$$\max_{\bar{\Omega}_T} u \leq FT + B.$$

Now, we define

$$w_2(x,t) = -(Ft + B) - u(x,t),$$

which satisfies

$$\begin{cases} w_{2t} - \Delta w_2 = -f(x,t) - F \leq 0, \\ w_2|_\Gamma = -\phi(x,t) - (Ft + B) \leq 0. \end{cases}$$

We thus apply the weak maximum principle to obtain

$$\max_{\bar{\Omega}_T} w_2 = \max_{\Gamma_T} w_2 \leq 0,$$

therefore

$$\max_{\bar{\Omega}_T} (-u) \leq FT + B.$$

We thus obtained

$$\max_{\bar{\Omega}_T} |u| \leq FT + B.$$

Problem 15. Assume u solves $u_t - \Delta u = 0$, prove the following statements

- (a) If $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth convex function, then $v = \phi(u)$ satisfies

$$v_t - \Delta v \leq 0.$$

- (b) Prove $v = |Du|^2 + u_t^2$ also satisfies the above inequality.

Solution; For (a), one computes directly for $v = \phi(u)$,

$$\begin{aligned} v_t - \Delta v &= \phi'(u)u_t - \phi'(u)\Delta u - \phi''(u)|\nabla u|^2 \\ &= -\phi''(u)|\nabla u|^2 \leq 0. \end{aligned}$$

For (b), we compute for $v = |Du|^2 + u_t^2$,

$$\begin{aligned} & v_t - \Delta v \\ & = 2Du \cdot Du_t + 2u_t u_{tt} - 2|Du_t|^2 - 2u_t \Delta u_t - 2|D^2 u|^2 - 2Du \cdot D(\Delta u) \\ & = 2Du \cdot D(u_t - \Delta u) + 2u_t \partial_t (u_t - \Delta u) - 2|Du_t|^2 - 2|D^2 u|^2 \\ & = -2|Du_t|^2 - 2|D^2 u|^2 \leq 0. \end{aligned}$$