STABILITY AND BIFURCATION OF DYNAMICAL SYSTEMS

Scope:

- To remind basic notions of Dynamical Systems and Stability Theory;
- To introduce fundaments of Bifurcation Theory, and establish a link with Stability Theory;
- To give an outline of the Center Manifold Method and Normal Form theory.

Outline:

1. General definitions

2. Fundaments of Stability Theory

3. Fundaments of Bifurcation Theory

4. Multiple bifurcations from a known path

5. The Center Manifold Method (CMM)

6. The Normal Form Theory (NFT)

1. GENERAL DEFINITIONS

We give general definitions for a *N*-dimensional autonomous systems.

- Equations of motion:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)), \quad \mathbf{x} \in \mathbb{R}^N$$

where **x** are *state-variables*, {**x**} the *state-space*, and **F** the *vector field*.

- Orbits:

Let $\mathbf{x}_S(t)$ be the solution to equations which satisfies prescribed initial conditions:

$$\begin{cases} \dot{\mathbf{x}}_S(t) = \mathbf{F}(\mathbf{x}_S(t)) \\ \mathbf{x}_S(0) = \mathbf{x}^0 \end{cases}$$

The set of all the values assumed by $\mathbf{x}_S(t)$ for $t > 0$ is called an *orbit* of the dynamical system. Geometrically, an orbit is a curve in the phase-space, originating from $\mathbf{x}^0$. The set of all orbits is the *phase-portrait* or *phase-flow*.

- Classifications of orbits:

Orbits are classified according to their time-behavior.

- *Equilibrium* (or *fixed*-) *point*: it is an orbit $\mathbf{x}_S(t) =: \mathbf{x}_E$ independent of time (represented by a point in the phase-space);
- *Periodic orbit*: it is an orbit $\mathbf{x}_S(t) =: \mathbf{x}_P(t)$ such that $\mathbf{x}_P(t+T) = \mathbf{x}_P(t)$, with *T* the period (it is a closed curve, called *cycle*);
- *Quasi-periodic orbit*: it is an orbit $\mathbf{x}_S(t) =: \mathbf{x}_Q(t)$ such that, given an arbitrary small $\varepsilon > 0$, there exists a time τ for which $\left|\mathbf{x}_Q(t+\tau) - \mathbf{x}_Q(t)\right| \leq \varepsilon$ holds for any *t*; (it is a curve that densely fills a 'tubular' space);
- *Non-periodic orbit*: orbit $\mathbf{x}_S(t)$ with no special properties.

The first three are *recurrent states*; the last one a *non-recurrent state*.

2. FUNDAMENTS OF STABILITY THEORY

- **Stability of orbits**

Basic idea: An orbit is *stable* if all orbits, originating close to it, remain confined in a small neighborhood; otherwise, it *unstable*. This notion specializes as follows.

- Stability of an equilibrium point (Liapunov):

  $\mathbf{x}_E$ is *stable* if, for every neighborhood $\mathcal{U}$ of $\mathbf{x}_E$, there exists a neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of $\mathbf{x}_E$, such that an orbit $\mathbf{x}(t)$ starting in $\mathcal{V}$ remains in $\mathcal{U}$ for all $t \geq 0$. If , in addition, $\mathbf{x}(t) \to \mathbf{x}_E$ as $t \to \infty$, then $\mathbf{x}_E$ is *asymptotically stable*.

- Stability of a general orbit (orbital stability):

  $\mathbf{x}_S(t)$ is *orbitally stable* if all orbits, originating from nearby initial points, remain 'close' to it, irrespectively of their time-parametrization.

■ Quantitative analysis: the variational equation

- Orbit $\mathbf{x}_S(t)$ to be analyzed:

$$\dot{\mathbf{x}}_S(t) = \mathbf{F}(\mathbf{x}_S(t))$$

- Perturbed motion:

$$\mathbf{x}(t) = \mathbf{x}_S(t) + \delta\mathbf{x}(t)$$

- *Equations for the perturbed motion*, Taylor- expanded:

$$\begin{aligned}\dot{\mathbf{x}}_S(t) + \delta\dot{\mathbf{x}}(t) &= \mathbf{F}(\mathbf{x}_S(t) + \delta\mathbf{x}(t)) \\ &= \mathbf{F}(\mathbf{x}_S(t)) + \mathbf{F}_{\mathbf{x}}(\mathbf{x}_S(t))\delta\mathbf{x}(t) + \mathrm{O}(\left|\delta\mathbf{x}(t)\right|^2)\end{aligned}$$

i.e.:

$$\delta\dot{\mathbf{x}}(t) = \mathbf{J}_S(t)\delta\mathbf{x}(t) + \mathrm{O}(\left|\delta\mathbf{x}(t)\right|^2), \qquad \mathbf{J}_S(t) := \mathbf{F}_{\mathbf{x}}(\mathbf{x}_S(t))$$

➢ By linearizing in $\delta\mathbf{x}(t)$:

$$\delta\dot{\mathbf{x}}(t) = \mathbf{J}_S(t)\delta\mathbf{x}(t)$$

This is called *the variational equation* (based on $\mathbf{x}_S(t)$); it is generally non-autonomous, since $\mathbf{x}_S(t)$ depends on t. Special cases:

- $x_S(t)$ is an equilibrium point $\mathbf{x}_E$:

$$\delta\dot{\mathbf{x}}(t) = \mathbf{J}_E\delta\mathbf{x}(t), \qquad \mathbf{J}_E := \mathbf{F}_\mathbf{x}(\mathbf{x}_E)$$

  in which $\mathbf{J}_E$ has *constant coefficients.*

- $x_S(t)$ is a periodic orbit $\mathbf{x}_P(t)$ of period T:

$$\delta\dot{\mathbf{x}}(t) = \mathbf{J}_P(t)\delta\mathbf{x}(t), \qquad \mathbf{J}_P(t) := \mathbf{F}_\mathbf{x}(\mathbf{x}_P(t))$$

  in which $\mathbf{J}_P(t) = \mathbf{J}_P(t+T)$ has *periodic coefficients* (Floquet Theory).

