ELEMENTARY EXAMPLES OF PERTURBATION ANALYSIS

Scope:

- To introduce analytical tools for solving weakly nonlinear problems;
- To illustrate the Multiple Scale Method, to be systematically used ahead for bifurcation analysis.

■ Outline

1. A quasi-linear algebraic problem, admitting a simple root

2. A quasi-linear algebraic problem, admitting a double root

3. Introducing a perturbation parameter

4. Multiparameter systems

5. Linear Algebraic Eigenvalue Problems

6. Nonlinear Algebraic Eigenvalue Problems

7. Initial Value Problems: straightforward expansions

8. The Multiple Scale Method: basic aspects

9. The Multiple Scale Method: advanced topics

1. PERTURBING A SIMPLE ROOT OF AN ALGEBRAIC EQUATION

■ **Example: a nonlinear algebraic equation:**

$$x - \varepsilon x^3 - 1 = 0, \qquad 0 \le \varepsilon \ll 1, \;\; x \in \mathbb{R}$$

- This is a linear equation, x-1=0, perturbed by a nonlinear term, $-\varepsilon x^3$;
- ε is the *perturbation parameter;* the linear unperturbed equation is the *generating equation*.
- The unperturbed equation admits the (unique) root x_0=1, called the *generating solution.* We want to find the (unique) root x=$x(\varepsilon)$ of the perturbed equation which tends to x_0 when $\varepsilon \to 0$.
- We expand the unknown solution $x(\varepsilon)$ in Mac Laurin series:

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$$

and want to find the coefficient of the series $x_k := (1/k!)\,\mathrm{d}^k x / \mathrm{d}\varepsilon^k \,|_{\varepsilon=0}$

- By substituting the expansion in the equation, and collecting terms with the same powers of ε:

$$\left(x_0 - 1\right) + \varepsilon\left(x_1 - x_0^3\right) + \varepsilon^2\left(x_2 - 3x_0^2 x_1\right) + \ldots = 0$$

- Since this expression must hold $\forall\varepsilon$, the coefficients of ε^k must vanish separately for any *k*:

$$\varepsilon^0 : \; x_0 = 1$$

$$\varepsilon^1 : \; x_1 = x_0^3$$

$$\varepsilon^2 : \; x_2 = 3x_0^2 x_1$$

....

- These are called *the perturbation* equations. They are a sequence of *linear* equations, in the unknowns $x_0, x_1, x_2, \cdots,$ having the same operator. They can be solved in chain:

$$x_0 = 1, \;\; x_1 = 1, \;\; x_2 = 3, \; \ldots$$

- The series, consequently, reads :

$$x = 1 + \varepsilon + 3\varepsilon^2 + \ldots$$

- **Note:** the procedure gives an asymptotic expression just for one root of the cubic equation. Indeed, the remaining two roots, which tend to $\pm\infty$ when $\varepsilon \to 0$, cannot be found as perturbation of the (finite) root x_0.

- **Comments**
  - In the problem studied, since $x_1 \neq 0$, an order-ε perturbation *of the generating equation entails a modification of the same order of its root* (normal sensitivity, $x - x_0 = O(\varepsilon)$).
  - There exist problems in which the first derivative vanishes , $x_1 = 0$ (low-sensitivity $x - x_0 = O(\varepsilon)$); the Mac Laurin expansion still works.
  - There exist degenerate problems in which *the sensitivity of x with respect to* ε *is infinite* (high-sensitivity, the function is not analytical at $\varepsilon = 0$). Mac Laurin series cannot be used! An example is given ahead.

2. PERTURBING A MULTIPLE ROOT OF AN ALGEBRAIC EQUATION

■ **Example: a double-zero root**

$$x-\varepsilon x^3-1=0, \qquad 0\le\varepsilon\ll 1,\ \ x\in\mathbb{R}$$

- When $\varepsilon \neq 0$, the cubic equation admits two roots of large modulus. To find them, we introduce the transformation $x=1/y$; consequently:

$$y^3-y^2+\varepsilon=0$$

- The generator system ($\varepsilon=0$) admits the simple root $y=1$ (already studied) and the *double-zero* root $y=0$.
- *The standard method fails for double roots*. Indeed, by expanding *y*:

$$y=y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots$$

the perturbation equations follow:

$$\varepsilon^0 : \ y_0^3 - y_0^2 = 0$$

$$\varepsilon^1 : \ y_1(3y_0^2 - 2y_0) = -1$$

....

in which the ε-order equation cannot be solved.

