STATIC BIFURCATIONS OF LOW-DIMENSIONAL SYSTEMS

Scope:

- To study static bifurcations of simple systems, exhibiting behaviors commonly encountered in more complex systems.
- To introduce the concept of *imperfections* and *robustness of a bifurcation*.
- To show how the Multiple Scale Method works as a reduction method, alternative to the Center Manifold Method.

Outline:

1. **Codimension-1 static bifurcations**
2. **Imperfection sensitivity**
3. **Multiple scale analysis of a sample systems**

1. CODIMENSION-1 STATIC BIFURCATIONS

■ **One-dimensional, one-parameter dynamical system**

$$\dot{x} = F(x,\mu) \quad x \in \mathbb{R}, \mu \in \mathbb{R}$$

➢ Critical equilibrium point $(x,\mu)=(0,0)$:

$$F(0,0)=0, \qquad J := F_x(0,0)=0$$

➢ Equation of motion expanded around the critical point:

$$\dot{x} = \frac{1}{2} F_{xx}^0 x^2 + \frac{1}{6} F_{xxx}^0 x^3 + \cdots$$
$$+\mu\left(F_\mu^0 + F_{x\mu}^0 x + \cdots\right) + \frac{1}{2}\mu^2\left(F_{\mu\mu}^0 + F_{x\mu\mu}^0 x + \cdots\right) + \cdots$$

■ Generic case: fold bifurcation

$$F_{\mu}^{0} \neq 0 \quad F_{xx}^{0} \neq 0$$

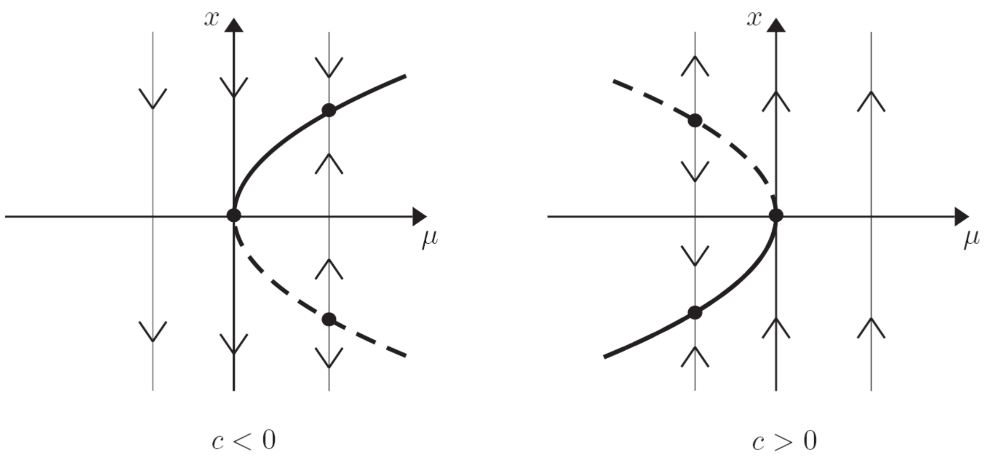
At the lower-order, the equation reads:

$$\dot{x} = F_{\mu}^{0}\mu + \frac{1}{2}F_{xx}^{0}\,x^{2}$$

or, after a change of variable:

$$\dot{x} = \mu + cx^{2}$$

The equation describes a *fold*(or *saddle-node*) *bifurcation*. The critical point is called a *turning* or *limit point*; here a *catastrophic* bifurcation takes place.



