7. INITIAL VALUE PROBLEMS: THE STRAIGHTFORWARD EXPANSION

We show that the *straightforward expansion method*, successfully applied in static problems, does not work in initial value problems.

■ Example: The Raileigh-Duffing oscillator

$$\begin{cases} \ddot{x}(t)+\omega^2 x(t)-\mu\dot{x}(t)+b\dot{x}^3(t)+cx^3(t)=0 \\ x(0)=a_0, \quad \dot{x}(0)=0 \end{cases}$$

➢ Rescaling:

$$\mu \to \varepsilon\mu\,, \; x \to \varepsilon^{1/2}x\,.$$

➢ Series expansion:

$$x(t;\varepsilon)=x_0(t)+\varepsilon x_1(t)+\varepsilon^2 x_2(t)+\cdots$$

- Perturbation equations:

$$\varepsilon^0: \begin{cases} \ddot{x}_0(t) + \omega^2 x_0(t) = 0 \\ x_0\left(0\right) = a_0,\ \dot{x}_0\left(0\right) = 0 \end{cases}$$

$$\varepsilon^1: \begin{cases} \ddot{x}_1(t) + \omega_0 x_1(t) = \mu \dot{x}_0(t) - b\dot{x}_0^3(t) - c x_0^3(t) \\ x_1\left(0\right) = 0,\ \dot{x}_1\left(0\right) = 0 \end{cases}$$

....

- Generating solution:

$$x_0 = (a_0 / 2)\, \mathrm{e}^{i\omega t} + c.c.$$

- ε -order equation:

$$\begin{cases} \ddot{x}_1(t) + \omega_0 x_1(t) = f_1 \mathrm{e}^{i\omega t} + f_3 \mathrm{e}^{3i\omega t} + c.c. \\ x_1\left(0\right) = 0,\ \dot{x}_1\left(0\right) = 0 \end{cases}$$

where:

$$f_1 := \frac{1}{2}\left(i\mu\omega - \frac{3}{4} i\omega^3 b a_0^2 - \frac{3}{4} c a_0^2\right), \quad f_3 := \frac{a_0^2}{8}\left(i\omega^3 b - c\right)$$

- Solution to the ε -order equation:

$$x_1(t) = \underbrace{\frac{1}{2} a_1 \mathrm{e}^{i(\omega t+\vartheta_1)}}_{\text{complementary solution}} \underbrace{- i\frac{1}{2\omega} f_1 t\, \mathrm{e}^{i\omega t}}_{\text{secular term}} \underbrace{- \frac{1}{8\omega^2} f_3 \mathrm{e}^{3i\omega t}}_{\text{non-secular term}} + c.c.$$

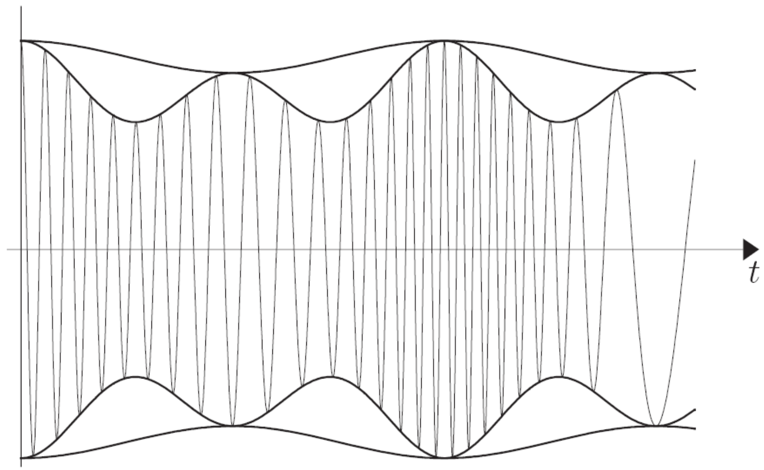
The secular term diverges in time. The series is *not uniformly valid*, since $\mathrm{O}(\varepsilon x_1 / x_0) \geq 1$, i.e. x_1 is not a small correction of x_0. Therefore, the straightforward method has not practical utility in initial value problems.

8. THE MULTIPLE SCALE METHOD: BASIC ASPECTS

The MSM is probably the most powerful perturbation method able to furnish uniformly valid expansions for oscillatory problems.

■ Basic idea

The response of a weekly nonlinear system, can be considered as a *periodic signal slowly modulated on slower scales*. Example in nature: the temperature in a fixed site varies periodically on a daily-scale, but it is modulated, *in amplitude and phase*, on a yearly-scale, and, in turn, on a century-scale.