- To solve the problem, we use *fractional power* series expansion. By putting $y = \mathrm{O}(\varepsilon^{1/\nu})$ and matching the lowest power of y, i.e. $y^2 = \mathrm{O}(\varepsilon^{2/\nu})$, with the known-term ε, $\nu = 2$ follows. Therefore we take:

$$y = \varepsilon^{1/2} y_1 + \varepsilon y_2 + \varepsilon^{3/2} y_3 + \cdots$$

and we get the following perturbation equations:

$$\varepsilon \quad : y_1^2 = 1$$

$$\varepsilon^{1/2} : 2 y_1 y_2 = y_1^3$$

$$\varepsilon^{3/2} : 2 y_1 y_3 = 3 y_1^2 y_2 - y_2^2$$

....

- The first equation is *nonlinear* in y_1, while the other ones are *linear* in $y_2, y_3, \cdots$. At order-ε *two* solutions are found, i.e. $y_1 = \pm 1$. For *each* of them, the successive equations furnish *one* solution, i.e.: $y_2 = 1/2,\ y_3 = \pm 5/8, \cdots$. Thus:

$$y = \pm\varepsilon^{1/2} + \frac{1}{2}\varepsilon \pm \frac{5}{8}\varepsilon^{3/2} + \cdots$$

- By coming back to the original variable $x = 1/y$ and expanding in series, we finally obtain:

$$x = \pm\frac{1}{\sqrt{\varepsilon}} - \frac{1}{2} \mp \frac{3}{8}\sqrt{\varepsilon} + \cdots$$

- **Note:** The expansion leads to a *quadratic* equation that furnishes *two* roots, which are perturbations of the *unique* double-zero root. The higher-order equations just improve the approximation of each perturbed root.

3. INTRODUCING IN THE EQUATION A PERTURBATION PARAMETER

- How to introduce a perturbation parameter in the equation?
- Sometimes a small parameter ε naturally appears (e.g.: aspect ratios of slender bodies, damping ratios of slightly damped systems, frequency ratios of weakly coupled systems).
- In other cases, the perturbation parameter can be introduced artificially by rescaling the state variables **x** as , $\hat{\mathbf{x}} := \varepsilon^{\alpha}\mathbf{x}$ for a suitable α>0. Then, ε measures the ‘smallness’ of the state vector **x**. Asymptotic solution, are valid in a small neighborhood of the state-space origin.
- If the system contains a parameter μ, this should also be rescaled as $\hat{\mu} = \varepsilon^{\beta}\mu$, for some β>0.
- As a general rule, when ε has been artificially introduced, *it can be always eliminated at the end of the procedure*, by an inverse rescaling.

■ Example: a weakly nonlinear algebraic problem

- A nonlinear algebraic system:

$$\mathbf{Ax} + \mathbb{B}\mathbf{x}^3 = \mu\mathbf{b}$$

where:

$$(\mathbf{x},\mathbf{b}) \in \mathbb{R}^N, \quad \mathbf{A} \in \mathbb{R}^N \times \mathbb{R}^N, \quad \mathbb{B} \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$$

and μ is a parameter. We want to find $\mathbf{x}=\mathbf{x}(\mu)$.

- We assume: $\|\mathbf{A}\| = O(1)$, $\|\mathbb{B}\| = O(1)$, $\|\mathbf{b}\| = O(1)$, so that no small parameters naturally appear.
- In order that $\mathbb{B}\mathbf{x}^3 \ll \mathbf{Ax} = O(\mu\mathbf{b})$, we have to rescale $\mathbf{x}$ and μ at the same order, e.g.:

$$\mathbf{x} = \varepsilon^{1/2}\hat{\mathbf{x}}, \quad \mu = \varepsilon^{1/2}\mu, \quad \text{with } \|\hat{\mathbf{x}}\| = O(1), \hat{\mu} = O(1),$$