■ **Non-generic case:** ***bifurcations from a known path***

- We introduce, as further assumption, that the system admits the *trivial equilibrium path* $x_T=0\ \forall\,\mu$ (fundamental path), i.e.:

$$F(0,\mu)=0\ \ \forall\mu$$

- By successive differentiations and evaluation at $\mu=0$, it follows:

$$F_{\mu}^{0}=F_{\mu\mu}^{0}=\cdots=0$$

- Equation reduces to:

$$\dot{x}=\frac{1}{2}F_{xx}^{0}\,x^{2}+\frac{1}{6}F_{xxx}^{0}\,x^{3}+\ldots+F_{x\mu}^{0}\,x\mu+\frac{1}{2}F_{x\mu\mu}^{0}\,x\mu^{2}+\ldots$$

- We analyze two cases:

(a) Transcritical bifurcation: $F_{xx}^{0}\neq 0$ (non-symmetric systems)

(b) Fork bifurcation : $F_{xx}^{0}=0,\ F_{xxx}^{0}\neq 0$ (symmetric systems)

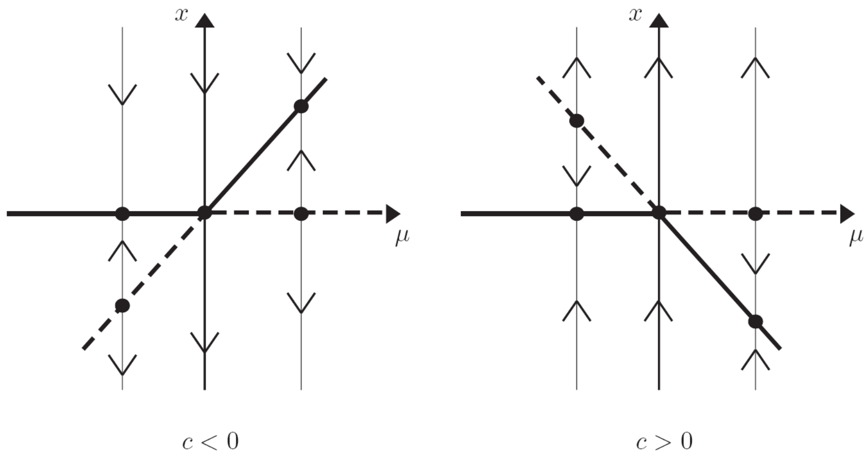
- Transcritical bifurcation:

$$F^0_{x\mu} > 0, \quad F^0_{xx} \neq 0$$

At the lower-order the equation is equivalent to:

$$\dot{x} = \mu x + c x^2$$

Therefore two equilibria exist at the same μ: $x_T = 0, \quad x_{NT} = -\mu / c$, which coalesce at μ=0. This is called a *transcritical bifurcation*.



- ❑ **Note:** An *exchange of stability* occurs at the bifurcation, between the fundamental and bifurcated paths.

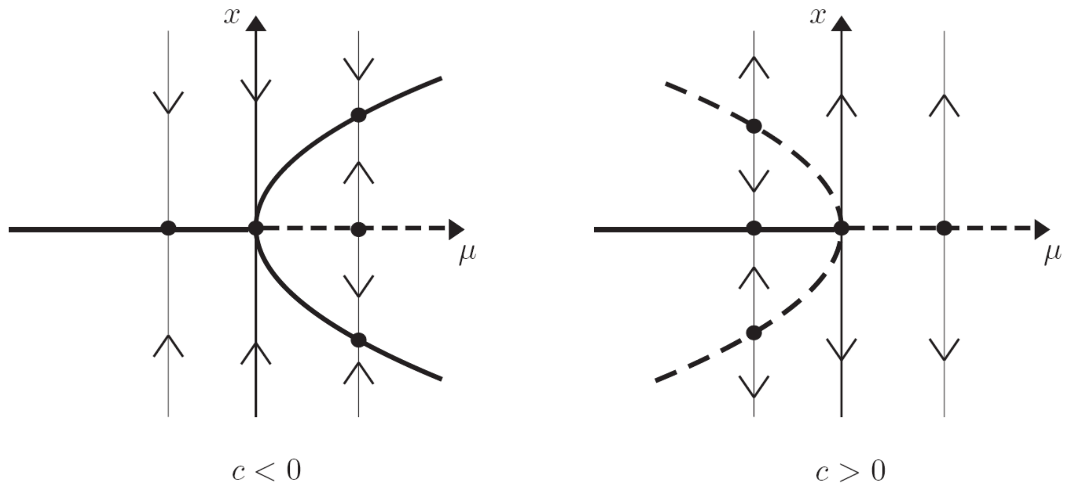
- Fork bifurcation:

$$F^0_{x\mu} > 0, \quad F^0_{xx} = 0, \quad F^0_{xxx} \neq 0$$

At the lower-order, the equation is equivalent to:

$$\dot{x} = \mu x + c x^3$$

One *or* three equilibria exist at the same μ: $x_T = 0 \; \forall \mu$ and $x_{NT} = \pm\sqrt{-\mu / c}$ for $\mu/c<0$. This is a *fork bifurcation*, *super-critical* if $c<0$, *sub-critical* if $c>0$.



❑ **Note:** An *exchange of stability* occurs at the bifurcation point.

2. **IMPERFECTION SENSITIVITY**

We assume that the system, additionally, depends on a small *imperfection parameter* η, accounting for uncertainties in modeling, i.e:

$$\dot{x} = \tilde{F}(x, \mu; \eta) \quad x \in \mathbb{R}, \mu \in \mathbb{R}, \eta \in \mathbb{R}$$

By expanding for small η and retaining only the leading-order term:

$$\begin{aligned}\dot{x} &= \tilde{F}(x, \mu; 0) + \eta[\tilde{F}_\eta(0,0;0) + x\tilde{F}_{x\eta}(0,0;0) + \mu\tilde{F}_{\mu\eta}(0,0;0) + \cdots] + \cdots \\ &= F(x, \mu) + \eta\tilde{F}_\eta(0,0;0) + \mathrm{O}(\eta x, \eta\mu)\end{aligned}$$

Therefore, the imperfections *just add a constant* to the bifurcation equations of the relevant perfect system.

The perfect bifurcation is said:

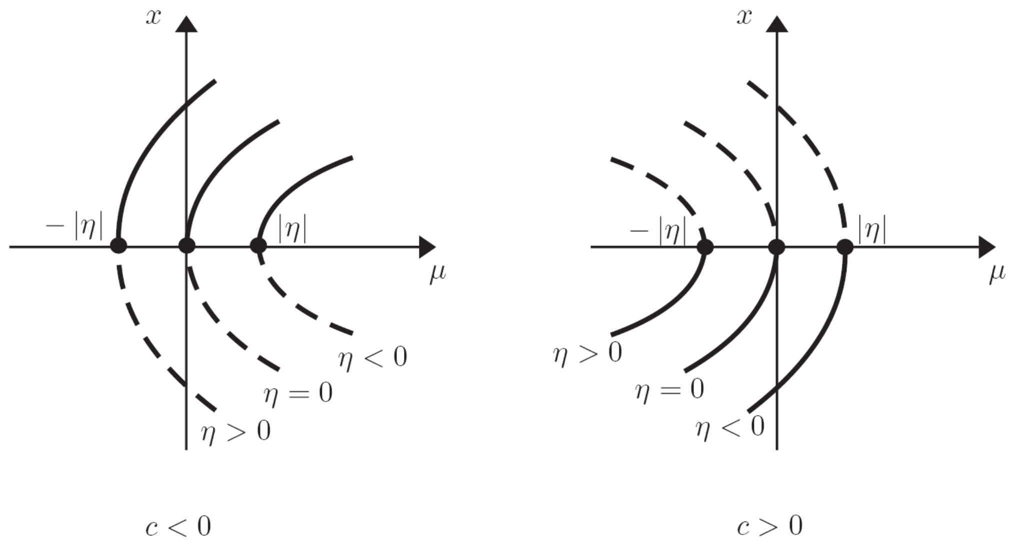
- *structurally stable*, if it persists under imperfections;
- *structurally unstable*, if it does not persist under imperfections.

