4. RESONANT DOUBLE-HOPF BIFURCATIONS OF 1:1 AND 1:3 TYPE

- *Internal resonance* occurs when the critical frequencies are *rationally linearly dependent*. E.g., two frequencies are internally resonant if $k_1\omega_1 + k_1\omega_2 = 0$, with $k_i \in \mathbb{Z}$, i.e. $\omega_2 = r\omega_1, r \in \mathbb{Q}$.

- The frequencies are *nearly-resonant* when $k_1\omega_1 + k_1\omega_2 = O(\varepsilon)$. In this case one has to introduce a *small mistuning as a further control parameter*.

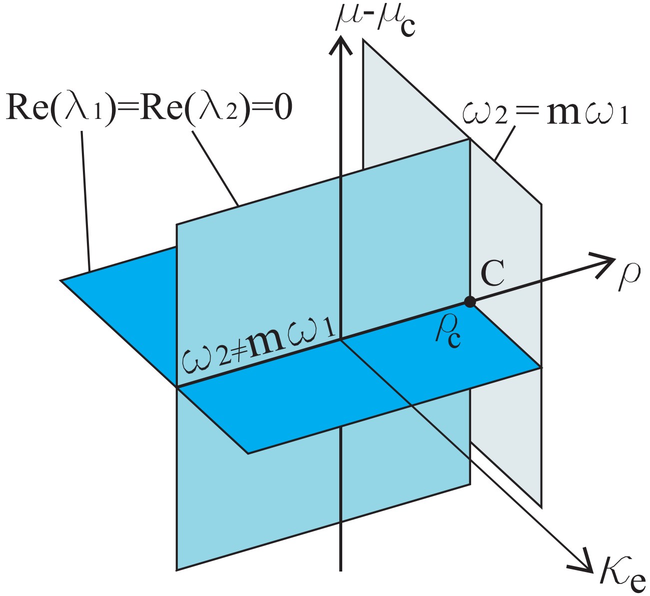
EXAMPLE: RAYLEIGH-DUFFING COUPLED OSCILLATORS IN 1:1 OR 1:3 INTERNAL RESONANCE

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega_1^2 x + b_1\dot{x}^3 + cx^3 - b_0(\dot{y} - \dot{x})^3 = 0 \\ \ddot{y} - \nu\dot{y} + \omega_2^2 y + b_2\dot{y}^3 + cy^3 + b_0(\dot{y} - \dot{x})^3 = 0 \end{cases}$$

where:

$$\omega_2 := \hat{\omega}_2 + \varepsilon\sigma, \quad \hat{\omega}_2 := r\omega_1, \quad \sigma = \mathrm{O}(1)$$

in which the *detuning* σ is the *third bifurcation parameter*.



Parameter space and bifurcation loci

- **Note:** The *internal resonance has no effects on the linear stability of an equilibrium point*, but it affects the nonlinear behavior.

- Perturbation equations:

By following the same steps of the non-resonant case, we get:

$$\varepsilon^0 : \begin{cases} d_0^2\, x_0 + \omega_1^2 x_0 = 0 \\ d_0^2\, y_0 + \hat{\omega}_2^2\, y_0 = 0 \end{cases}$$

$$\varepsilon^1 : \begin{cases} d_0^2\, x_1 + \omega_1^2 x_1 = -2\, d_0 d_1 x_0 + \mu\, d_0 x_0 - b_1 (d_0 x_0)^3 - c x_0^3 + b_0 (d_0 y_0 - d_0 x_0)^3 \\ d_0^2\, y_1 + \hat{\omega}_2^2\, y_1 = -2\, d_0 d_1 y_0 + \nu\, d_0 y_0 - b_2 (d_0 y_0)^3 - c y_0^3 - b_0 (d_0 y_0 - d_0 x_0)^3 \\ \qquad\qquad -2 \hat{\omega}_2 \sigma\, y_0 \end{cases}$$

- Generating solution:

$$\begin{cases} x_0 = A_1 \left(t_1, t_2, \ldots \right) e^{i \omega_1 t_0} + c.c. \\ y_0 = A_2 \left(t_1, t_2, \ldots \right) e^{i \hat{\omega}_2 t_0} + c.c. \end{cases}$$

- ε -order:
  - equations:

    The harmonics $(\omega_1, \hat{\omega}_2; 3\omega_1, 3\hat{\omega}_2, \hat{\omega}_2 \pm 2\omega_1, 2\hat{\omega}_2 \pm \omega_1)$ arise:

$$
\begin{cases}
\mathrm{d}_0^2\, x_1 + \omega_1^2 x_1 = f_{1,1}\,\mathrm{e}^{i\omega_1 t_0} + f_{1,2}\,\mathrm{e}^{i\hat{\omega}_2 t_0} + f_{1,30}\,\mathrm{e}^{3i\omega_1 t_0} + f_{1,03}\,\mathrm{e}^{3i\hat{\omega}_2 t_0} \\
\qquad + f_{1,21}\,\mathrm{e}^{i(2\omega_1+\hat{\omega}_2)t_0} + f_{1,\bar{2}1}\,\mathrm{e}^{i(\hat{\omega}_2-2\omega_1)t_0} \\
\qquad + f_{1,12}\,\mathrm{e}^{i(2\hat{\omega}_2+\omega_1)t_0} + f_{1,\bar{1}2}\,\mathrm{e}^{i(2\hat{\omega}_2-\omega_1)t_0} + c.c. \\
\mathrm{d}_0^2\, y_1 + \hat{\omega}_2^2 y_1 = f_{2,1}\,\mathrm{e}^{i\omega_1 t_0} + f_{2,2}\,\mathrm{e}^{i\hat{\omega}_2 t_0} + f_{2,30}\,\mathrm{e}^{3i\omega_1 t_0} + f_{2,03}\,\mathrm{e}^{3i\hat{\omega}_2 t_0} \\
\qquad + f_{2,21}\,\mathrm{e}^{i(2\omega_1+\hat{\omega}_2)t_0} + f_{2,\bar{2}1}\,\mathrm{e}^{i(\hat{\omega}_2-2\omega_1)t_0} \\
\qquad + f_{2,12}\,\mathrm{e}^{i(2\hat{\omega}_2+\omega_1)t_0} + f_{2,\bar{1}2}\,\mathrm{e}^{i(2\hat{\omega}_2-\omega_1)t_0} + c.c.
\end{cases}
$$