■ Introducing independent time-scales

- Rayleigh-Duffing oscillator :

$$\begin{cases} \ddot{x}(t)+\omega^2 x(t)-\mu\dot{x}(t)+b\dot{x}^3(t)+cx^3(t)=0 \\ x(0)=a_0, \quad \dot{x}(0)=0 \end{cases}$$

➢We assume that the variable $x(t)$ depends *on several independent time scales*, defined as:

$$t_0 := t, \quad t_1 := \varepsilon t, \quad t_2 := \varepsilon^2 t, \ \cdots$$

➢Rules for derivatives:

Since $x(t) = x(t_0(t), t_1(t), t_2(t), \cdots)$, the chain rule furnishes:

$$\dot{x}(t) = \frac{\partial x}{\partial t_0} + \varepsilon \frac{\partial x}{\partial t_1} + \varepsilon^2 \frac{\partial x}{\partial t_2} + \cdots$$

Hence, formally:

$$\mathrm{D} = \mathrm{d}_0 + \varepsilon^1 \mathrm{d}_1 + \varepsilon^2 \mathrm{d}_2 + \cdots = \sum_{k=0}^{\infty} \varepsilon^k \mathrm{d}_k, \qquad \mathrm{D} := \frac{\mathrm{d}}{\mathrm{d}t}, \quad \mathrm{d}_k := \frac{\partial}{\partial t_k}$$

Similarly, for second-order derivative:

$$D^2 = (\mathrm{d}_0 + \varepsilon^1 \mathrm{d}_1 + \varepsilon^2 \mathrm{d}_2 + \cdots)^2 = \mathrm{d}_0^2 + 2\varepsilon \mathrm{d}_0 \mathrm{d}_1 + \varepsilon^2 (\mathrm{d}_1^2 + 2\mathrm{d}_0 \mathrm{d}_2) + \cdots$$

- Rescaling:

$$\mu \to \varepsilon\mu, \quad x \to \varepsilon^{1/2} x$$

- Series expansion:

$$x(t;\varepsilon) = x_0(t_0, t_1, t_2, \cdots) + \varepsilon x_1(t_0, t_1, t_2, \cdots) + \cdots$$

- Perturbation equations:

$$\varepsilon^0:\begin{cases} \mathrm{d}_0^2 x_0+\omega^2 x_0=0 \\ x_0(0)=a_0,\ \ \mathrm{d}_0 x_0(0)=0 \end{cases}$$

$$\varepsilon^1:\begin{cases} \mathrm{d}_0^2 x_1+\omega^2 x_1=-2\,\mathrm{d}_0\mathrm{d}_1 x_0+\mu\,\mathrm{d}_0 x_0-b(\mathrm{d}_0 x_0)^3-c x_0^3 \\ x_1(0)=0,\ \ \mathrm{d}_0 x_1(0)=-\mathrm{d}_1 x_0(0) \end{cases}$$

$$\varepsilon^2:\begin{cases} \mathrm{d}_0^2 x_2+\omega^2 x_2=-(2\,\mathrm{d}_0\mathrm{d}_2 x_0+\mathrm{d}_1^2 x_0+2\,\mathrm{d}_0\mathrm{d}_1 x_1) \\ \qquad\qquad +\mu(\mathrm{d}_1 x_0+\mathrm{d}_0 x_1)-3b(\mathrm{d}_0 x_0)^2(\mathrm{d}_1 x_0+\mathrm{d}_0 x_1)-3c x_0^2 x_1 \\ \qquad x_2(0)=0,\ \ \mathrm{d}_0 x_2(0)=-\mathrm{d}_1 x_1(0)-\mathrm{d}_2 x_0(0) \end{cases}$$

………

where $x_k(0)$ is a shortening for $x_k(0,0,\cdots)$.

- **Note:** the perturbation equations are *partial differential equations*, although the original equations are ordinary differential equations.

■ Introducing a time-dependent amplitude

➢ Generating solution:

$$x_0 = a(t_1, t_2, \cdots)\cos(\omega t_0 + \theta(t_1, t_2, \cdots))$$
$$= A(t_1, t_2, \ldots)\mathrm{e}^{i\omega t_0} + c.c.$$

where:

$$A(t_1, t_2, \ldots) := \frac{1}{2} a(t_1, t_2, \ldots)\mathrm{e}^{i\theta(t_1, t_2, \ldots)}$$

❑ **Note:** x_0 is periodic on the fast t_0- scale, and modulated on the slower scales by a complex quantity A or, equivalently, by two real unknowns, a and θ.