- Imperfect fold bifurcation:

$$\dot{x} = \mu + cx^2 + \eta$$

➢ Bifurcation diagram:

By projecting the equilibra $x_E = \pm\sqrt{-(\mu+\eta)/c}$ on the (μ,x)-plane:

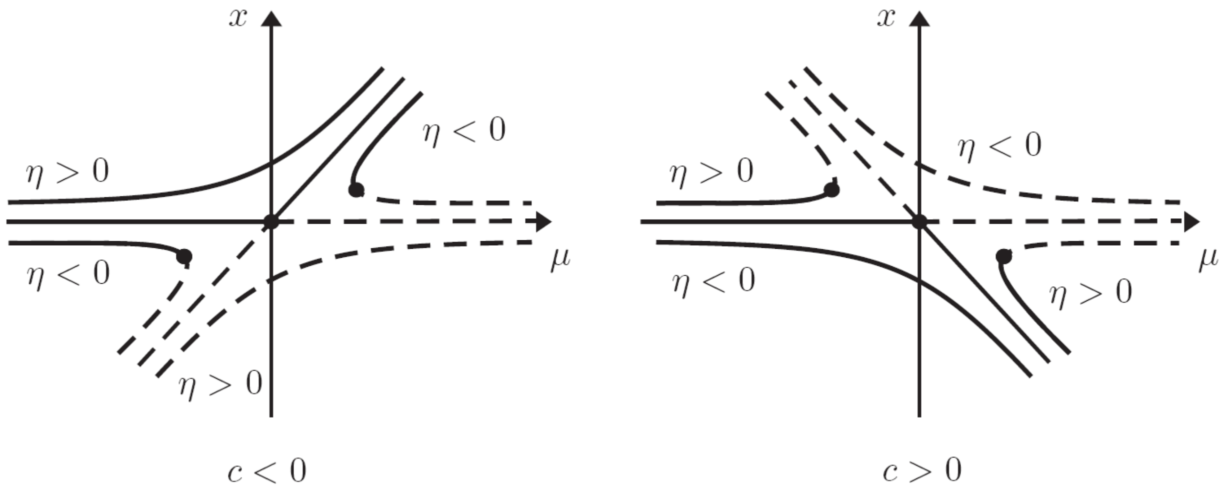


➢ The bifurcation diagrams relevant to different η's are all equivalent. Therefore, the fold bifurcation is *structurally stable*.

- Imperfect transcritical bifurcation:

$$\dot{x} = \mu x + cx^2 + \eta$$

  - Bifurcation diagram:

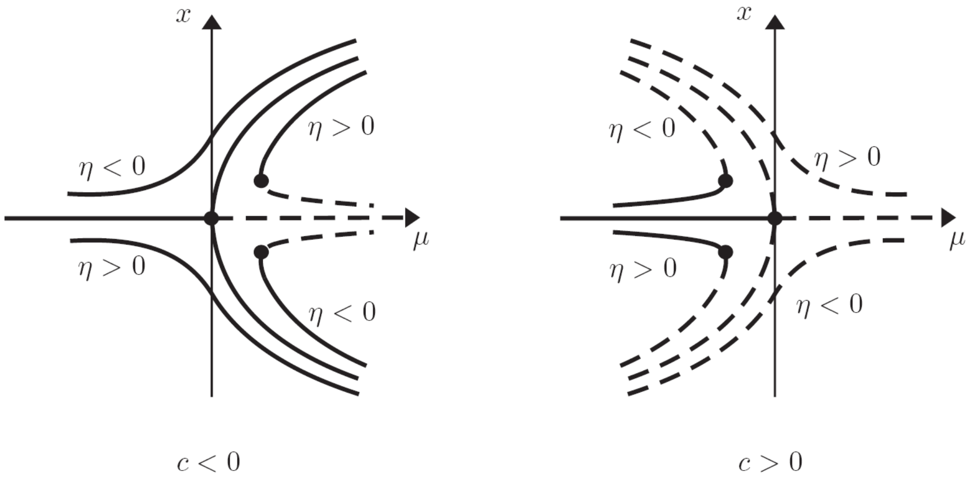


  - The bifurcation diagram exhibits fold bifurcations that, for a suitable sign of η, *reduce* the maximum stable value of μ (catastrophic bifurcation).
  - The transcritical bifurcation *is not structurally stable* in a one-parameter family. Also the fundamental path is destroyed.