- The rescaled equations become:

$$\mathbf{A}\hat{\mathbf{x}} + \varepsilon \mathbb{B}\hat{\mathbf{x}}^{\mathbf{3}} = \hat{\mu}\mathbf{b}$$

- Series expansion:

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_0 + \varepsilon \hat{\mathbf{x}}_1 + \varepsilon^2 \hat{\mathbf{x}}_2 + \ldots$$

- Perturbation equations:

$$\varepsilon^0 : \mathbf{A}\hat{\mathbf{x}}_0 = \hat{\mu}\mathbf{b}$$
$$\varepsilon^1 : \mathbf{A}\hat{\mathbf{x}}_1 = -\mathbb{B}\hat{\mathbf{x}}_0^3$$
$$\varepsilon^2 : \mathbf{A}\hat{\mathbf{x}}_2 = -3\mathbb{B}\hat{\mathbf{x}}_0^2\,\hat{\mathbf{x}}_{\mathbf{1}}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

- By solving in chain:

$$\hat{\mathbf{x}}_0 = \hat{\mu}\mathbf{A}^{-1}\mathbf{b}, \quad \hat{\mathbf{x}}_1 = -\hat{\mu}^3\mathbf{A}^{-1}\mathbb{B}\left(\mathbf{A}^{-1}\mathbf{b}\right)^3, \quad \cdots$$

➢ The series expansion furnishes:

$$\hat{\mathbf{x}} = \hat{\mu}\mathbf{A}^{-1}\mathbf{b} - \varepsilon\hat{\mu}^{3}\mathbf{A}^{-1}\mathbb{B}\left(\mathbf{A}^{-1}\mathbf{b}\right)^{3} + \cdots$$

➢ By coming back to the original variables:

$$\mathbf{x} = \mu\mathbf{A}^{-1}\mathbf{b} - \mu^{3}\mathbf{A}^{-1}\mathbb{B}\left(\mathbf{A}^{-1}\mathbf{b}\right)^{3} + \cdots$$

- **Note:** one can *formally* come back to the original variables by dropping the hat and letting ε=1. We will use this short method ahead.

4. MULTIPARAMETER SYSTEMS

Very often systems depend on *several* parameters $\boldsymbol{\mu} \in \mathbb{R}^M$, instead of a unique parameter ε. We show how to bring back this problem to a one-parameter problem.

- Let us assume that a solution $\mathbf{x}(\boldsymbol{\mu}_0)$ is known at $\boldsymbol{\mu} = \boldsymbol{\mu}_0$. We want to determine $\mathbf{x}(\boldsymbol{\mu})$ inside a ball of radius ε and center $P_0 := \boldsymbol{\mu}_0$ in the M-dimensional parameter space.
- We choose to explore the ball *along selected curves* of parametric equations $\boldsymbol{\mu} = \boldsymbol{\mu}(\varepsilon)$. By Taylor-expanding these equations we get:

$$\boldsymbol{\mu} = \boldsymbol{\mu}_0 + \varepsilon \boldsymbol{\mu}_1 + \varepsilon^2 \boldsymbol{\mu}_2 + \cdots$$

  where $\boldsymbol{\mu}_k := (1/k!)(\mathrm{d}^k \boldsymbol{\mu} / \mathrm{d}\varepsilon^k)_{\varepsilon=0}$ are *known* quantities.
- Usually, straight lines are sufficient to the scope, i.e., $\boldsymbol{\mu}_k = 0 \;\; \forall k > 1$.
- By coming back to $\boldsymbol{\mu}$, the solution is described in the whole ball.

■ A nonlinear algebraic system

- A nonlinear, parameter-dependent, algebraic system:

$$\mathbf{A}(\boldsymbol{\mu})\,\mathbf{x} + \mathbb{B}\mathbf{x}^3 = \mathbf{b}$$

- We assume that $(\mathbf{x},\boldsymbol{\mu}) = (\mathbf{x}_0,\boldsymbol{\mu}_0)$ is an *exact* solution of the *nonlinear* problem , i.e.:

$$\mathbf{A}_0\mathbf{x}_0 + \mathbb{B}\mathbf{x}_0^3 = \mathbf{b}, \qquad \mathbf{A}_0 := \mathbf{A}(\boldsymbol{\mu}_0)$$