    where:

$$f_{1,1} := -2i\omega_1 \mathrm{d}_1 A_1 + i\omega_1 \mu A_1 - 3[c + i\omega_1^3 (b_0 + b_1)] A_1^2 \bar{A}_1 - 6ib_0 \omega_1 \omega_2^2 A_1 A_2 \bar{A}_2$$

$$f_{2,2} := -2i\omega_2 \mathrm{d}_1 A_2 + \omega_2 (i\nu - 2\sigma) A_2 - 3[c + i\omega_2^3 (b_0 + b_2)] A_2^2 \bar{A}_2 - 6ib_0 \omega_1^2 \omega_2 A_1 \bar{A}_1 A_2$$

$$f_{1,2} := 6ib_0 \omega_1^2 \omega_2 A_1 A_2 \bar{A}_1 + 3ib_0 \omega_2^3 A_2^2 \bar{A}_2, \quad f_{2,1} := 3ib_0 \omega_1^3 A_1^2 \bar{A}_1 + 6ib_0 \omega_1 \omega_2^2 A_1 A_2 \bar{A}_2$$

$$f_{1,30} := [-c + i\omega_1^3 (b_0 + b_1)] A_1^3, \quad f_{2,30} := -ib_0 \omega_1^3 A_1^3$$

$$f_{1,03} := -ib_0 \omega_2^3 A_2^3, \quad f_{2,03} := [-c + i\omega_2^3 (b_0 + b_2)] A_2^3$$

$$f_{1,21} = -f_{2,21} := -3ib_0 \omega_1^2 \omega_2 A_1^2 A_2, \quad f_{1,\bar{2}1} = -f_{2,\bar{2}1} := -3ib_0 \omega_1^2 \omega_2 \bar{A}_1^2 A_2$$

$$f_{1,12} = -f_{2,12} := 3ib_0 \omega_1 \omega_2^2 A_1 A_2^2, \quad f_{1,\bar{1}2} = -f_{2,\bar{1}2} := -3ib_0 \omega_1 \omega_2^2 \bar{A}_1 A_2^2$$

- Zeroing secular terms:

In a first-order analysis it does not need to compute all the f-coefficients, but only the resonant ones. By inspection:

$$\begin{cases} f_{1,1} + \delta_{r1}(f_{1,2} + f_{1,2\bar{1}} + f_{1,\bar{1}2}) + \delta_{r3} f_{1,\bar{2}1} = 0 \\ f_{2,2} + \delta_{r1}(f_{2,1} + f_{2,2\bar{1}} + f_{2,\bar{1}2}) + \delta_{r3} f_{2,30} = 0 \end{cases}$$

where δ_{rk} is the Kronecker symbol ($\delta_{rk} = 1$ if $r = k$, $\delta_{rk} = 0$ if $r \neq k$).

■ **The r=1 case**

The complex AME read:

$$\begin{cases} \mathrm{d}_1 A_1 = \dfrac{1}{2}\mu A_1 + \dfrac{3}{2}[i\dfrac{c}{\omega_1} - (b_1 + b_0)\omega_1^2]A_1^2\overline{A}_1 - 3b_0\omega_1^2 A_1 A_2 \overline{A}_2 \\ \qquad +3b_0\omega_1^2 A_1\overline{A}_1 A_2 + \dfrac{3}{2}b_0\omega_1^2 A_1^2\overline{A}_2 - \dfrac{3}{2}b_0\omega_1^2\overline{A}_1 A_2^2 + \dfrac{3}{2}b_0\omega_1^2 A_2^2\overline{A}_2 \\ \mathrm{d}_1 A_2 = (\dfrac{1}{2}\nu + i\sigma)A_2 + \dfrac{3}{2}[i\dfrac{c}{\omega_1} - (b_2 + b_0)\omega_1^2]A_2^2\overline{A}_2 - 3b_0\omega_1^2 A_1\overline{A}_1 A_2 \\ \qquad +3b_0\omega_1^2 A_1 A_2\overline{A}_2 + \dfrac{3}{2}b_0\omega_1^2 A_1^2\overline{A}_1 - \dfrac{3}{2}b_0\omega_1^2 A_1^2\overline{A}_2 + \dfrac{3}{2}b_0\omega_1^2\overline{A}_1 A_2^2 \end{cases}$$

in which $\hat{\omega}_2 = \omega_1$ has been considered. By absorbing the parameter ε, using the polar representation and separating the real and imaginary parts, four real bifurcation equations follow:

$$
\begin{cases}
\dot{a}_1 = \frac{1}{2}\mu a_1 - \frac{3}{8}(b_0 + b_1)\omega_1^2 a_1^3 - \frac{3}{8} b_0 \omega_1^2 [2 + \cos(2\theta_1 - 2\theta_2)] a_1 a_2^2 \\
\qquad + \frac{9}{8} b_0 \omega_1^2 a_1^2 a_2 \cos(\theta_1 - \theta_2) + \frac{3}{8} b_0 \omega_1^2 a_2^3 \cos(\theta_1 - \theta_2) \\
\dot{a}_2 = \frac{1}{2}\nu a_2 - \frac{3}{8}(b_0 + b_2)\omega_1^2 a_2^3 - \frac{3}{8} b_0 \omega_1^2 [2 + \cos(2\theta_1 - 2\theta_2)] a_1^2 a_2 \\
\qquad + \frac{3}{8} b_0 \omega_1^2 a_1^3 \cos(\theta_1 - \theta_2) + \frac{9}{8} b_0 \omega_1^2 a_1 a_2^2 \cos(\theta_1 - \theta_2) \\
a_1 \dot{\theta}_1 = \frac{3}{8}\frac{c}{\omega_1} a_1^3 + \frac{3}{8} b_0 \omega_1^2 a_1^2 a_2 \sin(\theta_1 - \theta_2) + \frac{3}{8} b_0 \omega_1^2 a_2^3 \sin(\theta_1 - \theta_2) \\
\qquad - \frac{3}{8} b_0 \omega_1^2 a_1 a_2^2 \sin(2\theta_1 - 2\theta_2) \\
a_2 \dot{\theta}_2 = \sigma a_2 + \frac{3}{8}\frac{c}{\omega_1} a_2^3 + \frac{3}{8} b_0 \omega_1^2 a_1^3 \sin(\theta_1 - \theta_2) + \frac{3}{8} b_0 \omega_1^2 a_1 a_2^2 \sin(\theta_1 - \theta_2) \\
\qquad - \frac{3}{8} b_0 \omega_1^2 a_1^2 a_2 \sin(2\theta_1 - 2\theta_2)
\end{cases}
$$