➢ By enforcing the initial conditions, it follows:

$$a(0) = a_0, \quad \theta(0) = 0$$

➢ ε-order equations:

Since:

$$x_0 = A\left(t_1, t_2, \ldots\right) \mathrm{e}^{i\omega t_0} + c.c.$$

$$\mathrm{d}_0 \mathrm{d}_1 x_0 = i\omega \mathrm{d}_1 A \, \mathrm{e}^{i\omega t_0} + c.c.,$$

$$x_0^3 = (A \, \mathrm{e}^{i\omega t_0} + \overline{A} \, \mathrm{e}^{-i\omega t_0})^3 = A^3 \, \mathrm{e}^{3i\omega t_0} + 3A^2 \overline{A} \, \mathrm{e}^{i\omega t_0} + c.c.$$

$$(\mathrm{d}_0 x_0)^3 = (i\omega A \, \mathrm{e}^{i\omega t_0} - i\omega \overline{A} \, \mathrm{e}^{-i\omega t_0})^3$$

$$= -i\omega^3 A^3 \, \mathrm{e}^{3i\omega t_0} + 3i\omega^3 A^2 \overline{A} \, \mathrm{e}^{i\omega t_0} + c.c.$$

then:

$$\begin{cases} \mathrm{d}_0^2 x_1 + \omega^2 x_1 = f_1 \, \mathrm{e}^{i\omega t_0} + f_3 \, \mathrm{e}^{3i\omega t_0} + c.c. \\ x_1\left(0\right) = 0, \;\; \mathrm{d}_0 x_1(0) = -\mathrm{d}_1 A(0) + c.c. \end{cases}$$

where:

$$f_1 := -2i\omega \mathrm{d}_1 A + i\omega\mu A - 3\left(i\omega^3 b + c\right) A^2 \overline{A}, \quad f_3 := \left(i\omega^3 b - c\right) A^3$$

➢Eliminating secular terms:

The resonant forcing-term of frequency-ω would lead to secular terms $t_0 \exp(i\omega t_0)$ to appear in the solution. To remove them, $f_1 = 0$ must be enforced, i.e.:

$$\mathrm{d}_1 A = \frac{1}{2}\mu A + \frac{3}{2}\left(-\omega^2 b + i\frac{c}{\omega}\right)A^2\bar{A}$$

This is a *nonlinear differential equation* governing the modulation on the t_1-scale; it is called the (first-order) *Amplitude Modulation Equation* (AME).

➢Real form of AME:

Since $\mathrm{d}_1 A = (1/2)\left(\mathrm{d}_1 a + ia\,\mathrm{d}_1 \vartheta\right) \mathrm{e}^{i\theta}$, by separating real and imaginary parts, the complex AME furnishes two real equations:

$$\begin{cases} \mathrm{d}_1 a = \dfrac{1}{2} a \left(\mu - \dfrac{3}{4} \omega^2 b a^2 \right) \\ a\, \mathrm{d}_1 \vartheta = \dfrac{3}{8} \dfrac{c}{\omega} a^3 \end{cases}$$

to be sided by $a(0) = a_0, \quad \theta(0) = 0$.

- Coming back to the original variables:

By truncating the analysis at this order (*first-order perturbation solution*):

- The dependence on $t_2, t_3, \cdots$ must be ignored, i.e. $a = a(t_1), \theta = \theta(t_1)$.
- By multiplying the real AME's by $\varepsilon^{3/2}$ and using $\varepsilon^{1/2} a \rightarrow a, \varepsilon\mu \rightarrow \mu, \varepsilon\, \mathrm{d}_1 \rightarrow \mathrm{D}$, the perturbation parameter is reabsorbed and return to the true time *t* is performed.

➢Reduced dynamical system:

The two real AME can be solved in sequence; first, the amplitude $a(t)$ is drawn by integrating:

$$\dot{a} = \frac{1}{2} a \left(\mu - \frac{3}{4} \omega^2 b a^2 \right)$$

which represents a one-dimensional *reduced dynamical system*, of type $\dot{a} = F(a, \mu)$.

Successively, the phase $\theta(t)$ is determined by integrating:

$$a\dot{\vartheta} = \frac{3}{8} \frac{c}{\omega} a^3$$

- ❑ **Note:** the real amplitude-equation captures the essential dynamics of the system; the phase equation describes a complementary aspect.

■ **Steady-solutions**

➢ The AME admit two steady solutions $a(t) = \text{const} =: a_s$:

o Trivial solution:

$$a_s = 0 \quad \forall \mu, \forall \vartheta$$

which describes the trivial equilibrium path x=0 ∀ μ.

o Periodic solution:

$$\mu = \frac{3}{4}\omega^2 b a_s^2, \quad \vartheta = \kappa t \qquad \kappa =: \frac{3}{8} c a_s^2$$

which describes the limit cycle $x(t) = a_0 \cos\left[\left(\omega + \kappa\right)t\right]$, where κ is the frequency correction, and $\Omega = \omega + \kappa$ the nonlinear frequency.