We will confine ourselves to equilibrium points.

■ Stability of an equilibrium point

We study the stability of an equilibrium point $\mathbf{x}_E$.

- Linearized stability:

We first study *linearized stability* of $\mathbf{x}_E$, by ignoring the reminder $O(|\delta\mathbf{x}(t)|^2)$ in the equation for perturbed motion.

➢ Variational equation:

$$\delta\dot{\mathbf{x}}(t) = \mathbf{J}_E \delta\mathbf{x}(t)$$

➢ Eigenvalue problem:

$$(\mathbf{J}_E - \lambda\mathbf{I})\mathbf{u} = \mathbf{0}$$

➢ Discussion:

- if *all the eigenvalues* λ have negative real parts, then $\delta\mathbf{x}(t) \to 0$ as $t \to \infty$; therefore $\mathbf{x}_E$ is *asymptotically stable*;
- if *at least one eigenvalue* λ has positive real part, then $\delta\mathbf{x}(t) \to \infty$ as $t \to \infty$; therefore $\mathbf{x}_E$ is *unstable*;
- if *all the eigenvalues* λ have non-positive real parts, and at least one of them has zero real part, then, the asymptotic behavior of $\delta\mathbf{x}(t)$, as $t \to \infty$, depends on the eigenspace associated with Re(λ)=0, namely: (a) if it is complete (simple or semi-simple roots λ), $\delta\mathbf{x}(t)$ is oscillatory, hence $\mathbf{x}_E$ is *neutrally stable*; (b) if this is incomplete (non-semi-simple roots λ), $\delta\mathbf{x}(t) \to \infty$, hence $\mathbf{x}_E$ is *unstable*.

- Hyperbolic and non-hyperbolic equilibrium points:

  We introduce the following definition, based on the type of the eigenvalues of $\mathbf{J}_E$:

  - the equilibria at which all the eigenvalues λ have non-zero real parts are *hyperbolic points*;
  - the equilibria at which at least one of the eigenvalues λ has zero real part are *non-hyperbolic points* (also named *critical*).

- **Note:** the most interesting case of non-hyperbolic point, is that in which the eigenvalues have either negative and zero real parts (*neutrally stable linear system*).

- Nonlinear stability of *hyperbolic points*:

Since the remainder term $O(|\delta \mathbf{x}(t)|^2)$ in the nonlinear equation

$$\delta \dot{\mathbf{x}}(t) = \mathbf{J}_S(t)\delta \mathbf{x}(t) + O(|\delta \mathbf{x}(t)|^2)$$

can be made as small as we wish, by selecting a sufficiently small neighborhood of $\mathbf{x}_E$, results for linear system apply also to nonlinear system. Therefore:

> *A hyperbolic point is asymptotically stable if all the eigenvalues of the Jacobian matrix $\boldsymbol{J}_E$ have negative real parts; is unstable if at least one eigenvalue has positive real part.*

- Nonlinear stability of *non-hyperbolic points*:

Nonlinear terms decide the true character of the equilibrium, stable or unstable. Therefore, linear stability analysis fails to give an answer, and a nonlinear analysis is necessary.

➢ Example:

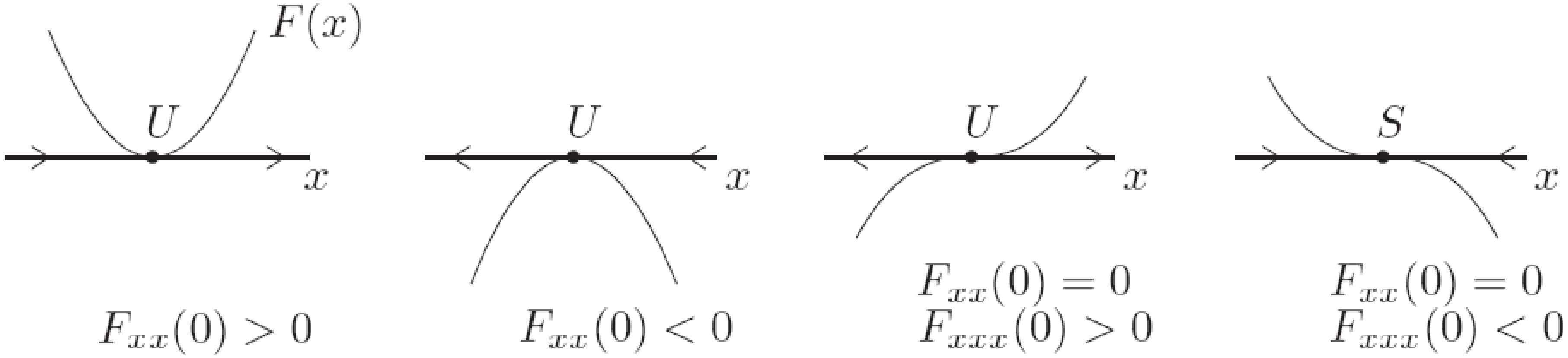
A one-dimensional system admitting a non-hyperbolic equilibrium point x_E=0 is considered:

$$\dot{x} = F(x), \quad x \in \mathbb{R}, \quad F(0) = F_x(0) = 0$$

By letting x=x_E+δx and expanding in series, the equation reads:

$$\delta\dot{x} = \cancel{F(0)} + \cancel{F_x(0)}\,\delta x + \frac{1}{2}F_{xx}(0)\delta x^2 + \frac{1}{6}F_{xxx}(0)\delta x^3 + O(\delta x^4)$$

The phase-portrait is illustrated in the figure.



3. FUNDAMENTS OF BIFURCATION THEORY

Bifurcation theory considers *families of systems* depending on parameters. Its aim is to divide the parameter space in regions in which the system has qualitatively similar behaviors. At the separating boundaries, sudden alteration of the dynamics takes place. They are called *bifurcations*.

■ Parameter-dependent systems

- Autonomous dynamical system:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \boldsymbol{\mu}), \quad \mathbf{x} \in \mathbb{R}^N, \boldsymbol{\mu} \in \mathbb{R}^M$$

where $\boldsymbol{\mu}$ are parameters.

- Phase-portrait:

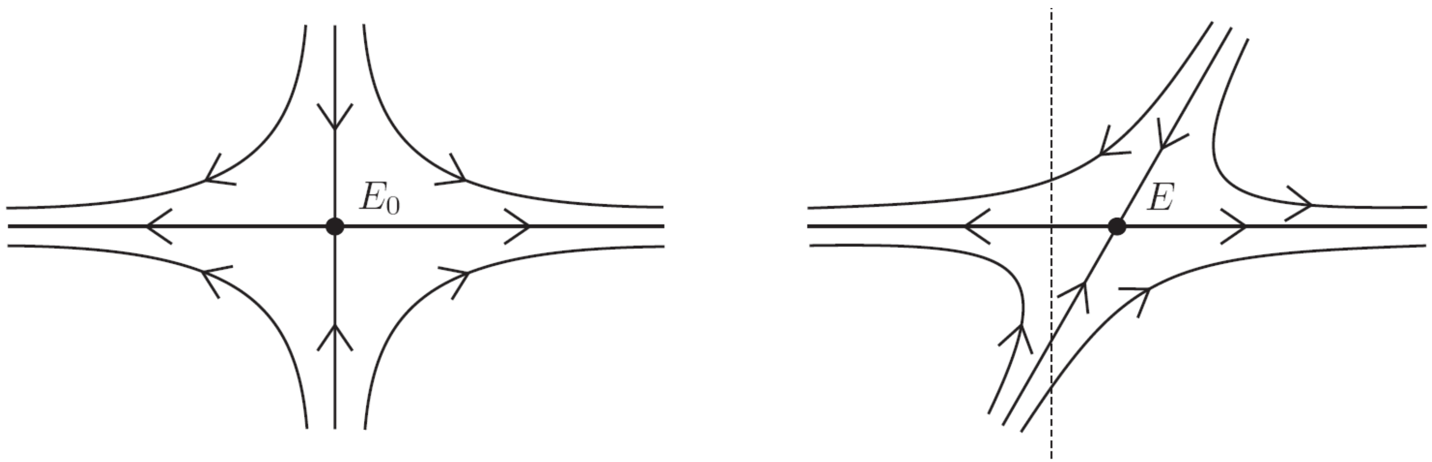
Since orbits $\mathbf{x}_S = \mathbf{x}_S(t, \boldsymbol{\mu})$ depend on parameters, when these latter are (quasi-statically) varied, the whole phase-portrait is modified.

■ Structural Stability

A phase-portrait is *robust*, or *structurally stable*, if small perturbations of the vector-field (as $\varepsilon\mathbf{G}(\mathbf{x},\boldsymbol{\mu})$, with $\varepsilon \ll 1$) *do not qualitatively change* it, but only entails smooth deformations.

- **Note:** Stability and Structural Stability should not be confused. The former refers to a *selected orbit*, and depends on the phase-portraits surrounding it; the latter refer on the *whole phase-portrait*.

➢ Example of robust dynamics:



$\boldsymbol{\mu} = \boldsymbol{\mu}_0$ $\qquad$ $\boldsymbol{\mu} = \boldsymbol{\mu}_0 + \delta\boldsymbol{\mu}$

■ Bifurcation: general definition

- A bifurcation is *a qualitative change of dynamics*. It occurs at a bifurcation value $\boldsymbol{\mu} = \boldsymbol{\mu}_c$ of the parameters at which structural stability is lost. Therefore, the dynamics at $\boldsymbol{\mu}_c + \delta\boldsymbol{\mu}$, with $|\delta\boldsymbol{\mu}|$ arbitrary small, is *topologically inequivalent* from that at $\boldsymbol{\mu}_c$.

- **Note:** at a bifurcation, stability of equilibria changes, or the number of equilibria and/or periodic orbits change.

• Example: saddle-node bifurcation:

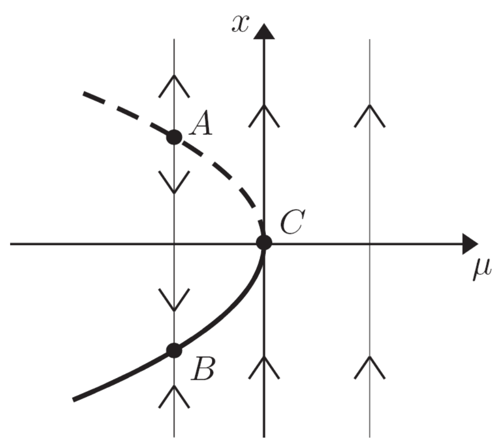
- Dynamical system:

$$\begin{cases} \dot{x} = \mu + x^2 \\ \dot{y} = -y \end{cases}$$

- Equilibrium:

$$x_E = \pm\sqrt{-\mu}, \quad y_E = 0$$

- Equilibrium path (or *bifurcation diagram*):

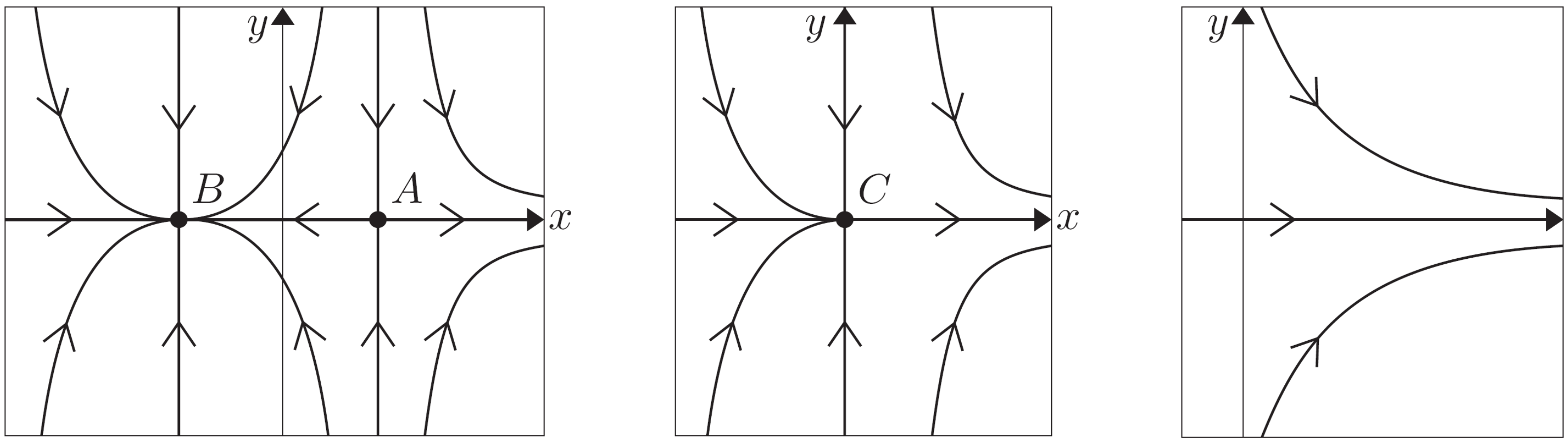


- Jacobian matrix at equilibrium:

$$\mathbf{J}_E = \begin{bmatrix} \pm 2\sqrt{-\mu} & 0 \\ 0 & -1 \end{bmatrix}$$

The equilibrium at μ=0 is not-hyperbolic; here a bifurcation occurs.

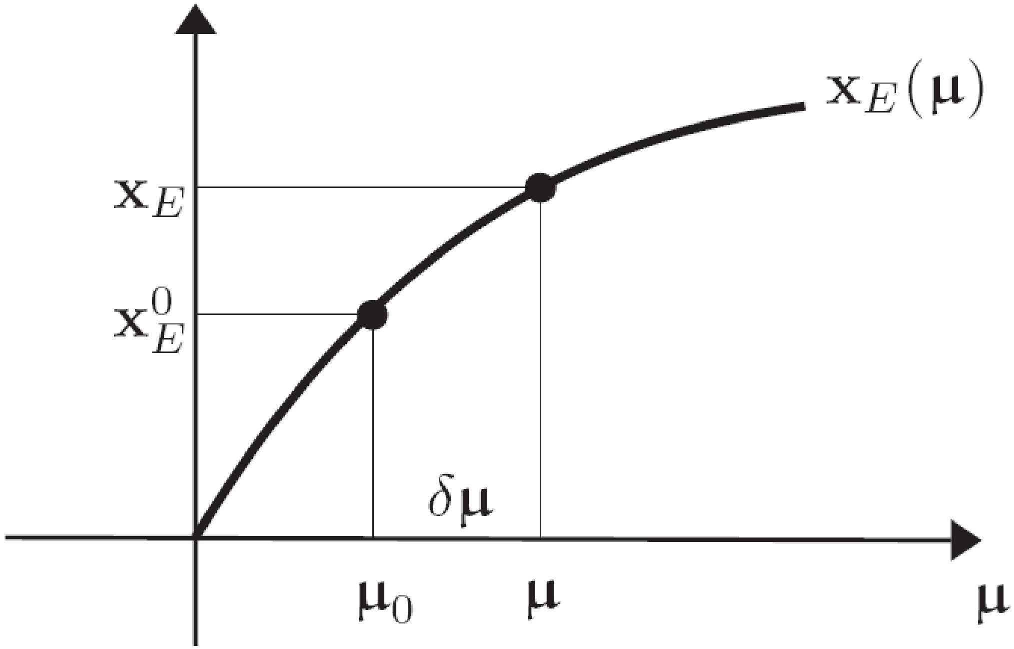
- Phase-portraits:



$\mu < 0$ $\mu = 0$ $\mu > 0$

- **Bifurcation of equilibrium points**:

Let $\mathbf{x}_E = \mathbf{x}_E(\boldsymbol{\mu})$ be an equilibrium path, i.e. $\mathbf{F}(\mathbf{x}_E(\boldsymbol{\mu}), \boldsymbol{\mu}) = 0 \; \forall \boldsymbol{\mu}$ and let $\mathbf{x}_E^0 := \mathbf{x}_E(\boldsymbol{\mu}_0)$ be the equilibrium at $\boldsymbol{\mu} = \boldsymbol{\mu}_0$. We want to analyze changes of the system dynamics, when the parameters $\boldsymbol{\mu}_0$ are perturbed to $\boldsymbol{\mu}_0 + \delta\boldsymbol{\mu}$.



➢Linear dynamics around the perturbed equilibrium point $\mathbf{x}_E(\boldsymbol{\mu})$:

$$\mathbf{x}(t) = \mathbf{x}_E(\boldsymbol{\mu}) + \delta\mathbf{x}(t)$$

$$\cancel{\dot{\mathbf{x}}_E(\boldsymbol{\mu})} + \delta\dot{\mathbf{x}}(t) = \cancel{\mathbf{F}(\mathbf{x}_E(\boldsymbol{\mu}),\boldsymbol{\mu})} + \mathbf{F}_{\mathbf{x}}(\mathbf{x}_E(\boldsymbol{\mu}),\boldsymbol{\mu})\delta\mathbf{x}(t) + \cancel{\mathrm{O}(|\delta\mathbf{x}(t)|^2)}$$

$$\delta\dot{\mathbf{x}}(t) = \mathbf{J}_E(\boldsymbol{\mu})\delta\mathbf{x}(t), \qquad \mathbf{J}_E(\boldsymbol{\mu}) := \mathbf{F}_{\mathbf{x}}(\mathbf{x}_E(\boldsymbol{\mu}),\boldsymbol{\mu})$$

➢Series expansion of the Jacobian matrix at the perturbed point $\mathbf{x}_E(\boldsymbol{\mu})$:

$$\mathbf{J}_E(\boldsymbol{\mu}) = \mathbf{F}_{\mathbf{x}}(\mathbf{x}_E(\boldsymbol{\mu}_0),\boldsymbol{\mu}_0)$$
$$+[\mathbf{F}_{\mathbf{xx}}(\mathbf{x}_E(\boldsymbol{\mu}_0),\boldsymbol{\mu}_0)\mathbf{x}_{E\boldsymbol{\mu}}(\boldsymbol{\mu}_0) + \mathbf{F}_{\mathbf{x}\boldsymbol{\mu}}(\mathbf{x}_E(\boldsymbol{\mu}_0),\boldsymbol{\mu}_0)]\delta\boldsymbol{\mu} + \cancel{\mathrm{O}(|\delta\boldsymbol{\mu}|^2)}$$

or, in short:

$$\mathbf{J}_E =: \mathbf{J}_E^0 + \mathbf{J}_{E\boldsymbol{\mu}}^0\delta\boldsymbol{\mu}$$

The local flow around $\mathbf{x}_E(\boldsymbol{\mu})$ is governed by the eigenvalues of $\mathbf{J}_E$; these are perturbations of the eigenvalues of $\mathbf{J}_E^0$.

- Hyperbolic and non-hyperbolic equilibria

  - If $\mathbf{x}_E^0$ is hyperbolic, the sign of the real part of the eigenvalues of $\mathbf{J}_E^0$ does not change under sufficiently small perturbations $\delta\boldsymbol{\mu}$, so that the dynamics remain substantially unaltered. We conclude that *the local phase-portrait at a hyperbolic equilibrium point is structurally stable.*

  - If $\mathbf{x}_E^0$ is non-hyperbolic, one or more eigenvalues of $\mathbf{J}_E^0$ have zero real parts. Therefore, arbitrary small perturbations $\delta\boldsymbol{\mu}$ may lead to eigenvalues of $\mathbf{J}_E$ with (small) positive or negative real parts, thus strongly changing the dynamics. We conclude that *the local phase-portrait at a non- hyperbolic equilibrium point is structurally unstable.*

- By summarizing:

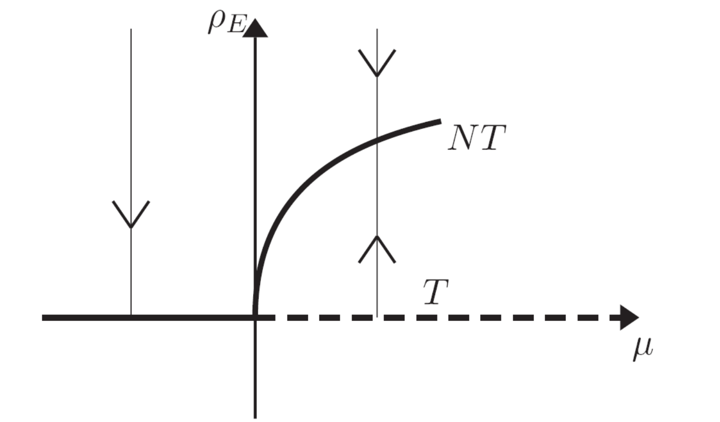
  Bifurcation of equilibrium occurs at *non-hyperbolic* points. If the associated linear system is neutrally stable, then generic perturbations entail loss of stability of the equilibrium.

- Example: Hopf bifurcation
  - Dynamical system:

$$\begin{cases} \dot{\rho} = \rho(\mu - \rho^2) \\ \dot{\theta} = 1 \end{cases}$$

  - Equilibria:

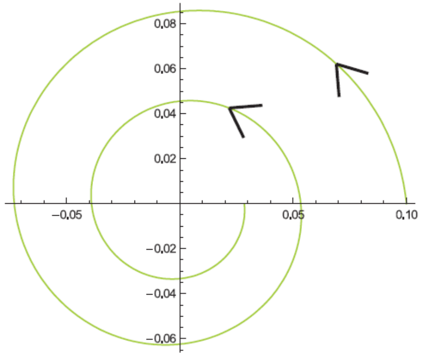
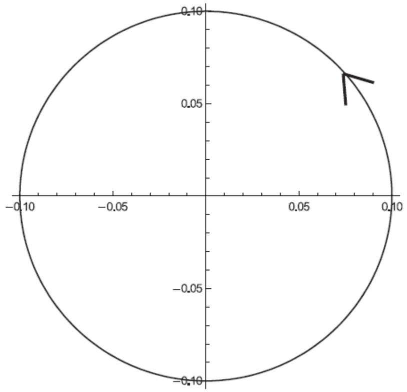
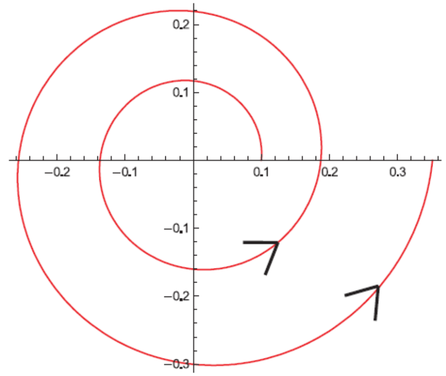
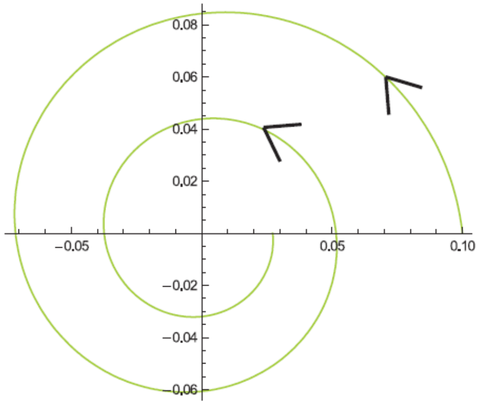
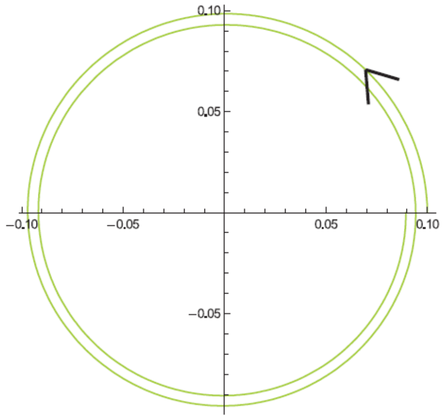
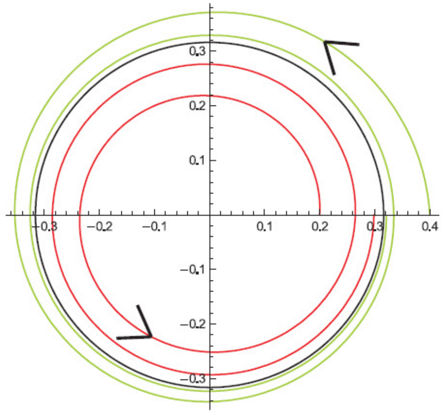
$$\begin{cases} \rho = 0 \ \ \forall \mu \quad (T) \\ \rho = \sqrt{\mu} \quad (NT) \end{cases}$$



  - Jacobian:

$$J_E = \mu - 3\rho^2 = \begin{cases} \mu & \text{on } T \\ \text{-}2\mu & \text{on } NT \end{cases}$$

➢Phase-portraits:

<table>
<tr><th colspan="3">Linearized system</th></tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr><td>stable focus</td><td>center</td><td>unstable focus</td></tr>
<tr><th colspan="3">Nonlinear system</th></tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr><td>Stable focus</td><td>Stable focus</td><td>unst. focus + limit cycle</td></tr>
<tr><td>$\mu<0$</td><td>$\mu=0$</td><td>$\mu>0$</td></tr>
</table>

- Comments:

  - The dynamics around *hyperbolic* points is robust. It is qualitatively the same for linear and nonlinear systems, and does not suffer changes of parameters.

  - The dynamics around *non-hyperbolic* points is not robust. Adding nonlinearities, and/or changing the parameters, may lead to strong changes.

4. MULTIPLE BIFURCATIONS FROM A KNOWN PATH

- **Bifurcations from a known path**

We consider an autonomous dynamical system depending on parameters:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \boldsymbol{\mu}), \quad \mathbf{x} \in \mathbb{R}^N, \boldsymbol{\mu} \in \mathbb{R}^M$$

We assume to know an equilibrium path (named *fundamental path*) $\mathbf{x}_E = \mathbf{x}_E(\boldsymbol{\mu})$; we want:

(a) to find the values $\boldsymbol{\mu}_c$ of the parameters for which a bifurcation takes place (analysis of *critical behavior*);

(b) to study the dynamics of the nonlinear system for values of $\boldsymbol{\mu}$ close to $\boldsymbol{\mu}_c$ (*post-critical behavior*).

For task (a), a linearized analysis is required; for task (b), a nonlinear analysis is necessary.

- **Local form of the equation of motion**

  - It is convenient to introduce *local coordinates*:

$$\tilde{\mathbf{x}}(t,\boldsymbol{\mu}) := \mathbf{x}(t) - \mathbf{x}_E(\boldsymbol{\mu})$$

Here $\tilde{\mathbf{x}}(t,\boldsymbol{\mu})$ denotes, for a given $\boldsymbol{\mu}$, the deviation of the actual state from the corresponding equilibrium. Often, the dependence on $\boldsymbol{\mu}$ is understood, and $\tilde{\mathbf{x}}$ is written as $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(t)$.

  - The corresponding *local form* of the equations read:

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{F}(\tilde{\mathbf{x}}(t) + \mathbf{x}_E(\boldsymbol{\mu}), \boldsymbol{\mu}) := \tilde{\mathbf{F}}(\tilde{\mathbf{x}}(t), \boldsymbol{\mu})$$

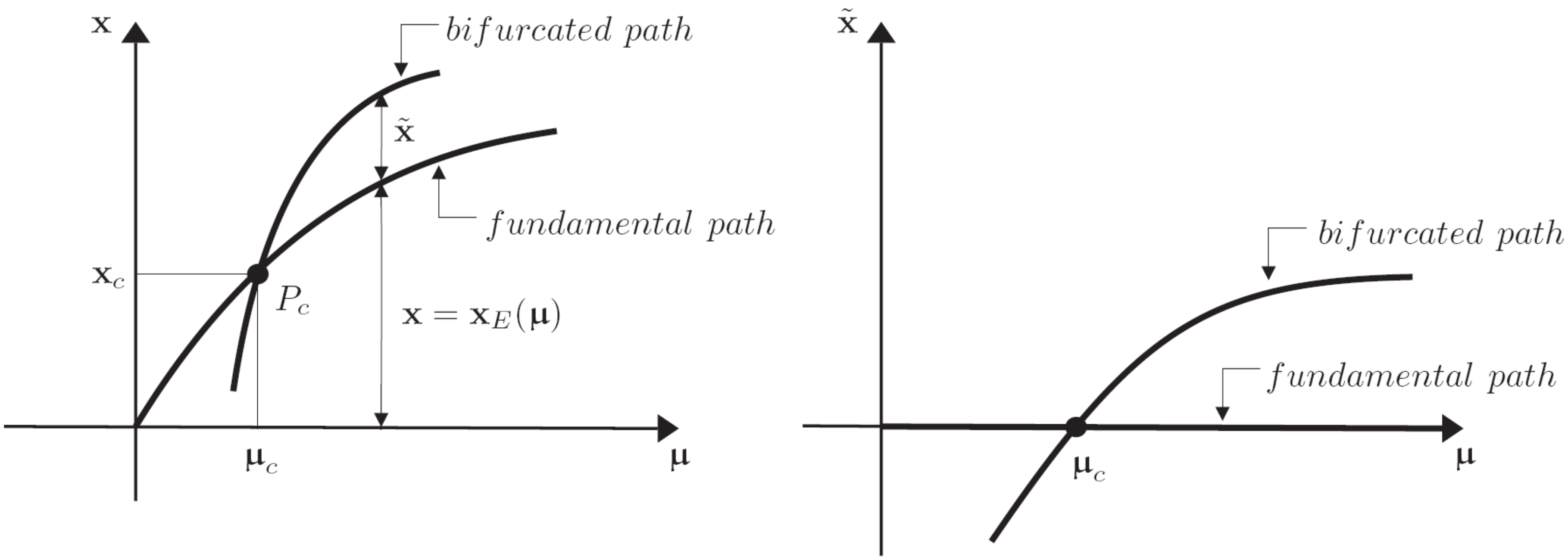
or, by omitting the tilde:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \boldsymbol{\mu}), \qquad \mathbf{F}(\mathbf{0}, \boldsymbol{\mu}) = \mathbf{0}\ \forall \boldsymbol{\mu}$$

➢ In the new variables, the fundamental path appears as trivial:

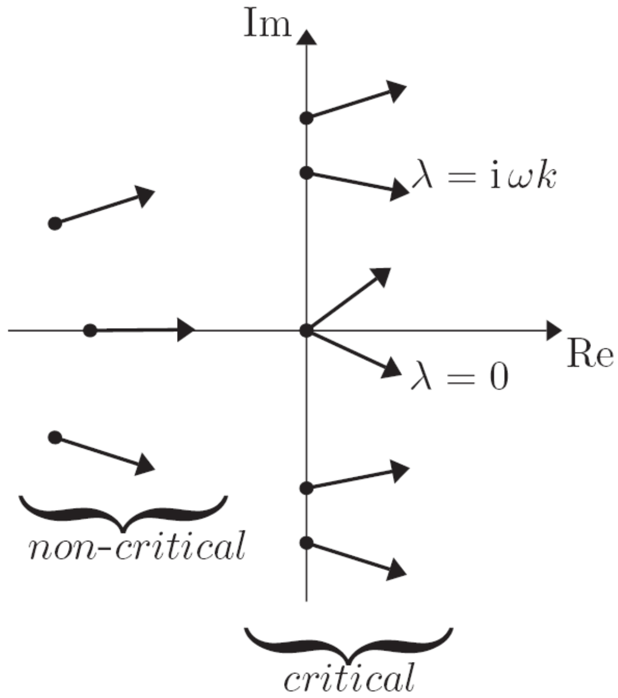
$$\tilde{\mathbf{x}}_E = \mathbf{0}\ \forall \boldsymbol{\mu}$$

➢ Geometrical meaning of the local coordinates:



■ **Static and dynamic bifurcations**

We consider a bifurcation point $\boldsymbol{\mu} = \boldsymbol{\mu}_c$, at which the Jacobian matrix $\mathbf{F}_{\mathbf{x}}^c := \mathbf{F}_{\mathbf{x}}(\mathbf{x}_E(\boldsymbol{\mu}_c), \boldsymbol{\mu}_c)$ admits $N_c := N_z + 2\,N_i$ (non-hyperbolic or *critical*) eigenvalues with zero real part, ($\lambda_k = 0, 0, \cdots; \pm i\omega_1, \pm i\omega_2, \cdots .$), the remaining (non-critical) having negative real part (example in Fig: N_z=2, N_i =2).



- Transversality condition

  We assume that the critical eigenvalues cross the imaginary axis with non-zero velocity (i.e. $(\partial / \partial \boldsymbol{\mu}) \operatorname{Re}(\lambda_k) \neq \mathbf{0}$) .

We distinguish:

- *static bifurcation* (also said *divergence* bifurcation), if the critical eigenvalues are all zero, $\lambda_k = 0, k = 1, 2, \cdots N_z$;
- *dynamic bifurcation* (or Hopf bifurcation): if the critical eigenvalues are all purely imaginary, $\lambda_j = \pm i\omega_j, \quad j = 1, 2, \cdots, N_i$;
- *static-dynamic bifurcation*: if the critical eigenvalues are either zero and purely imaginary, $\lambda_k = 0, \lambda_j = \pm i\omega_j, \quad k = 1, 2, \cdots N_z, j = 1, 2, \cdots, N_i$.

If N_z+N_i =1, the bifurcation is called *simple*; if N_z+N_i>1, the bifurcation is called *multiple*.

- **Resonant dynamic bifurcations**

If the critical eigenvalues $\lambda_j = \pm i\omega_j$ are *rationally dependent*, i.e. if there exist some sets of integer numbers k_{rj} such that:

$$\sum_{j=1}^{N_c} k_{rj}\omega_j = 0, \quad k_{rj} \in \mathbb{Z}, \quad r = 1, 2, \cdots, R$$

then the (multiple) bifurcation is said *resonant*. If no such numbers exist, the bifurcation is said *non-resonant*.

- **Note:** resonance conditions do not play any role in determining if a point is, or not, of bifurcation. However, as it will be shown ahead, the resonances strongly affect the *nonlinear* dynamics.

- **Linear codimension of a bifurcation**

In the parameter-space, bifurcations take place on manifolds on which some relations (*constraints*) among the parameters is satisfied, namely:

  - if the bifurcation is *non-resonant*:

$$\mathrm{Re}(\lambda_k) = 0, \quad k = 1, 2, \cdots, N_z + N_i$$

  - if the bifurcation is *resonant*:

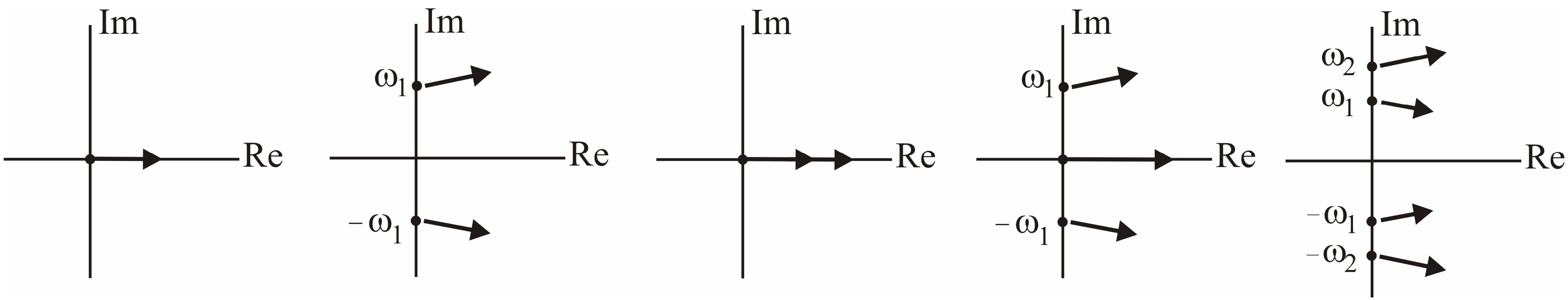
$$\begin{cases} \mathrm{Re}(\lambda_k) = 0, & k = 1, 2, \cdots, N_z + N_i \\ \sum_{k=1}^{N_c} k_{rj}\, \mathrm{Im}(\lambda_k) = 0, & r = 1, 2, \cdots, R \end{cases}$$

The number:

$$M := N_z + N_i + R$$

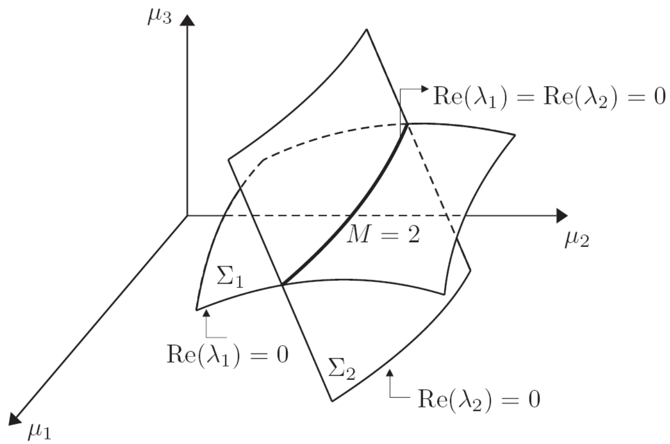
is the (linear) *codimension* of the bifurcation; it is the codimension of the manifold on which the multiple bifurcation occurs.

- Examples of low-codimension bifurcations:



Divergence	Hopf	Double-zero	Hopf-Diverg.	Double-Hopf
M=1	M=1	M=2	M=2	$\begin{cases} M = 2 \text{ non-res.} \\ M = 3 \text{ resonant} \end{cases}$
$\lambda_1 = 0$	$\mathrm{Re}(\lambda_1) = 0$	$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases}$	$\begin{cases} \mathrm{Re}(\lambda_1) = 0 \\ \lambda_2 = 0 \end{cases}$	$\begin{cases} \mathrm{Re}(\lambda_1) = 0 \\ \mathrm{Re}(\lambda_2) = 0 \end{cases}$ $\omega_2 / \omega_1 = r \in \mathbb{Q}$

- **The bifurcation parameters**
  - A bifurcation point $\boldsymbol{\mu}_c$ would appear as a rare singularity, if specific systems were considered. A small perturbation $\delta\boldsymbol{\mu}$ would cancel it.
  - In contrast, if a *family of systems* is considered, the bifurcating system *naturally appears* as a member of that family.



  - The lowest-dimensional family where to embed the bifurcation has dimension-*M*, and it is *transversal to the critical manifold*.
  - Any perturbation of the transversal manifold changes the bifurcation point, but cannot destroy the bifurcation. The *M* parameters of the family are the *bifurcation parameters*.

- **The (nonlinear) bifurcation analysis**

To analyze the system dynamics around a bifurcation point $\boldsymbol{\mu}=\boldsymbol{\mu}_c$, there are essentially two methods:

- The *Center Manifold Method* (CMM) and *Normal Form Theory* (NFT)
- The *Multiple Scale Method* (MSM) (or equivalent perturbation methods).

The CMM *reduces the dimension* of the system, leading to an equivalent system (bifurcation equations) which describes the asymptotic ($t \to \infty$) dynamics.

The NFT *reduces the complexity* of the bifurcation equations, giving them the simplest nonlinear form.

The MSM performs both the operations simultaneously. In addition *it filters the fast dynamics* from the bifurcation equations.