- **Note:** *the real-amplitude equations are coupled with the phase-equations .*

Since phases appear as a linear combination, we introduce a *phase-combination*:

$$\gamma := \theta_1 - \theta_2$$

and recombine the phase-equations according $\dot{\gamma} = \dot{\theta}_1 - \dot{\theta}_2$. We obtain:

➢ *three* RAME in the state-variables $\{a_1, a_2, \gamma\}$:

$$\begin{cases} \dot{a}_1 = \frac{1}{2}\mu a_1 - \frac{3}{8}(b_0 + b_1)\omega_1^2 a_1^3 - \frac{3}{8} b_0 \omega_1^2 [2 + \cos 2\gamma] a_1 a_2^2 \\ \quad + \frac{9}{8} b_0 \omega_1^2 a_1^2 a_2 \cos\gamma + \frac{3}{8} b_0 \omega_1^2 a_2^3 \cos\gamma \\ \dot{a}_2 = \frac{1}{2}\nu a_2 - \frac{3}{8}(b_0 + b_2)\omega_1^2 a_2^3 - \frac{3}{8} b_0 \omega_1^2 [2 + \cos 2\gamma] a_1^2 a_2 \\ \quad + \frac{3}{8} b_0 \omega_1^2 a_1^3 \cos\gamma + \frac{9}{8} b_0 \omega_1^2 a_1 a_2^2 \cos\gamma \\ a_1 a_2 \dot{\gamma} = -\sigma a_1 a_2 + \frac{3}{8}(\frac{c}{\omega_1} + b_0 \omega_1^2 \sin 2\gamma) a_1^3 a_2 + \frac{3}{8}(b_0 \omega_1^2 \sin 2\gamma - \frac{c}{\omega_1}) a_1 a_2^3 \\ \quad - \frac{3}{8} b_0 \omega_1^2 a_1^4 \sin\gamma - \frac{3}{4} b_0 \omega_1^2 a_1^2 a_2^2 \sin\gamma - \frac{3}{8} b_0 \omega_1^2 a_2^4 \sin\gamma \end{cases}$$

➢two phase-equations:

$$\begin{cases} a_1\dot{\theta}_1 = \frac{3}{8}\frac{c}{\omega_1}a_1^3 + \frac{3}{8}b_0\omega_1^2 a_1^2 a_2 \sin\gamma + \frac{3}{8}b_0\omega_1^2 a_2^3 \sin\gamma - \frac{3}{8}b_0\omega_1^2 a_1 a_2^2 \sin 2\gamma \\ a_2\dot{\theta}_2 = \sigma a_2 + \frac{3}{8}\frac{c}{\omega_1}a_2^3 + \frac{3}{8}b_0\omega_1^2 a_1^3 \sin\gamma + \frac{3}{8}b_0\omega_1^2 a_1 a_2^2 \sin\gamma - \frac{3}{8}b_0\omega_1^2 a_1^2 a_2 \sin 2\gamma \end{cases}$$

Once the RAME have been solved, the phase-equations can be integrated by quadrature.

- **Note:** while the RAME of a *non-resonant* system are pure-amplitude equations, those of a *resonant* system are mixed-amplitude-phase equations.

■ **The r=3 case**

In a similar way, the complex AME are found to be:

$$\begin{cases} \mathrm{d}_1 A_1 = \dfrac{1}{2}\mu A_1 + \dfrac{3}{2}[i\dfrac{c}{\omega_1} - (b_1 + b_0)\omega_1^2]A_1^2\overline{A}_1 - 27b_0\omega_1^2 A_1 A_2 \overline{A}_2 - \dfrac{9}{2}b_0\omega_1^2\overline{A}_1^2 A_2 \\ \mathrm{d}_1 A_2 = (\dfrac{1}{2}\nu + i\sigma)A_2 + \dfrac{1}{2}[i\dfrac{c}{\omega_1} - 27(b_2 + b_0)\omega_1^2]A_2^2\overline{A}_2 - 3b_0\omega_1^2 A_1 A_2 \overline{A}_1 - \dfrac{1}{6}b_0\omega_1^2 A_1^3 \end{cases}$$

in which $\hat{\omega}_2 = 3\omega_1$ has been substituted.

After parameter reabsorbing, and use of the polar representation, we obtain four real bifurcation equations:

$$\begin{cases} \dot{a}_1 = \frac{1}{2}\mu a_1 - \frac{3}{8}(b_0 + b_1)\omega_1^2 a_1^3 - \frac{27}{4} b_0 \omega_1^2 a_1 a_2^2 - \frac{9}{8} b_0 \omega_1^2 a_1^2 a_2 \cos(3\theta_1 - \theta_2) \\ \dot{a}_2 = \frac{1}{2}\nu a_2 - \frac{27}{8}(b_0 + b_2)\omega_1^2 a_2^3 - \frac{3}{4} b_0 \omega_1^2 a_1^2 a_2 - \frac{1}{24} b_0 \omega_1^2 a_1^3 \cos(3\theta_1 - \theta_2) \\ a_1\dot{\theta}_1 = \frac{3}{8}\frac{c}{\omega_1} a_1^3 + \frac{9}{8} b_0 \omega_1^2 a_1^2 a_2 \sin(3\theta_1 - \theta_2)] \\ a_2\dot{\theta}_2 = \sigma a_2 + \frac{1}{8}\frac{c}{\omega_1} a_2^3 - \frac{1}{24} b_0 \omega_1^2 a_1^3 \sin(3\theta_1 - \theta_2)] \end{cases}$$