- To solve the equation for $\boldsymbol{\mu}$ close to $\boldsymbol{\mu}_0$,we expand $\mathbf{A}$ in series; hence:

$$\begin{aligned}\mathbf{A}(\boldsymbol{\mu}) &= \mathbf{A}(\boldsymbol{\mu}_0) + \mathbf{A}'(\boldsymbol{\mu}_0)(\boldsymbol{\mu}-\boldsymbol{\mu}_0) + \frac{1}{2}\mathbf{A}''(\boldsymbol{\mu}_0)(\boldsymbol{\mu}-\boldsymbol{\mu}_0)^2 + \cdots \\ &= \mathbf{A}(\boldsymbol{\mu}_0) + \varepsilon\mathbf{A}'(\boldsymbol{\mu}_0)\,\boldsymbol{\mu}_1 + \varepsilon^2[\mathbf{A}'(\boldsymbol{\mu}_0)\,\boldsymbol{\mu}_2 + \frac{1}{2}\mathbf{A}''(\boldsymbol{\mu}_0)\,\boldsymbol{\mu}_1^2] + \cdots\end{aligned}$$

i.e.:

$$\mathbf{A} = \mathbf{A}_0 + \varepsilon\mathbf{A}_1 + \varepsilon^2\mathbf{A}_1^2 + \cdots$$

- The equation, therefore, reads:

$$(\mathbf{A}_0 + \varepsilon \mathbf{A}_1 + \varepsilon^2 \mathbf{A}_1^2 + \cdots)\mathbf{x} + \mathbb{B}\mathbf{x}^3 = \mathbf{b}$$

- By expanding also the unknown **x** as:

$$\mathbf{x} = \mathbf{x}_0 + \varepsilon \mathbf{x}_1 + \varepsilon^2 \mathbf{x}_2 + \ldots$$

the perturbation equations follow:

$$\varepsilon^1 : \mathbf{L}_0 \mathbf{x}_1 = -\mathbf{A}_1 \mathbf{x}_0$$
$$\varepsilon^2 : \mathbf{L}_0 \mathbf{x}_2 = -3\mathbb{B}\mathbf{x}_0 \mathbf{x}_1^2 - \mathbf{A}_1 \mathbf{x}_1 - \mathbf{A}_2 \mathbf{x}_0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

where the tangent operator $\mathbf{L}_0 := \mathbf{A}_0 + 3\mathbb{B}\mathbf{x}_0^2$ is assumed non-singular, i.e. det $\mathbf{L}_0 \neq 0$.

➢ Solution:

$$\mathbf{x}_1 = -\mathbf{L}_0^{-1}\mathbf{A}_1\mathbf{x}_0, \quad \mathbf{x}_2 = \mathbf{L}_0^{-1}\left[-3\mathbb{B}\mathbf{x}_0(\mathbf{L}_0^{-1}\mathbf{A}_1\mathbf{x}_0)^2 + \mathbf{A}_1(\mathbf{L}_0^{-1}\mathbf{A}_1\mathbf{x}_0) - \mathbf{A}_2\mathbf{x}_0\right], \quad \cdots$$

from which:

$$\mathbf{x} = \mathbf{x}_0 - \mathbf{L}_0^{-1}\mathbf{A}_0'\,\mathbf{x}_0(\varepsilon\boldsymbol{\mu}_1 + \varepsilon^2\boldsymbol{\mu}_2)$$
$$+\mathbf{L}_0^{-1}\left[-3\mathbb{B}\mathbf{x}_0(\mathbf{L}_0^{-1}\mathbf{A}_0'\,\mathbf{x}_0)^2 + \mathbf{A}_0'\,(\mathbf{L}_0^{-1}\mathbf{A}_0'\,\mathbf{x}_0) - \frac{1}{2}\mathbf{A}_0''\,\mathbf{x}_0\right]\varepsilon^2\boldsymbol{\mu}_1^2 + \cdots$$

➢ By reabsorbing ε, to within an error of order ε^2, one has:

$$\mathbf{x} = \mathbf{x}_0 - \mathbf{L}_0^{-1}\mathbf{A}_0'\,\mathbf{x}_0(\boldsymbol{\mu} - \boldsymbol{\mu}_0)$$
$$+\mathbf{L}_0^{-1}\left[-3\mathbb{B}\mathbf{x}_0(\mathbf{L}_0^{-1}\mathbf{A}_0'\,\mathbf{x}_0)^2 + \mathbf{A}_0'\,(\mathbf{L}_0^{-1}\mathbf{A}_0'\,\mathbf{x}_0) - \frac{1}{2}\mathbf{A}_0''\,\mathbf{x}_0\right](\boldsymbol{\mu} - \boldsymbol{\mu}_0)^2 + \cdots$$

- **Note:** T*he solution does not depend on the curve* chosen to explore the ball, but only on the difference between the actual point $\boldsymbol{\mu}$ and the initial point $\boldsymbol{\mu}_0$.

5. LINEAR ALGEBRAIC EIGENVALUE PROBLEMS

We analyze how the eigenvalues and eigenvectors of a matrix vary under perturbations of parameter(s). The derivatives of the eigenpairs with respect the parameter(s) are called *sensitivities*.

■ A linear free motion problem

- Free motion of a linear, lightly damped, single-d.o.f. oscillator:

$$\ddot{q} + 2\varepsilon\omega_0\dot{q} + \omega_0^2 q = 0$$

  where the damping $\varepsilon \ll 1$ is the perturbation parameter.

- In state-form, by letting $\mathbf{x} = (q, \dot{q})^{\mathrm{T}}$:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\varepsilon\omega_0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- By taking $\mathbf{x}(t)=\mathbf{u}\exp(\lambda t)$, an eigenvalue problem follows:

$$\left(\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}+\varepsilon\begin{pmatrix} 0 & 0 \\ 0 & -2\omega_0 \end{pmatrix}-\lambda\right)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- We want to find the eigenpairs $\lambda(\varepsilon)$,$\mathbf{u}(\varepsilon)$, as analytical functions of ε, by perturbing the eigenpairs $\lambda(0)$,$\mathbf{u}(0)$.To this end we expand *both* the eigenvectors *and* the eigenvalues in power series of ε:

$$\mathbf{u}(\varepsilon)=\mathbf{u}_0+\varepsilon\mathbf{u}_1+\varepsilon^2\mathbf{u}_2+\cdots$$
$$\lambda(\varepsilon)=\lambda_0+\varepsilon\lambda_1+\varepsilon^2\lambda_2+\cdots$$

- Since **u** is defined to within an arbitrary constant, we introduce a *normalization condition*, e.g. by requiring $u_1=1\,\forall\varepsilon$; this entails:

$$u_{10}=1,\quad u_{1k}=0\quad k=1,2,\cdots$$

- Perturbation equations:

$$\varepsilon^0: \begin{pmatrix} -\lambda_0 & 1 \\ -\omega_0^2 & -\lambda_0 \end{pmatrix} \begin{pmatrix} u_{10} \\ u_{20} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\varepsilon^1: \begin{pmatrix} -\lambda_0 & 1 \\ -\omega_0^2 & -\lambda_0 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} = \begin{pmatrix} \lambda_1 u_{10} \\ (2\omega_0 + \lambda_1) u_{20} \end{pmatrix}$$

$$\cdots\cdots\cdots$$

- Generating solution:

$$\lambda_0 = \pm i\omega_0, \; \mathbf{u}_0 = (1, \pm i\omega_0)^{\mathrm{T}}, \; \mathbf{v} = (\mp i\omega_0, 1)^{\mathrm{T}}$$

where **v** are left eigenvectors, satisfying the transpose conjugate problem $(\mathbf{A} - \lambda_0 \mathbf{I})^H \mathbf{v} = 0$ (*adjoint problem*) .

- Compatibility (or solvability) condition:
In order that the ε-order (singular) equation admits solution, its known term must belong to the range of the operator, i.e. it must be orthogonal to the kernel of the adjoint operator, namely:

$$(\pm i\omega_0, 1)\begin{pmatrix} \lambda_1 \\ \pm(2\omega_0 + \lambda_1)i\omega_0 \end{pmatrix} = 0$$

from which $\lambda_1 = -\omega_0$ follows.