❑ **Note:** The MSM *filters the fast dynamics*, so that *a periodic x-motion appears as an equilibrium a-position*

▪ Stability of steady-state solutions

➢ Introducing a perturbation:

To analyze stability of the steady solutions, we put:

$$a(t) = a_s + \delta a(t)$$

with $\delta a(t)$ a small perturbation superimposed to the steady amplitude.

➢ Variational equation:

By linearizing the equation in $\delta a(t)$, the *variational equation* follows:

$$\delta \dot{a}(t) = \frac{1}{2}(\mu - \frac{9}{4}\omega^2 b a_s^2)\,\delta a(t)$$

whose solution is:

$$\delta a(t) = \delta a(0)\exp[\frac{1}{2}(\mu - \frac{9}{4}\omega^2 b a_s^2)t]$$

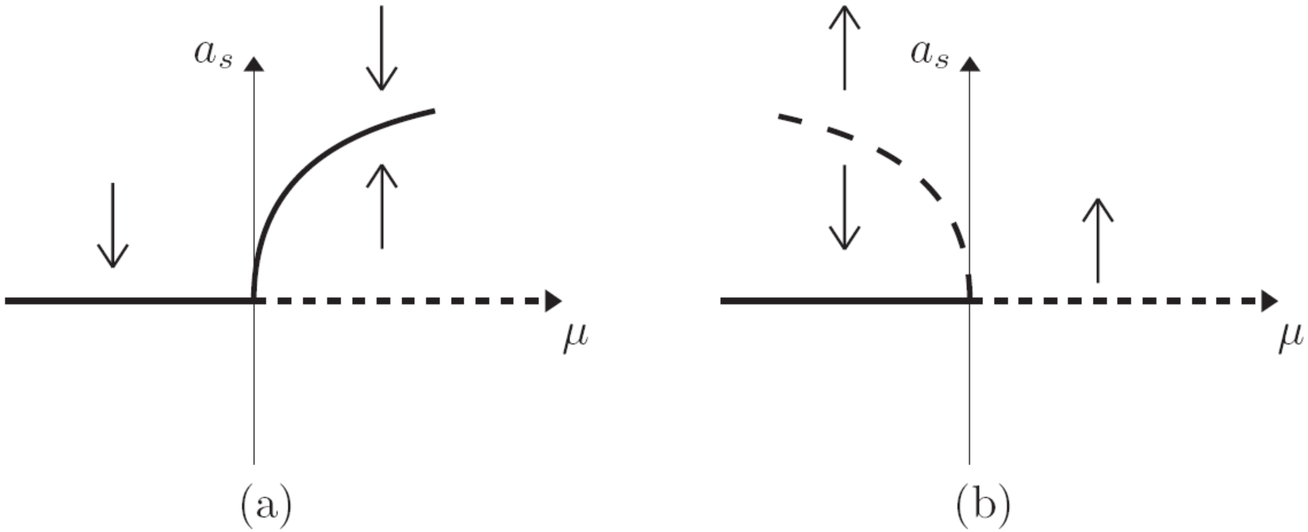
- Stability of the trivial solution:

By substituting $a_s = 0$ in the solution of the variational equation, it follows:

$$\delta a(t) = \delta a(0) \exp[\frac{1}{2}\mu t]$$

When $t \to \infty$:

- $\delta a(t) \to \delta a(0)$ if $\mu < 0$, i.e. the equilibrium is (asymptotically) stable;
- $\delta a(t) \to \infty$ if $\mu > 0$, i.e. the equilibrium is unstable.