- Imperfect fork bifurcation:

$$\dot{x} = \mu x + cx^3 + \eta$$

  - ➢ Bifurcation diagram:



  - o The bifurcation diagram exhibits fold bifurcations; therefore, the fork bifurcation *is structurally unstable*
  - o In the sub-critical case, imperfections of both signs *reduce* the maximum stable value of μ.
  - o In the super-critical case, imperfections have non-catastrophic character.

4.MULTIPLE SCALE ANALYSIS OF SAMPLE SYSTEMS

When a multi-dimensional system, undergoing a codimension-M static bifurcation, is considered, *a reduction process* must be applied, in order to get an M-dimensional bifurcation equation.

An example of reduction performed by the Center Manifold Method (CMM) for M=1 was already shown. The same example is now worked out by the Multiple Scale Method (MSM).

A new example relevant to M=2 is also shown.

■ A two-dimensional system, undergoing a simple divergence bifurcation

We consider the system already analyzed, with an imperfection η added:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} xy + cx^3 + \eta \\ bx^2 \end{pmatrix}$$

- Rescaling:

$$(x, y) \to (\varepsilon x, \varepsilon y), \quad \mu \to \varepsilon^2 \mu, \quad \eta \to \varepsilon^3 \eta$$

The equations become:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \varepsilon \begin{pmatrix} xy \\ bx^2 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \mu x + cx^3 + \eta \\ 0 \end{pmatrix}$$

- Series expansions:

$$\begin{pmatrix} x(t;\varepsilon) \\ y(t;\varepsilon) \end{pmatrix} = \begin{pmatrix} x_0(t_0,t_1,t_2,\cdots) \\ y_0(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon \begin{pmatrix} x_1(t_0,t_1,t_2,\cdots) \\ y_1(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} x_2(t_0,t_1,t_2,\cdots) \\ y_2(t_0,t_1,t_2,\cdots) \end{pmatrix} + \cdots$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{d}_0 + \varepsilon\,\mathrm{d}_1 + \varepsilon^2\,\mathrm{d}_2 + \cdots, \quad \mathrm{d}_k := \partial / \partial t_k, \quad t_k := \varepsilon^k t_k$$

- Perturbation equations:

$$\varepsilon^0 : \begin{cases} \mathrm{d}_0\, x_0 = 0 \\ \mathrm{d}_0\, y_0 + y_0 = 0 \end{cases}$$

$$\varepsilon^1 : \begin{cases} \mathrm{d}_0\, x_1 = -\mathrm{d}_1 x_0 + x_0 y_0 \\ \mathrm{d}_0\, y_1 + y_1 = -\mathrm{d}_1 y_0 + b x_0^2 \end{cases}$$

$$\varepsilon^2 : \begin{cases} \mathrm{d}_0\, x_2 = -\mathrm{d}_2 x_0 - \mathrm{d}_1 x_1 + (x_1 y_0 + x_0 y_1) + \mu x_0 + c x_0^3 + \eta \\ \mathrm{d}_0\, y_2 + y_2 = -\mathrm{d}_2 y_0 - \mathrm{d}_1 y_1 + 2b x_0 x_1 \end{cases}$$

………

- Generating solution:

$$\begin{cases} x_0 = a(t_1, t_2) \\ y_0 = k(t_1, t_2)\,\mathrm{e}^{-t_0} \end{cases}$$

By ignoring transient motions, the steady contribution only is retained:

$$\begin{cases} x_0 = a(t_1, t_2) \\ y_0 = 0 \end{cases}$$

❑ **Note:** the passive variable *y* *does not* enter the generating solution.