They suggest the following definition for the phase-combination:

$$\gamma := 3\theta_1 - \theta_2$$

➢ The RAME are:

$$\begin{cases} \dot{a}_1 = \frac{1}{2}\mu a_1 - \frac{3}{8}(b_0 + b_1)\omega_1^2 a_1^3 - \frac{27}{4} b_0 \omega_1^2 a_1 a_2^2 - \frac{9}{8} b_0 \omega_1^2 a_1^2 a_2 \cos\gamma \\ \dot{a}_2 = \frac{1}{2}\nu a_2 - \frac{27}{8}(b_0 + b_2)\omega_1^2 a_2^3 - \frac{3}{4} b_0 \omega_1^2 a_1^2 a_2 + \frac{1}{24}(b_0 + b_1)\omega_1^2 a_1^3 \cos\gamma \\ a_1 a_2 \dot{\gamma} = -\sigma a_1 a_2 + \frac{9}{8}\frac{c}{\omega_1} a_1^3 a_2 - \frac{1}{8}\frac{c}{\omega_1} a_1 a_2^3 + \frac{1}{24} b_0 \omega_1^2 a_1^4 \sin\gamma + \frac{27}{8} b_0 \omega_1^2 a_1^2 a_2^2 \sin\gamma \end{cases}$$

➢ The phase-equations are:

$$\begin{cases} a_1 \dot{\theta}_1 = \frac{3}{8}\frac{c}{\omega_1} a_1^3 + \frac{9}{8} b_0 \omega_1^2 a_1^2 a_2 \sin\gamma \\ a_2 \dot{\theta}_2 = \sigma a_2 + \frac{1}{8}\frac{c}{\omega_1} a_2^3 - \frac{1}{24} b_0 \omega_1^2 a_1^3 \sin\gamma \end{cases}$$

- Response (r =1,3 cases)

After integration, the RAME furnish $a_1(t), a_2(t), \gamma(t)$; successively, the phase equations give $\theta_1(t), \theta_2(t)$. The response read:

$$\begin{cases} x = a_1(t)\cos(\Phi_1(t)) + h.o.t. \\ y = a_2(t)\cos(\Phi_2(t)) + h.o.t. \end{cases}$$

where:

$$\Phi_1(t) := \omega_1 t + \theta_1(t), \quad \Phi_2(t) := \hat{\omega}_2 t + \theta_2(t)$$

are total phases.

■ Steady-state solutions and fixed points of RAME

- The RAME, are of the following type:

$$\begin{cases} \dot{a}_1 = F_1(a_1, a_2, \gamma) \\ \dot{a}_2 = F_2(a_1, a_2, \gamma) \\ a_1 a_2 \dot{\gamma} = G(a_1, a_2, \gamma) \end{cases}$$

and phase-equations are of the type:

$$\begin{cases} a_1 \dot{\theta}_1 = H_1(a_1, a_2, \gamma) \\ a_2 \dot{\theta}_2 = H_2(a_1, a_2, \gamma) \end{cases}$$

- ❑ **Note:** The RAME can be put in the standard form $\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z})$, with $\mathbf{z} := (a_1, a_2, \gamma)$, if and only if $a_1 \neq 0$, $a_2 \neq 0$ (complete solutions).
- ❑ **Note:** in incomplete solutions ($a_1 = 0$, and/or $a_2 = 0$), the phases of the zero-amplitudes remains undetermined; however, they are inessential.

- The fixed points $(a_{1s}, a_{2s}, \gamma_s) = \text{const}$ of RAME are solutions of:

$$\begin{cases} F_1(a_{1s}, a_{2s}, \gamma_s) = 0 \\ F_2(a_{1s}, a_{2s}, \gamma_s) = 0 \\ G(a_{1s}, a_{2s}, \gamma_s) = 0 \end{cases}$$

Consequently, the associated phases (if determined) are *linearly varying* functions:

$$\theta_{1s}(t) = \kappa_{1s} t + \theta_{1s}^0, \quad \theta_{2s}(t) = \kappa_{2s} t + \theta_{2s}^0$$

with $(\kappa_{1s}, \kappa_{2s}) = \text{const}$ the *frequency corrections.*

- For a complete solution, we prove that *the* (non-trivial) *fixed points of the RAME are periodic motions for the original system* (for incomplete solution, this is a trivial matter). Indeed:
    - a constant phase-difference:

      $$\gamma_s := r\theta_{1s} - \theta_{2s} = r(\kappa_{1s}t + \theta_{1s}^0) - (\kappa_{2s}t + \theta_{2s}^0) = \text{const} \qquad r = 1, 3$$

      entails a relation between frequency corrections and initial phases:

      $$r\kappa_{1s} - \kappa_{2s} = 0, \quad r\theta_{1s}^0 - \theta_{2s}^0 = \gamma_s$$

    - consequently, since $\hat{\omega}_2 = r\omega_1$, the total phases read:

      $$\Phi_1(t) := \omega_1 t + \theta_{1s}(t) = (\omega_1 + \kappa_{1s})t + \theta_{1s}^0$$

      $$\Phi_2(t) := \hat{\omega}_2 t + \theta_{2s}(t) = (\hat{\omega}_2 + \kappa_{2s})t + \theta_{2s}^0 = [r(\omega_1 + \kappa_{1s})t + \theta_{2s}^0]$$

      i.e. *the nonlinear frequencies Ω_k are in the same integer ratio r as the linear frequencies ω_k:*

      $$\Omega_{1s} := \omega_1 + \kappa_{1s}, \quad \Omega_2 := \hat{\omega}_2 + \kappa_{2s} = r\Omega_{1s}$$

The steady response, therefore, is periodic, and it reads:

$$\begin{cases} x = a_1(t)\cos(\Omega_{1s}t + \theta_{1s}^0) + h.o.t. \\ y = a_2(t)\cos[r(\Omega_{1s}t + \theta_{1s}^0) - \gamma_s] + h.o.t. \end{cases}$$

- **Note:** the phase difference γ_s is given by the solution; however, an initial phase, e.g. θ_{1s}^0 remains undetermined, since the limit cycle can be traveled starting from any of its points.

■ Finding the fixed points of RAME

- In the r=1 case, the RAME admit:

  - (T) the trivial solution:

$$a_{1T} = a_{2T} = 0, \quad \forall \gamma_T, \quad \forall(\mu, \nu, \sigma)$$

with the phase-difference γ being undetermined.

  - (P) a number of *bimodal* (or complete) periodic solutions:

$$a_{1P} = a_{1P}(\mu, \nu, \sigma), \quad a_{2P} = a_{2P}(\mu, \nu, \sigma), \quad \gamma_P = \gamma_P(\mu, \nu, \sigma)$$

with associated, determined phases θ_{1P} and θ_{2P}.

- In the r=3 case the RAME admit:

  - (T) the trivial solution:

$$a_{1T} = a_{2T} = 0, \quad \forall \gamma_T, \quad \forall(\mu, \nu, \sigma)$$

  - (M) a *mono-modal* (incomplete) periodic solution:

$$a_{1M} = 0, \quad a_{2M} = a_{2M}(\nu), \quad \forall \gamma_M$$

  with:

$$\theta_{2M} = \theta_{2M}(\sigma, \nu), \quad \forall \theta_{1M}$$

  - (P) one or more *bimodal* (complete) periodic solutions:

$$a_{1P} = a_{1P}(\mu, \nu, \sigma), \quad a_{2P} = a_{2P}(\mu, \nu, \sigma), \quad \gamma_P = \gamma_P(\mu, \nu, \sigma)$$

  with associated phases θ_{1P} and θ_{2P}.

■ Stability of steady solutions

It needs to distinguish:

- *The steady-solution is complete* (s=P): since all quantities are determined, and the RAME can be put in the normal form $\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z})$, with $\mathbf{z} := (a_1, a_2, \gamma)$, stability is governed by the variational equation:

$$\delta\dot{\mathbf{z}} = \mathbf{J}_P \delta\mathbf{z}$$

  A zero eigenvalue of $\mathbf{J}_P$ denotes a branching of a new *periodic* solution; a pair of purely imaginary eigenvalues denotes a branching of a *quasi-periodic* solution (i.e. a periodically modulated periodic motion).

- *The steady-solution is incomplete* (s=T,M): since γ_s is undetermined, and the RAME are *not* in standard form, use of the (not reduced) AME must be made. Examples are given below.

- Stability of the trivial solution (r=1,3 cases)

The variation of the AME, based on $A_{1T} = A_{2T} = 0$, reads:

$$\begin{cases} \delta \dot{A}_1 = \frac{1}{2}\mu \delta A_1 \\ \delta \dot{A}_2 = (\frac{1}{2}\nu + i\sigma)\delta A_2 \end{cases}$$

whose solution is:

$$\delta A_1 = \delta \hat{A}_1 \exp(\frac{1}{2}\mu t), \quad \delta A_2 = \delta \hat{A}_2 \exp[(\frac{1}{2}\nu + i\sigma)t]$$

with $\delta \hat{A}_1, \delta \hat{A}_2$ constants. The trivial solution is therefore stable when $\mu < 0, \nu < 0$.

- Stability of the mono-modal solution (r=3 case)

  - The variation of the AME, based on:

$$A_{1M}=0, \quad A_{2M}=A_{2M}:=\frac{1}{2}a_{2M}\exp[i(\kappa_{2M}t+\theta_{2M}^{0})]$$

  assumes the following (uncoupled) form:

$$\begin{cases} \delta\dot{A}_1=(R_1+R_2\frac{1}{4}a_{2M}^2)\delta A_1 \\ \delta\dot{A}_2=(C_1+C_2\frac{1}{4}a_{2M}^2)\delta A_2+C_3\frac{1}{4}a_{2M}^2\exp[2i(\kappa_{2M}t+\theta_{2M}^{0})]\delta\bar{A}_2 \end{cases}$$

  where $R_j\in\mathbb{R}, C_j:=R_j+iI_j\in\mathbb{C}$ are coefficients.

- **Note:** due to the presence of the frequency correction κ_{2M}, $A_{2M}\neq\text{const}$; consequently, the second variational equation depends on time.

- *To render the second equation autonomous*, a change of variable is performed:

$$\delta A_2 = \delta B_2 \exp[i(\alpha t + \beta)]$$

with α, β to be determined. By requiring the coefficients are independent of time, it follows: $\alpha = \kappa_{2M}$; moreover $\beta = \theta_{2M}^0$ is taken for simplicity.

- In the new variables, the variational equations read:

$$\begin{cases} \delta \dot{A}_1 = (R_1 + R_2 \frac{1}{4} a_{2M}^2) \delta A_1 \\ \delta \dot{B}_2 = (C_1 + C_2 \frac{1}{4} a_{2M}^2 - i\kappa_{2M}) \delta B_2 + C_3 \frac{1}{4} a_{2M}^2 \delta \bar{B}_2 \end{cases}$$

- Since the equations are linear, a Cartesian representation is better suited:

$$\delta A_1 = p_1 + iq_1, \quad \delta B_2 = p_2 + iq_2$$

leading to four real variational equations:

$$\begin{pmatrix} \dot{p}_1 \\ \dot{q}_1 \\ \dot{p}_2 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} J_{11} & 0 & & \\ 0 & J_{22} & & \\ & & J_{33} & J_{34} \\ & & J_{43} & J_{44} \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{pmatrix}$$

where:

$$J_{11} = J_{22} := R_1 + \frac{R_2}{4} a_{2M}^2$$

$$J_{33} = R_1 + (R_2 + R_3)\frac{a_{2M}^2}{4}, \quad J_{34} = -I_1 + (-I_2 + I_3)\frac{a_{2M}^2}{4} + \kappa_{2M}$$

$$J_{33} = I_1 + (I_2 + I_3)\frac{a_{2M}^2}{4} - \kappa_{2M}, \quad J_{34} = R_1 + (R_2 - R_3)\frac{a_{2M}^2}{4} \quad ,$$

The eigenvalues (four real, or two real and two complex conjugate), govern the stability of the *M*-solution.

5. THE 1:2 RESONANT DOUBLE-HOPF BIFURCATION

To study the 1:2 resonant case we again consider the system of the previous section, but modify the degree of the coupling term from 3 to 2, in order that the resonance manifests itself at lower order.

EXAMPLE: RAYLEIGH-DUFFING OSCILLATORS WITH QUADRATIC COUPLING

$$\begin{cases} \ddot{x} - \mu \dot{x} + \omega_1^2 x + b_1 \dot{x}^3 + c x^3 - b_0 (\dot{y} - \dot{x})^2 = 0 \\ \ddot{y} - \nu \dot{y} + \omega_2^2 y + b_2 \dot{y}^3 + c y^3 + b_0 (\dot{y} - \dot{x})^2 = 0 \end{cases}$$

- Rescaling

$$(\mu, \nu) \to (\varepsilon\mu, \varepsilon\nu), \qquad (x, y) \to (\varepsilon x, \varepsilon y)$$

from which:

$$\begin{cases} \ddot{x}+\omega_1^2 x-\varepsilon[\mu\dot{x}+b_0(\dot{y}-\dot{x})^2]+\varepsilon^2(b_1\dot{x}^3+cx^3)=0 \\ \ddot{y}+\omega_2^2 y-\varepsilon[\nu\dot{y}-b_0(\dot{y}-\dot{x})^2]+\varepsilon^2(b_2\dot{x}^3+cx^3)=0 \end{cases}$$

- Detuning:

$$\omega_2 := 2\omega_1+\varepsilon\sigma, \quad \sigma = \mathrm{O}(1)$$

- Series expansions:

$$\begin{pmatrix} x(t;\varepsilon) \\ y(t;\varepsilon) \end{pmatrix} = \begin{pmatrix} x_0(t_0,t_1,t_2,\cdots) \\ y_0(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon \begin{pmatrix} x_1(t_0,t_1,t_2,\cdots) \\ y_1(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} x_2(t_0,t_1,t_2,\cdots) \\ y_2(t_0,t_1,t_2,\cdots) \end{pmatrix} + \cdots$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{d}_0+\varepsilon\,\mathrm{d}_1+\varepsilon^2\,\mathrm{d}_2+\cdots, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \mathrm{d}_0^2+2\varepsilon\,\mathrm{d}_0\,\mathrm{d}_1+\varepsilon^2\left(\mathrm{d}_1^2+2\,\mathrm{d}_0\,\mathrm{d}_2\right)+\cdots$$

where $t_k := \varepsilon^k t_k$ and $\mathrm{d}_k := \partial/\partial t_k$.

- Perturbation equations:

$$\varepsilon^0 : \begin{cases} d_0^2 x_0 + \omega_1^2 x_0 = 0 \\ d_0^2 y_0 + 4\omega_1^2 y_0 = 0 \end{cases}$$

$$\varepsilon^1 : \begin{cases} d_0^2 x_1 + \omega_1^2 x_1 = -2 d_0 d_1 x_0 + \mu d_0 x_0 + b_0 (d_0 y_0 - d_0 x_0)^2 \\ d_0^2 y_1 + 4\omega_1^2 y_1 = -2 d_0 d_1 y_0 + \nu d_0 y_0 - b_0 (d_0 y_0 - d_0 x_0)^2 - 4\omega_1 \sigma y_0 \end{cases}$$

$$\varepsilon^2 : \begin{cases} d_0^2 x_2 + \omega_1^2 x_2 = -(2 d_0 d_2 x_0 + d_1^2 x_0 + 2 d_0 d_1 x_1) - b_1 (d_0 x_0)^3 - c x_0^3 \\ \qquad +2b_0 (d_0 y_0 - d_0 x_0)(d_0 y_1 + d_1 y_0 - d_0 x_1 - d_1 x_0) + \mu (d_1 x_0 + d_0 x_1) \\ d_0^2 y_2 + 4\omega_1^2 y_2 = -(2 d_0 d_2 y_0 + d_1^2 y_0 + 2 d_0 d_1 y_1) - b_2 (d_0 y_0)^3 - c y_0^3 \\ \qquad -2b_0 (d_0 y_0 - d_0 x_0)(d_0 y_1 + d_1 y_0 - d_0 x_1 - d_1 x_0) + \nu (d_1 y_0 + d_0 y_1) \\ \qquad -4\omega_1 \sigma y_1 - \sigma^2 y_0 \end{cases}$$

………

- Generating solution:

$$\begin{cases} x_0 = A_1\left(t_1, t_2, \ldots\right) e^{i\omega_1 t_0} + c.c. \\ y_0 = A_2\left(t_1, t_2, \ldots\right) e^{2i\omega_1 t_0} + c.c. \end{cases}$$

- ε -order:
  - equations:

    The harmonics $(0, \omega_1, 2\omega_1, 3\omega_1, 4\omega_1)$ arise:

$$\begin{cases} d_0^2\, x_1 + \omega_1^2 x_1 = f_{1,0} + f_{1,1}\, e^{i\omega_1 t_0} + (f_{1,2}\, e^{2i\omega_1 t_0} + f_{1,3}\, e^{3i\omega_1 t_0} + f_{1,4}\, e^{4i\omega_1 t_0} + c.c.) \\ d_0^2\, y_1 + 4\omega_1^2 y_1 = f_{2,0} + f_{2,1}\, e^{i\omega_1 t_0} + (f_{2,2}\, e^{2i\omega_1 t_0} + f_{2,3}\, e^{3i\omega_1 t_0} + f_{2,4}\, e^{4i\omega_1 t_0} + c.c.) \end{cases}$$

where:

$$f_{1,0} = -f_{2,0} := 2b_0\omega_1^2 (A_1\bar{A}_1 + 4A_2\bar{A}_2)$$

$$f_{1,1} := -2i\omega_1\, d_1\, A_1 + i\mu\omega_1 A_1 - 4b_0\omega_1^2 \bar{A}_1 A_2, \quad f_{2,1} := 4b_0\omega_1^2 \bar{A}_1 A_2$$

$$f_{1,2} := -b_0\omega_1^2 A_1^2, \quad f_{2,2} := -4i\omega_1\, d_1\, A_2 + 2\omega_1(i\nu - 2\sigma)A_2 + b_0\omega_1^2 A_1^2$$

$$f_{1,3} = -f_{2,3} := 4b_0\omega_1^2 A_1 A_2, \quad f_{1,4} = -f_{2,4} := -4b_0\omega_1^2 A_2^2$$

- Elimination of secular terms requires $f_{1,1}=0, f_{2,2}=0$, i.e.:

$$\mathrm{d}_1 A_1 = \frac{1}{2}\mu A_1 + 2ib_0\omega_1\bar{A}_1 A_2, \quad \mathrm{d}_1 A_2 = (\frac{1}{2}\nu + i\sigma)A_2 - \frac{1}{4}ib_0\omega_1 A_1^2$$

- Solution:

$$x_1 = 2b_0(A_1\bar{A}_1 + 4A_2\bar{A}_2) + (\frac{1}{3}b_0 A_1^2\,\mathrm{e}^{2i\omega_1 t_0} - \frac{1}{2}b_0 A_1 A_2\,\mathrm{e}^{3i\omega_1 t_0} + \frac{4}{15}b_0 A_2^2\,\mathrm{e}^{4i\omega_1 t_0} + c.c.)$$

$$y_1 = -\frac{1}{2}b_0(A_1\bar{A}_1 + 4A_2\bar{A}_2) + (\frac{4}{3}b_0\bar{A}_1 A_2\,\mathrm{e}^{i\omega_1 t_0} + \frac{4}{5}b_0 A_1 A_2\,\mathrm{e}^{3i\omega_1 t_0} - \frac{1}{3}b_0 A_2^2\,\mathrm{e}^{4i\omega_1 t_0} + c.c.)$$

➢ zeroing of the secular terms:

By zeroing the coefficients of the harmonics ω_1 (in the x_2-equation) and $2\omega_1$ (in the y_2-equation), and accounting for:

$$\mathrm{d}_1^2 A_1 = \frac{1}{2}\mu\,\mathrm{d}_1 A_1 + 2ib_0\omega_1(A_2\,\mathrm{d}_1\bar{A}_1 + \bar{A}_1\,\mathrm{d}_1 A_2)$$

$$= \frac{1}{4}\mu^2 A_1 + b_0\omega_1(2i\mu + i\nu - 2\sigma)\bar{A}_1 A_2 + \frac{1}{2}b_0^2\omega_1^2 A_1^2\bar{A}_1 + 4b_0^2\omega_1^2 A_1 A_2\bar{A}_2$$

$$\mathrm{d}_1^2 A_2 = (\frac{1}{2}\nu + i\sigma)\,\mathrm{d}_1 A_2 - \frac{1}{2}ib_0\omega_1 A_1\,\mathrm{d}_1 A_1$$

$$= (\frac{1}{4}\nu^2 + i\nu\sigma - \sigma^2)A_2 + \frac{1}{4}b_0\omega_1(\sigma - i\mu - \frac{1}{2}i\nu)A_1^2 + b_0^2\omega_1^2 A_1\bar{A}_1 A_2$$

it follows:

$$\mathrm{d}_2 A_1 = -\frac{i\mu^2}{8\omega_1}A_1 - b_0\mu\bar{A}_1A_2 + \left(\frac{3ic}{2\omega_1} - \frac{2}{3}ib_0^2\omega_1 - \frac{3}{2}b_1\omega_1^2\right)A_1^2\bar{A}_1 - \frac{67}{15}ib_0^2\omega_1A_1A_2\bar{A}_2$$

$$\mathrm{d}_2 A_2 = -\frac{i\nu^2}{16\omega_1}A_2 - \frac{1}{32}b_0(6\mu + \nu - 2i\sigma)A_1^2 - \frac{61}{30}ib_0^2\omega_1A_1\bar{A}_1A_2$$
$$+\left(\frac{3ic}{4\omega_1} - \frac{12}{5}ib_0^2\omega_1 - 6b_2\omega_1^2\right)A_2^2\bar{A}_2$$

- Reconstitution:

By using $\dot{A}_k = \varepsilon\,\mathrm{d}_1 A_k + \varepsilon^2\,\mathrm{d}_2 A_k \quad (k = 1, 2)$ and reabsorbing ε (after multiplication of the equations by ε, and use of the inverse transformations $\varepsilon A_k \to A_k$, $\varepsilon(\mu,\nu,\sigma) \to (\mu,\nu,\sigma)$) , the complex bifurcation equations read:

$$
\begin{cases}
\dot{A}_1 = \frac{\mu}{2}(1 - \frac{i\mu}{4\omega_1})A_1 + b_0(2i\omega_1 - \mu)\bar{A}_1 A_2 \\
\quad +(\frac{3ic}{2\omega_1} - \frac{2}{3}ib_0^2\omega_1 - \frac{3}{2}b_1\omega_1^2)A_1^2\bar{A}_1 - \frac{67}{15}ib_0^2\omega_1 A_1 A_2 \bar{A}_2 \\
\dot{A}_2 = (\frac{1}{2}\nu + i\sigma - \frac{i\nu^2}{16\omega_1})A_2 - \frac{1}{32}b_0(8i\omega_1 + 6\mu + \nu - 2i\sigma)A_1^2 \\
\quad -\frac{61}{30}ib_0^2\omega_1 A_1 \bar{A}_1 A_2 + (\frac{3ic}{4\omega_1} - \frac{12}{5}ib_0^2\omega_1 - 6b_2\omega_1^2)A_2^2\bar{A}_2
\end{cases}
$$

Using the polar representation for the amplitudes and introducing the *phase-combination*:

$$\gamma := 2\theta_1 - \theta_2$$

one obtains:

➢three RAME:

$$
\begin{cases}
\dot{a}_1 = \frac{1}{2}\mu a_1 + b_0(\omega_1 \sin\gamma - \frac{\mu}{2}\cos\gamma)a_1 a_2 - \frac{3}{8}b_1\omega_1^2 a_1^3 \\
\dot{a}_2 = \frac{1}{2}\nu a_2 + \frac{1}{8}b_0[(\omega_1 - \frac{\sigma}{4})\sin\gamma - (\frac{3}{4}\mu + \frac{\nu}{8})\cos\gamma]a_1^2 - \frac{3}{2}b_2\omega_1^2 a_2^3 \\
a_1 a_2 \dot{\gamma} = (-\sigma - \frac{\mu^2}{4\omega_1} + \frac{\nu^2}{16\omega_1})a_1 a_2 + b_0(2\omega_1\cos\gamma + \mu\sin\gamma)a_1 a_2^2 + \\
\quad + \frac{1}{8}b_0[(\omega_1 - \frac{\sigma}{4})\cos\gamma + (\frac{3}{4}\mu + \frac{\nu}{8})\sin\gamma]a_1^3 \\
\quad - (\frac{3c}{16\omega_1} + \frac{49}{30}b_0^2\omega_1)a_1 a_2^3 + (\frac{3c}{4\omega_1} + \frac{7}{40}b_0^2\omega_1)a_1^3 a_2
\end{cases}
$$

- two phase-modulation equations:

$$
\begin{cases}
a_1\dot{\theta}_1 = -\dfrac{\mu^2}{8\omega_1}a_1 + b_0(\omega_1\cos\gamma + \dfrac{1}{2}\mu\sin\gamma)a_1a_2 \\
\qquad -\dfrac{67}{60}b_0^2\omega_1 a_1 a_2^2 + (\dfrac{3c}{8\omega_1} - \dfrac{1}{6}b_0^2\omega_1)a_1^3 \\
a_2\dot{\theta}_2 = (\sigma - \dfrac{\nu^2}{16\omega_1})a_2 + \dfrac{1}{8}b_0[(\dfrac{\sigma}{4} - \omega_1)\cos\gamma - (\dfrac{3}{4}\mu + \dfrac{\nu}{8})\sin\gamma]a_1^2 \\
\qquad -\dfrac{61}{120}b_0^2\omega_1 a_1^2 a_2 - \dfrac{3}{5}b_0^2\omega_1 a_2^3
\end{cases}
$$

■ Steady-state solutions

- The RAME admit:

  - (T) the trivial solution:

$$a_{1T} = a_{2T} = 0, \quad \forall \gamma_T, \quad \forall(\mu, \nu, \sigma)$$

  - (M) a *mono-modal* (incomplete) periodic solution:

$$a_{1M} = 0, \quad a_{2M} = a_{2M}(\nu), \quad \forall \gamma_M$$

with:

$$\theta_{2M} = \theta_{2M}(\sigma, \nu), \quad \forall \theta_{1M}$$

  - (P) one or more *bimodal* (complete) periodic solutions:

$$a_{1P} = a_{1P}(\mu, \nu, \sigma), \quad a_{2P} = a_{2P}(\mu, \nu, \sigma), \quad \gamma_P = \gamma_P(\mu, \nu, \sigma)$$

with associated phases θ_{1P} and θ_{2P}.

■ **Stability of steady solutions**

- Stability of the periodic solution (s=P):

$$\delta \dot{\mathbf{z}} = \mathbf{J}_P \delta \mathbf{z}$$

- Stability of the trivial solution (s=T):

$$\begin{cases} \delta \dot{A}_1 = \dfrac{\mu}{2}(1 - \dfrac{i\mu}{4\omega_1})\delta A_1 \\ \delta \dot{A}_2 = (\dfrac{1}{2}\nu + i\sigma - \dfrac{i\nu^2}{16\omega_1})\delta A_2 \end{cases}$$

from wich:

$$\delta A_1 = \delta \hat{A}_1 \exp[\frac{\mu}{2}(1 - \frac{i\mu}{4\omega_1})t], \quad \delta A_2 = \delta \hat{A}_2 \exp[(\frac{1}{2}\nu + i\sigma - \frac{i\nu^2}{16\omega_1})t]$$

The trivial solution is stable when $\mu < 0, \nu < 0$.

- Stability of the mono-modal solution (s=M)

  - By accounting for:

$$A_{1M} = 0, \quad A_{2M} = A_{2M} := \frac{1}{2} a_{2M} \exp[i(\kappa_{2M} t + \theta_{2M}^0)]$$

the (uncoupled) variational equations read:

$$\begin{cases} \delta \dot{A}_1 = (C_1 + I_1 \frac{1}{4} a_{2M}^2) \delta A_1 + C_2 \frac{1}{2} a_{2M} \exp[i(\kappa_{2M} t + \theta_{2M}^0)] \delta \bar{A}_1 \\ \delta \dot{A}_2 = (C_3 + C_4 \frac{1}{4} a_{2M}^2) \delta A_2 + C_5 \frac{1}{4} a_{2M}^2 \exp[2i(\kappa_{2M} t + \theta_{2M}^0)] \delta \bar{A}_2 \end{cases}$$

  - By introducing the change of variable:

$$\delta A_1 = \delta B_1 \exp[i(\alpha_1 t + \beta_1)], \quad \delta A_2 = \delta B_2 \exp[i(\alpha_2 t + \beta_2)]$$

and requiring the coefficients are independent of time, it follows:

$$\alpha_1 = \frac{1}{2} \kappa_{2M}, \quad \alpha_2 = \kappa_{2M}; \quad \beta_1 = \frac{1}{2} \theta_{2M}^0, \quad \beta_2 = \theta_{2M}^0$$

- In the new variables, the variational equations become:

$$\begin{cases} \delta\dot{B}_1 = (C_1 + I_1 \frac{1}{4} a_{2M}^2 - \frac{1}{2} i\kappa_{2M})\delta B_1 + C_2 \frac{1}{2} a_{2M} \delta\overline{B}_1 \\ \delta\dot{B}_2 = (C_3 + C_4 \frac{1}{4} a_{2M}^2 - i\kappa_{2M})\delta B_2 + C_5 \frac{1}{4} a_{2M}^2 \delta\overline{B}_2 \end{cases}$$

- By letting:

$$\delta A_1 = p_1 + iq_1, \quad \delta B_2 = p_2 + iq_2$$

they assume the form:

$$\begin{pmatrix} \dot{p}_1 \\ \dot{q}_1 \\ \dot{p}_2 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} & & \\ J_{21} & J_{22} & & \\ & & J_{33} & J_{34} \\ & & J_{43} & J_{44} \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{pmatrix}$$

The eigenvalues of **J** decide on stability.