- General solution of the ε-order equation:

$$\begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ -\omega_0 \end{pmatrix} + c\begin{pmatrix} 1 \\ \pm i\omega_0 \end{pmatrix} \quad \forall c$$

Due to normalization, $c = 0$ must be taken.

- By the series expansion, truncated at the first-order, we finally have:

$$\lambda = \pm i\omega_0 - \varepsilon\omega_0 + \ldots, \quad \mathbf{u} = (1, \pm i\omega_0 - \varepsilon\omega_0)^{\mathrm{T}}$$

according with the series expansion of the exact solution:

$$\lambda(\varepsilon) = -\varepsilon\omega_0 \pm i\omega_0\sqrt{1-\varepsilon^2}\ , \ \mathbf{u}(\varepsilon) = (1, \lambda(\varepsilon))^{\mathrm{T}}$$

6. NONLINEAR ALGEBRAIC EIGENVALUE PROBLEMS

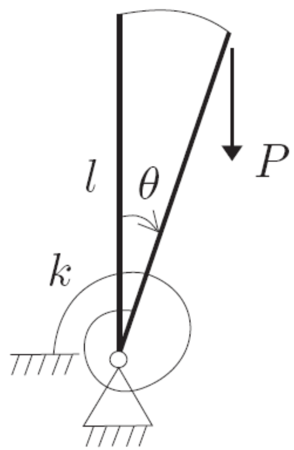
We want to solve *nonlinear homogeneous problems* depending on a parameter, also called *nonlinear eigenvalue problems*.

■ **A static bifurcation problem**

Let us consider the static system in the figure (reverse pendulum, elastically restrained). The equilibrium equation reads:

$$\theta - \mu \sin\theta = 0$$

where θ is the rotation and $\mu := Pl/k$ the non-dimensional load parameter.



➢ We look for non trivial solution to the equation, $\theta = \theta(\mu)$,
or $\theta = \theta(\varepsilon), \mu = \mu(\varepsilon)$ (*buckling problem*).

➢ We expand $\sin\theta \simeq \theta + \theta^3/6 + \cdots$ and introduce a perturbation parameter via the rescaling $\theta \to \varepsilon^{1/2}\theta$. Thus the equation becomes:

$$(1-\mu)\theta + \frac{1}{6}\varepsilon\mu\theta^3 + \cdots = 0$$

➢ As for linear problems, we expand *both* the state variable and the load parameter as:

$$\theta(\varepsilon) = \theta_0 + \varepsilon\theta_1 + \cdots, \qquad \mu(\varepsilon) = \mu_0 + \varepsilon\mu_1 + \cdots$$

➢ The following perturbation equations are obtained:

$$\varepsilon^0: \ (1\text{-}\mu_0)\theta_0 = 0$$

$$\varepsilon^1: \ (1\text{-}\mu_0)\theta_1 = \mu_1\theta_0 - \mu_0\frac{\theta_0^3}{6}$$

....

- Normalization condition:
  We denote by a the amplitude of θ, and therefore we require $\theta(\varepsilon) = a \quad \forall \varepsilon$, from which:

$$\theta_0 = a, \quad \theta_k = 0 \quad k = 1, 2, \cdots$$

- Generating solution:

$$\mu_0 = 1 \ , \ \theta_0 = a \ .$$

- Solvability condition and ε-order solution:
  The ε-order equation can be solved if and only if:

$$\mu_1 = (1/6)a^2$$

  The equation is solved by $\theta_1 = c \quad \forall c$; normalization entails $c = 0$.

- By coming back to the original (not rescaled) variable θ, we have:

$$\theta(\varepsilon) = \varepsilon^{1/2} a, \quad \mu(\varepsilon) = 1 + \frac{1}{6}\varepsilon a^2 + \cdots$$

and reabsorbing ε via $\varepsilon^{1/2} a \to a$:

$$\theta = a, \quad \mu = 1 + \frac{1}{6} a^2 + \cdots$$

or, in Cartesian form:

$$\mu = 1 + \frac{1}{6} \theta^2 + \cdots$$

- Bifurcation diagram:

  At $\mu = \mu_0$ a *bifurcation* takes place; μ_0 is the bifurcation (or *critical*) *load* (*T:* trivial, *NT:* non-trivial solution).