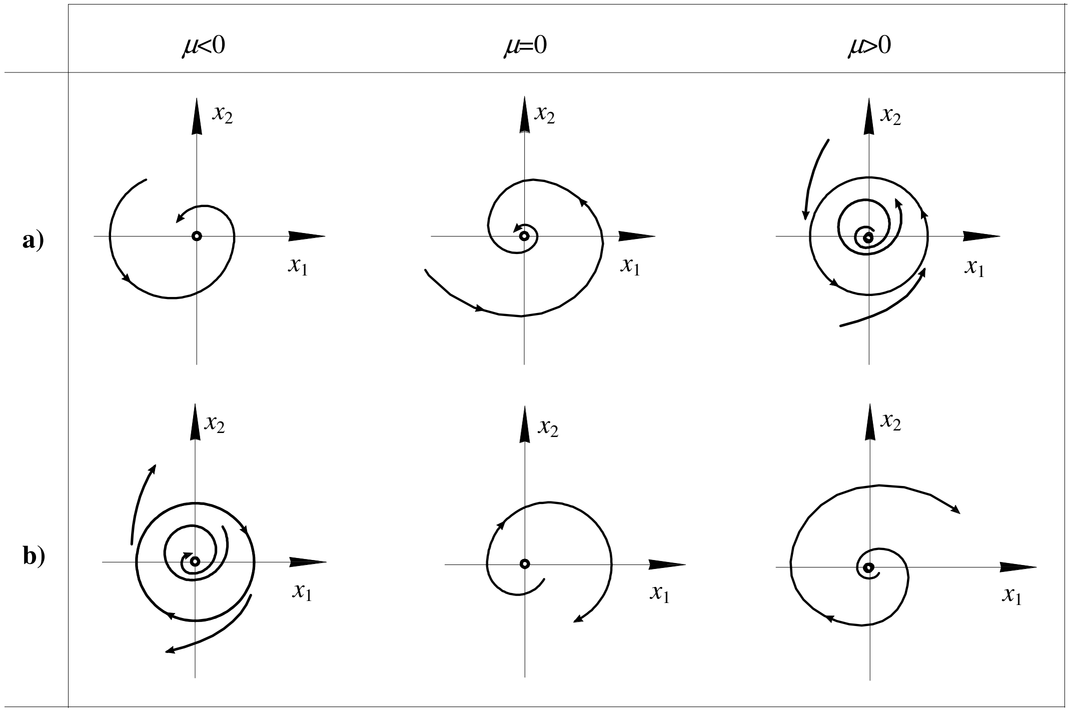
Bifurcation diagrams and orbits for (a) supercritical and (b) subcritical Hopf bifurcations.

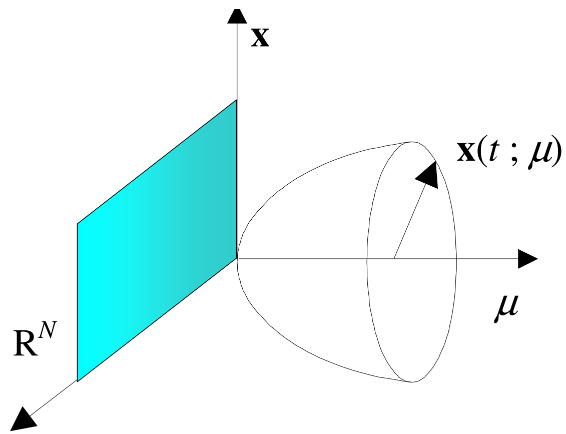
- Stability of the periodic solution:

  By substituting $a_s = \sqrt{4\mu/(3b\omega^2)}$ in the solution of the variational equation, it follows:

$$\delta a(t) = \delta a(0)\exp(-\mu).$$

  Hence, *the limit-cycle is stable if the bifurcation is supercritical* **(a)** and *unstable if the bifurcation is subcritical* **(b)**.





9. THE MULTIPLE SCALE METHOD: ADVANCED TOPICS

Usually, a first-order solution is sufficient to describe limit-cycles and their stability. However, there exist problems in which a higher-order solution is necessary to describe *qualitatively new aspects*. We illustrate how to get a second-order solution for the Rayleigh-Duffing oscillator.

■ Moving to higher-orders

➢ ε-order perturbation equation:

$$\begin{cases} \mathrm{d}_0^2 x_1 + \omega^2 x_1 = f_1 \, \mathrm{e}^{i\omega t_0} + f_3 \, \mathrm{e}^{3i\omega t_0} + c.c. \\ x_1(0) = 0, \; \mathrm{d}_0 x_1(0) = -\mathrm{d}_1 A(0) + c.c. \end{cases}$$

where $f_1 = 0$ to avoid secular terms. By solving it:

$$x_1 = B(t_1, t_2, ...)\,\mathrm{e}^{i\omega t_0} + \frac{1}{8}\left(\frac{c}{\omega^2} - i\omega b\right)A^3\,\mathrm{e}^{3i\omega t_0} + c.c.$$

where $B(t_1, t_2, ...)$ is an arbitrary function of the slower scales, constrained to satisfy the initial condition.

- To simplify the analysis, <u>we ignore this arbitrary function</u>, by letting B=0. Indeed, $B(t_1, t_2, ...)\,\mathrm{e}^{i\omega t_0}$ $+c.c.$ repeats the generating solution.
- Since the initial conditions cannot be enforced at any order, we will enforce them, as a whole, on the final solution (although this is an inconsistent method).

➢ ε^2-order perturbation equation:

$$\mathrm{d}_0^2\, x_2 + \omega^2 x_2 = -(2\,\mathrm{d}_0\mathrm{d}_2 x_0 + \mathrm{d}_1^2 x_0 + 2\,\mathrm{d}_0\mathrm{d}_1 x_1)$$
$$+\mu(\mathrm{d}_1 x_0 + \mathrm{d}_0 x_1) - 3b(\mathrm{d}_0 x_0)^2(\mathrm{d}_1 x_0 + \mathrm{d}_0 x_1) - 3c x_0^2 x_1$$

By ignoring the non-resonant terms (*NRT*), the various contributions are:

$$d_0\, d_2\, x_0 = i\omega d_2\, A\, e^{i\omega t_0} + c.c. + NRT, \quad d_1^2\, x_0 = d_1^2\, A\, e^{i\omega t_0} + c.c. + NRT,$$

$$d_0\, d_1\, x_1 = NRT,$$

$$d_1\, x_0 = d_1\, A\, e^{i\omega t_0} + c.c. + NRT, \quad d_0\, x_1 = +NRT,$$

$$d_0\, x_0 = i\omega A\, e^{i\omega t_0} + c.c.,$$

$$(d_0\, x_0)^2\, d_1\, x_0 = \omega^2 (2A\bar{A}\, d_1\, A - A^2\, d_1\, \bar{A})\, e^{i\omega t_0} + c.c. + NRT,$$

$$(d_0\, x_0)^2\, d_0\, x_1 = i\omega^3 [-\frac{3}{8}\left(\frac{c}{\omega^2} - i\omega b\right) A^3 \bar{A}^2]\, e^{i\omega t_0} + c.c. + NRT,$$

$$x_0^2 x_1 = [\frac{1}{8}\left(\frac{c}{\omega^2} - i\omega b\right) A^3 \bar{A}^2 +]\, e^{i\omega t_0} + c.c. + NRT$$

where $d_1\, A$ is known from the first-order AME, and $d_1^2\, A \equiv d_1(d_1\, A)$ is evaluated by differentiation:

$$\mathrm{d}_1 A = \frac{1}{2}\mu A + \frac{3}{2}\left(-\omega^2 b + i\frac{c}{\omega}\right)A^2\overline{A}$$

$$\mathrm{d}_1^2 A = \frac{1}{4}\mu^2 A - 3\mu(b\omega^2 - i\frac{c}{\omega})A^2\overline{A} + \frac{9}{4}(3b^2\omega^4 - \frac{c^2}{\omega^2} - 4ibc\omega)A^3\overline{A}^2$$

The ε^2-order perturbation equation reads:

$$\mathrm{d}_0^2 x_2 + \omega^2 x_2 = [-2i\omega \mathrm{d}_2 A + \frac{1}{4}\mu^2 A - \frac{3}{2}i\frac{c}{\omega}\mu A^2\overline{A}$$
$$+(\frac{45}{24}\frac{c^2}{\omega^2} - \frac{9}{8}b^2\omega^4 - 3ibc\omega)A^3\overline{A}^2]\mathrm{e}^{i\omega t_0} + c.c. + NRT$$

➢Elimination of the secular terms:

$$\mathrm{d}_2 A = -i\frac{1}{8\omega}\mu^2 A - \frac{3}{4}\frac{c}{\omega^2}\mu A^2\overline{A} + (-i\frac{45}{48}\frac{c^2}{\omega^3} + i\frac{9}{16}b^2\omega^3 - \frac{3}{2}bc)A^3\overline{A}^2$$

which governs the evolution of A on the t_2-scale.

- **The reconstitution method**

To come back to the true time t, the t_1- and t_2-derivatives of A are recombined as follows:

$$\dot{A} = \varepsilon \, \mathrm{d}_1 A + \varepsilon^2 \, \mathrm{d}_2 A + \cdots$$

$$= \varepsilon[\frac{1}{2}\mu A + \frac{3}{2}\left(-\omega^2 b + i\frac{c}{\omega}\right)A^2\bar{A}]$$

$$+\varepsilon^2[-i\frac{1}{8\omega}\mu^2 A - \frac{3}{4}\frac{c}{\omega^2}\mu A^2\bar{A} + (-i\frac{45}{48}\frac{c^2}{\omega^3} + i\frac{9}{16}b^2\omega^3 - \frac{3}{2}bc)A^3\bar{A}^2] + \cdots$$

This is *the second-order AME*. By multiplying both members by $\varepsilon^{1/2}$ and using $\varepsilon^{1/2}A \to A, \varepsilon\mu \to \mu$, the perturbation parameter is reabsorbed. Finally, by letting $A = a/2\exp(i\theta)$, the real form follows:

$$\begin{cases} \dot{a} = \dfrac{1}{2}\left(\mu - \dfrac{3}{4}\omega^2 b a^2\right)a - \dfrac{3}{16}\dfrac{c}{\omega^2}(\mu + \dfrac{1}{2}\omega^2 b a^2)a^3 \\ a\dot{\vartheta} = \dfrac{3}{8}\dfrac{c}{\omega}a^3 - \dfrac{1}{8}\dfrac{\mu^2}{\omega}a + \dfrac{3}{256}(3b^2\omega^3 - 5\dfrac{c^2}{\omega^3})a^5 \end{cases}$$

■ **The response**

Once the AME have been solved, the solution to the Rayleigh-Duffing equations reads:

$$x(t) = a(t)\cos(\omega t + \theta(t)) +$$
$$\frac{1}{32}a^3(t)\{\frac{c}{\omega^2}\cos[3(\omega t + \theta(t)] + \omega b \sin[3(\omega t + \theta(t)]\} + \cdots$$

where $a(0), \vartheta(0)$ follow from the initial conditions $x(0) = a_0,\ \dot{x}(0) = 0$.

■ Limit cycles and their stability

- Limit-cycles

The limit-cycles are the fixed points $a(t) = \text{const} =: a_s$ of the (real) AME. They satisfy the following algebraic equation:

$$\frac{1}{2}\left(\mu - \frac{3}{4}\omega^2 b a^2\right) a - \frac{3}{16}\frac{c}{\omega^2}(\mu + \frac{1}{2}\omega^2 b a^2) a^3 = 0$$

which implicitly defines a curve on the (μ, a)-plane, for given values of the auxiliary parameters. By solving it:

$$\mu = \frac{3}{2}\frac{4b\omega^2 + bca_s^2}{8\omega^2 - 3ca_s^2}\omega^2 a_s^2$$

or, by expanding for small amplitudes:

$$\mu = \frac{3}{4}b\omega^2 a_s^2 + \frac{15}{32}bca_s^4 + \cdots$$

■ Numerical results

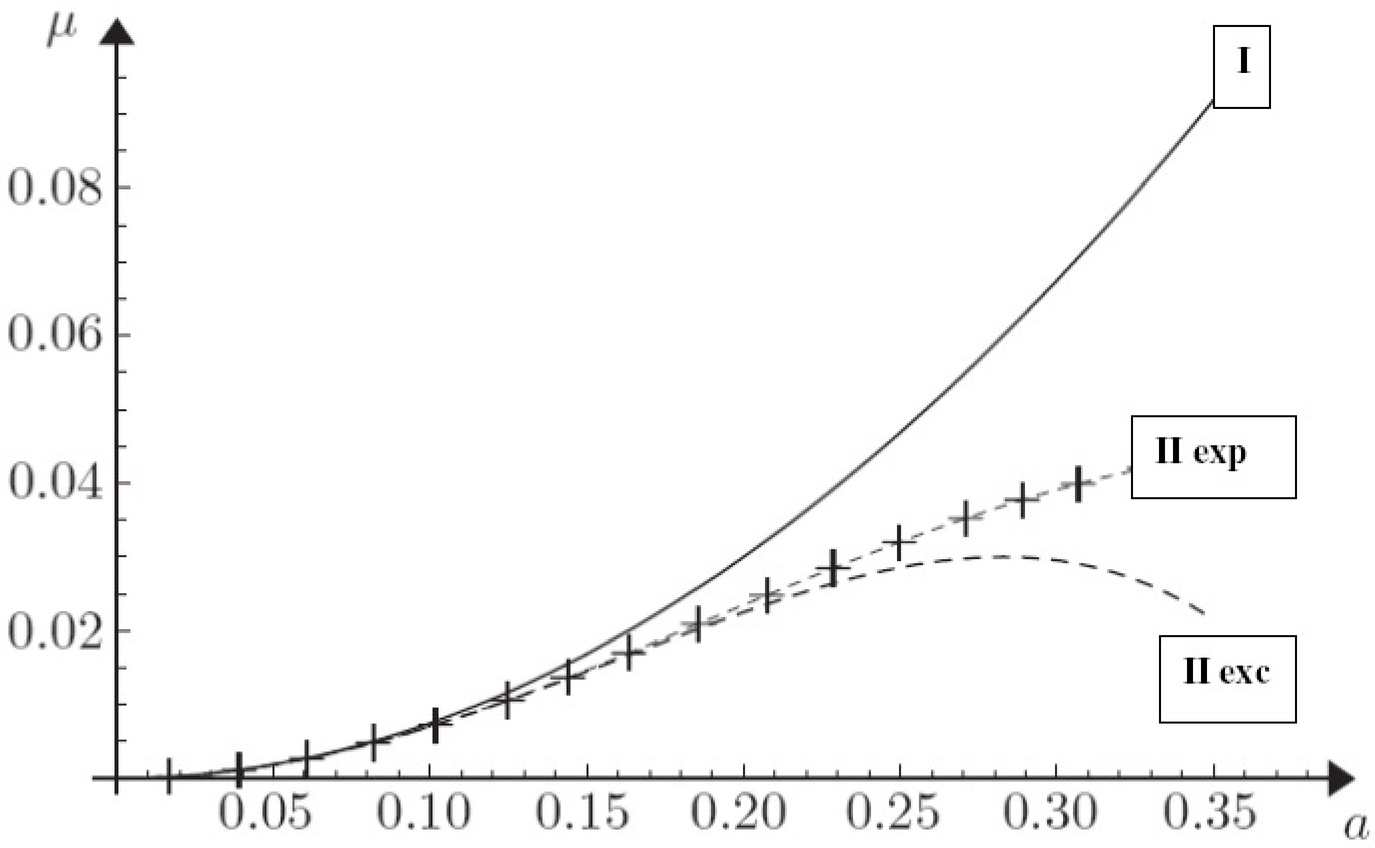
- The most interesting case is $c < 0$ (soft elastic nonlinearities), in which the second-order term subtracts in modulus from the first-order term.
- Since equilibrium of the Raileigh-Duffing oscillator is governed by:

$$\omega^2 x_E + c x_E^3 = 0$$

two nontrivial equilibrium points $x_E := \pm\omega / \sqrt{|c|}$ exist in this case.

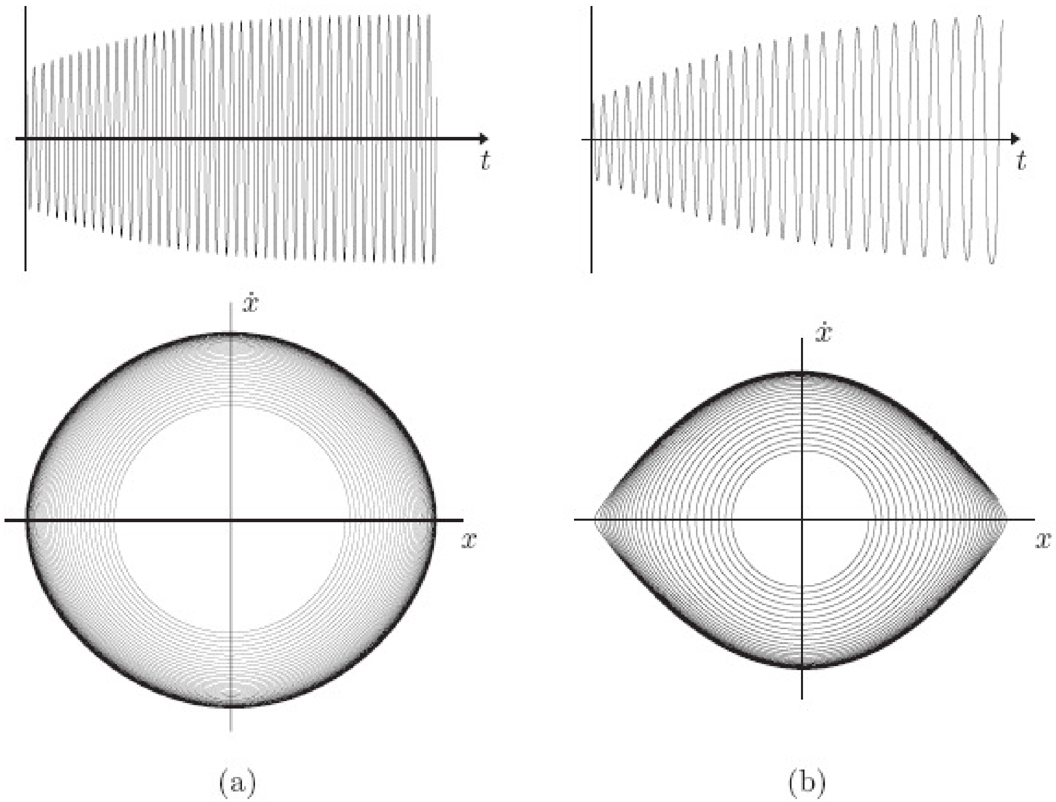
- The two points are connected by separatrices which prevent the limit cycle to increase unboundedly (as instead occurs in the case $c > 0$).
- These properties *are not* captured by the reduced dynamical system, since $x_E = \mathrm{O}(1)$ is too large for the asymptotic analysis ($x = \mathrm{O}(\varepsilon^{1/2})$) to hold.

- Bifurcation diagram



Bifurcation diagram for $\omega = 1, b = 1, c = -10$; --- exact second-order solution, --- expanded second-order solution, — first-order solution, + numerical solutions; heteroclinic bifurcation: a=0.316

- Numerical integrations of the original system:



Time-histories and orbits of the original system; $\omega = 1, b = 1, c = -10$: (a) $\mu = 0.02$; (b) $\mu = 0.035$

- **Note:** when the limit cycle collides with the non-trivial equilibrium points, it disappears. Such a phenomenon is called a *heteroclinic bifurcation*.