- ε-order:

  - equations:

$$\begin{cases} \mathrm{d}_0 x_1 = -\mathrm{d}_1 a \\ \mathrm{d}_0 y_1 + y_1 = ba^2 \end{cases}$$

  - elimination of secular terms:

$$\mathrm{d}_1 a = 0$$

  - solution:
    By omitting the complementary solutions:

$$\begin{cases} x_1 = 0 \\ y_1 = ba^2 \end{cases}$$

- **Note:** the link between passive and active coordinates is established at this order.

- ε^2-order:
  - equations:

$$\begin{cases} \mathrm{d}_0\, x_2 = -\mathrm{d}_2 a + \mu a + (b+c)a^3 + \eta \\ \mathrm{d}_0\, y_2 + y_2 = 0 \end{cases}$$

  - elimination of secular terms:

$$\mathrm{d}_2 a = \mu a + (b+c)a^3 + \eta$$

- By coming back to the original, not rescaled, variables, through:

$$\varepsilon a \to a, \quad \varepsilon^2 \mu \to \mu, \quad \varepsilon^3 \eta \to \eta, \quad \varepsilon^2\, \mathrm{d}_2 \to \mathrm{D}$$

the *bifurcation equation* follows:

$$\dot{a} = \mu a + (b+c)a^3 + \eta$$

This coincides with that furnished by the CMM, with the imperfection added.

■ A three-dimensional system, undergoing a double divergence bifurcation

We show as to apply the MSM to a multiple divergence bifurcation, referring to a M=2 case. The system is a direct generalization of the previous one, i.e.:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} xz + c_1 x^3 + \eta \\ yz + c_2 y^3 + \eta \\ b_1 x^2 + b_2 y^2 \end{pmatrix}$$

Here, **J** admits the (semi-simple) double eigenvalue λ=0 at $\boldsymbol{\mu}_c=(\mu_c,\nu_c)=(0,0)$.

In the CMM view, $\mathbf{x}_c = (x, y), \mathbf{x}_s = (z)$.

- Rescaling:
  After the rescaling $(x, y, z) \to (\varepsilon x, \varepsilon y, \varepsilon z)$, $\mu \to \varepsilon^2 \mu, \nu \to \varepsilon^2 \nu, \eta \to \varepsilon^3 \eta$ the equations read:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \varepsilon \begin{pmatrix} xz \\ yz \\ b_1 x^2 + b_2 y^2 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \mu x + c_1 x^3 + \eta \\ \nu y + c_2 y^3 + \eta \\ 0 \end{pmatrix}$$

- Series expansions:

$$\begin{pmatrix} x(t;\varepsilon) \\ y(t;\varepsilon) \\ z(t;\varepsilon) \end{pmatrix} = \begin{pmatrix} x_0(t_0, t_1, t_2, \cdots) \\ y_0(t_0, t_1, t_2, \cdots) \\ z_0(t_0, t_1, t_2, \cdots) \end{pmatrix} + \varepsilon \begin{pmatrix} x_1(t_0, t_1, t_2, \cdots) \\ y_1(t_0, t_1, t_2, \cdots) \\ z_1(t_0, t_1, t_2, \cdots) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} x_2(t_0, t_1, t_2, \cdots) \\ y_2(t_0, t_1, t_2, \cdots) \\ z_2(t_0, t_1, t_2, \cdots) \end{pmatrix} + \cdots$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{d}_0 + \varepsilon\, \mathrm{d}_1 + \varepsilon^2\, \mathrm{d}_2 + \cdots, \quad \mathrm{d}_k := \partial / \partial t_k, \quad t_k := \varepsilon^k t_k$$

- Perturbation equations:

$$\varepsilon^0 : \begin{cases} d_0\, x_0 = 0 \\ d_0\, y_0 = 0 \\ d_0\, z_0 + z_0 = 0 \end{cases}$$

$$\varepsilon^1 : \begin{cases} d_0\, x_1 = -d_1 x_0 + x_0 z_0 \\ d_0\, y_1 = -d_1 y_0 + y_0 z_0 \\ d_0\, z_1 + z_1 = -d_1 z_0 + b_1 x_0^2 + b_2 y_0^2 \end{cases}$$

$$\varepsilon^2 : \begin{cases} d_0\, x_2 = -d_2 x_0 - d_1 x_1 + (x_1 z_0 + x_0 z_1) + \mu x_0 + c_1 x_0^3 + \eta \\ d_0\, y_2 = -d_2 y_0 - d_1 y_1 + (y_1 z_0 + y_0 z_1) + \nu y_0 + c_2 y_0^3 + \eta \\ d_0\, z_2 + z_2 = -d_2 z_0 - d_1 z_1 + 2b_1 x_0 x_1 + 2b_2 y_0 y_1 \end{cases}$$

………

- Generating solution:

$$\begin{cases} x_0 = a_1(t_1, t_2) \\ y_0 = a_2(t_1, t_2) \\ z_0 = 0 \end{cases}$$

- ε-order:
  - equations:

$$\begin{cases} \mathrm{d}_0\, x_1 = -\mathrm{d}_1 a_1 \\ \mathrm{d}_0\, y_1 = -\mathrm{d}_1 a_2 \\ \mathrm{d}_0\, z_1 + z_1 = b_1 a_1^2 + b_2 a_2^2 \end{cases}$$

  - Secular terms:

$$\mathrm{d}_1 a_1 = 0, \quad \mathrm{d}_1 a_2 = 0$$

  - solution:

$$\begin{cases} x_1 = 0 \\ y_1 = 0 \\ z_1 = b_1 a_1^2 + b_2 a_2^2 \end{cases}$$

- ε^2-order:
  - equations:

$$\begin{cases} \mathrm{d}_0\, x_2 = -\mathrm{d}_2 a_1 + \mu a_1 + (b_1 + c_1) a_1^3 + b_2 a_1 a_2^2 + \eta \\ \mathrm{d}_0\, y_2 = -\mathrm{d}_2 a_2 + \nu a_2 + b_1 a_1^2 a_2 + (b_2 + c_2) a_2^3 + \eta \\ \mathrm{d}_0\, z_2 + z_2 = 0 \end{cases}$$

  - elimination of secular terms:

$$\begin{cases} \mathrm{d}_2 a_1 = \mu a_1 + (b_1 + c_1) a_1^3 + b_2 a_1 a_2^2 + \eta \\ \mathrm{d}_2 a_2 = \nu a_2 + b_1 a_1^2 a_2 + (b_2 + c_2) a_2^3 + \eta \end{cases}$$

- Bifurcation equations:

$$\begin{cases} \dot{a}_1 = a_1[\mu + (b_1 + c_1) a_1^2 + b_2 a_2^2] + \eta \\ \dot{a}_2 = a_2[\nu + b_1 a_1^2 + (b_2 + c_2) a_2^2] + \eta \end{cases}$$

- Steady-state solutions for the perfect (η=0) system::

$$(T) \; : \; a_1 = 0, a_2 = 0, \quad \forall(\mu, \nu) \quad \text{(Trivial)}$$

$$(M_1): \; a_1^2 > 0, a_2 = 0 \quad \text{(Mono-modal)}$$

$$(M_2): \; a_1 = 0, a_2^2 > 0 \quad \text{(Mono-modal)}$$

$$(B_{1,2}): \; a_1^2 > 0, \; a_2^2 > 0 \quad \text{(Bi-modal)}$$

Solutions (M_1), (M_2),(B) exist only in a sector of the (μ,ν)-parameter plane. In some sectors more solution can be in competition.

➢Example: