6. DEFECTIVE BIFURCATIONS: THE DOUBLE-ZERO CASE

- If the Jacobian matrix is *not-diagonalizable*, an incomplete set of critical eigenvectors exist. The bifurcation is said to be *defective*.

- For example: if $\lambda = 0$ is a double root, just *one* real eigenvector exists; if $\lambda = \pm i\omega$ is a double root, just *one* pair of complex conjugate eigenvectors exists.

- The basis for the state-space must be completed by *generalized eigenvectors*.

- Defective bifurcations require using special multiple scale algorithms, in which *fractional power expansions* of both state-variables and time-scales must be used.

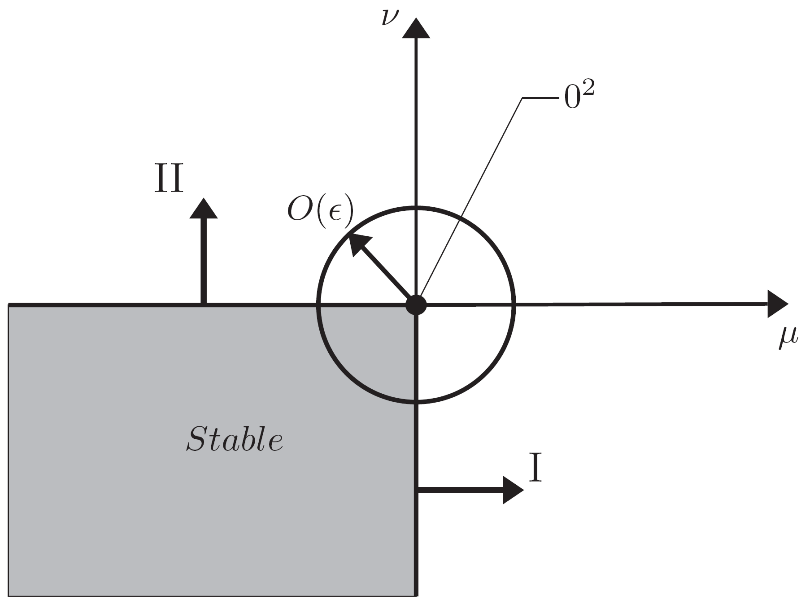
EXAMPLE: THE VAN DER POL-DUFFING OSCILLATOR UNDERGOING DOUBLE-ZERO BIFURCATION

$$\ddot{x} - \mu\dot{x} - \nu x + bx^2\dot{x} + cx^3 = 0$$

- Characteristic equation of the Jacobian matrix at the trivial equilibrium position:

$$\lambda^2 - \lambda\mu - \nu = 0$$

- Linear stability diagram:
  - positive ν -half-axis: $\lambda_{1,2} = \pm\sqrt{\nu}$
  - negative ν -half-axis $\lambda_{1,2} = \pm i\sqrt{|\nu|}$
  - whole μ -axis: $\lambda_{1,2} = 0, \mu$
  - A *double-zero bifurcation takes place at the origin* of the (μ, ν) -plane as a *degenerate Hopf bifurcation*, whose critical frequency approaches zero.



Linear stability diagram for the Van der Pol-Duffing oscillator, undergoing a double-zero bifurcation

- **Note:** The double-zero bifurcation 0^2 *is not* a double-divergence (0,0) bifurcation! It occurs at the intersection of a divergence and a Hopf manifold. It is a *static-dynamic interaction* phenomenon.

- **Note:** While in the $(0, \pm i\omega)$ case the Hopf boundary exists on both sides of the divergence boundary, in the 0^2 case *it dies at the intersection*.

EXAMPLE: A THREE-DIMENSIONAL DYNAMICAL SYSTEM UNDERGOING DOUBLE-ZERO BIFURCATION

We couple the Van der Pol-Duffing oscillator with a (stable) visco-elastic, non-inertial device:

$$\begin{cases} \ddot{x} - \mu\dot{x} - \nu x + bx^2\dot{x} - c_1(y-x)^3 = 0 \\ \dot{y} + ky + c_2(y-x)^3 = 0 \qquad k > 0 \end{cases}$$

- Rescaling:

$$(\mu,\nu) \to (\varepsilon\mu, \varepsilon\nu)\,, \quad (x,y) \to \varepsilon^{1/2}(x,y)$$

from which:

$$\begin{cases} \ddot{x} + \varepsilon[-\mu\dot{x} - \nu x + bx^2\dot{x} - c_1(y-x)^3] = 0 \\ \dot{y} + ky + \varepsilon c_2(y-x)^3 = 0 \end{cases}$$

■ Failure of the integer power expansion

We will show that integer power expansions *do not work* for defective system.

- Standard series expansions:

$$\begin{pmatrix} x(t;\varepsilon) \\ y(t;\varepsilon) \end{pmatrix} = \begin{pmatrix} x_0(t_0,t_1,t_2,\cdots) \\ y_0(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon \begin{pmatrix} x_1(t_0,t_1,t_2,\cdots) \\ y_1(t_0,t_1,t_2,\cdots) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} x_2(t_0,t_1,t_2,\cdots) \\ y_2(t_0,t_1,t_2,\cdots) \end{pmatrix} + \cdots$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{d}_0 + \varepsilon\,\mathrm{d}_1 + \varepsilon^2\,\mathrm{d}_2 + \cdots, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \mathrm{d}_0^2 + 2\varepsilon\,\mathrm{d}_0\,\mathrm{d}_1 + \varepsilon^2\left(\mathrm{d}_1^2 + 2\,\mathrm{d}_0\,\mathrm{d}_2\right) + \cdots$$

where $t_k := \varepsilon^k t_k$ and $\mathrm{d}_k := \partial / \partial t_k$.

- Perturbation equations:

$$\varepsilon^0: \begin{cases} \mathrm{d}_0^2 x_0 = 0 \\ \mathrm{d}_0 y_0 + k y_0 = 0 \end{cases}$$

$$\varepsilon: \begin{cases} \mathrm{d}_0^2 x_1 = -2\,\mathrm{d}_0\,\mathrm{d}_1 x_0 + \mu\,\mathrm{d}_0 x_0 + \nu x_0 - b x_0^2\,\mathrm{d}_0 x_0 + c_1 (y_0 - x_0)^3 \\ \mathrm{d}_0 y_1 + k y_1 = -\mathrm{d}_1 y_0 - c_2 (y_0 - x_0)^3 \end{cases}$$

……

- General solution of the zero-order equations:

$$x_0 = a(t_1, t_2, \cdots) + t_0 g_1(t_1, t_2, \cdots), \quad y_0 = g_2(t_1, t_2, \cdots)\,\mathrm{e}^{-k t_0}$$

To avoid secular terms, we take $g_1(t_1, t_2, \cdots) = 0$; since $y(t)$ decays, we take $g_2(t_1, t_2, \cdots) = 0$. Therefore, the generating solution is:

$$x_0 = a, \quad y_0 = 0$$

- ε -order equation:

$$\begin{cases} \mathrm{d}_0^2 x_1 = \nu a - c_1 a^3 \\ \mathrm{d}_0 y_1 + k y_1 = c_2 a^3 \end{cases}$$

- ε -order solution:

$$x_1 = (\nu a - c_1 a^3) t_0 + f(t_1, t_2) \quad \Rightarrow \quad \lim_{t_0 \to \infty} x_1 = \infty$$

Secular terms cannot be removed !!! The asymptotic expansions break down.

■ Employing fractional power expansions

We adopt (fractional) *powers series expansions of* $\varepsilon^{1/2}$ for the variables:

$$\begin{pmatrix} x(t;\varepsilon) \\ y(t;\varepsilon) \end{pmatrix} = \begin{pmatrix} x_0(t_0,t_1,\cdots) \\ y_0(t_0,t_1,\cdots) \end{pmatrix} + \varepsilon^{1/2}\begin{pmatrix} x_1(t_0,t_1,\cdots) \\ y_1(t_0,t_1,\cdots) \end{pmatrix} + \varepsilon\begin{pmatrix} x_2(t_0,t_1,\cdots) \\ y_2(t_0,t_1,\cdots) \end{pmatrix}$$
$$+\varepsilon^{3/2}\begin{pmatrix} x_3(t_0,t_1,\cdots) \\ y_3(t_0,t_1,\cdots) \end{pmatrix} + \varepsilon^{2}\begin{pmatrix} x_4(t_0,t_1,\cdots) \\ y_4(t_0,t_1,\cdots) \end{pmatrix} + \cdots$$

and *fractional time-scales*:

$$t_0 = t,\, t_1 = \varepsilon^{1/2} t\ ,\, t_2 = \varepsilon t\ ,\, t_3 = \varepsilon^{3/2} t,\, ,t_4 = \varepsilon^2 t \cdots$$

Chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{d}_0 + \varepsilon^{1/2}\,\mathrm{d}_1 + \varepsilon\,\mathrm{d}_2 + \varepsilon^{3/2}\,\mathrm{d}_3 + \varepsilon^2\,\mathrm{d}_4 + \cdots$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \mathrm{d}_0^2 + 2\varepsilon^{1/2}\,\mathrm{d}_0\,\mathrm{d}_1 + \varepsilon(\mathrm{d}_1^2 + 2\,\mathrm{d}_0\,\mathrm{d}_2) + 2\varepsilon^{3/2}(\mathrm{d}_0\,\mathrm{d}_3 + \mathrm{d}_1\,\mathrm{d}_2) + \varepsilon^2(\mathrm{d}_2^2 + 2\,\mathrm{d}_0\,\mathrm{d}_4 + 2\,\mathrm{d}_1\,\mathrm{d}_3) + \cdots$$

- Perturbation equations:

$$\varepsilon^0 \quad : \begin{cases} \mathrm{d}_0^2\, x_0 = 0 \\ \mathrm{d}_0\, y_0 + k y_0 = 0 \end{cases}$$

$$\varepsilon^{1/2} : \begin{cases} \mathrm{d}_0^2\, x_1 = -2\,\mathrm{d}_0\,\mathrm{d}_1\, x_0 \\ \mathrm{d}_0\, y_1 + k y_1 = -\mathrm{d}_1\, y_0 \end{cases}$$

$$\varepsilon \quad : \begin{cases} \mathrm{d}_0^2\, x_2 = -(\mathrm{d}_1^2 + 2\,\mathrm{d}_0\,\mathrm{d}_2) x_0 - 2\,\mathrm{d}_0\,\mathrm{d}_1\, x_1 + \mu\,\mathrm{d}_0\, x_0 + \nu x_0 - b x_0^2\,\mathrm{d}_0\, x_0 + c_1 (y_0 - x_0)^3 \\ \mathrm{d}_0\, y_2 + k y_2 = -\mathrm{d}_1\, y_0 - c_2 (y_0 - x_0)^3 \end{cases}$$

$$\varepsilon^{3/2} : \begin{cases} \mathrm{d}_0^2\, x_3 = -2(\mathrm{d}_0\,\mathrm{d}_3 + \mathrm{d}_1\,\mathrm{d}_2) x_0 - (\mathrm{d}_1^2 + 2\,\mathrm{d}_0\,\mathrm{d}_2) x_1 - 2\,\mathrm{d}_0\,\mathrm{d}_1\, x_2 + \mu(\mathrm{d}_1\, x_0 + \mathrm{d}_0\, x_1) + \nu x_1 \\ \qquad - b[x_0^2 (\mathrm{d}_1\, x_0 + \mathrm{d}_0\, x_1) + 2 x_0 x_1\,\mathrm{d}_0\, x_0] + 3 c_1 (y_0 - x_0)^2 (y_1 - x_1) \\ \mathrm{d}_0\, y_3 + k y_3 = -\mathrm{d}_3\, y_0 - \mathrm{d}_2\, y_1 - \mathrm{d}_1\, y_2 - 3 c_2 (y_0 - x_0)^2 (y_1 - x_1) \end{cases}$$

$$\varepsilon^2 \quad : \begin{cases} \mathrm{d}_0^2\, x_4 = -2(\mathrm{d}_0\,\mathrm{d}_4 + \mathrm{d}_1\,\mathrm{d}_3 + \mathrm{d}_2^2) x_0 - 2(\mathrm{d}_0\,\mathrm{d}_3 + \mathrm{d}_1\,\mathrm{d}_2) x_1 - (\mathrm{d}_1^2 + 2\,\mathrm{d}_0\,\mathrm{d}_2) x_2 - 2\,\mathrm{d}_0\,\mathrm{d}_1\, x_3 \\ \qquad + \mu(\mathrm{d}_2\, x_0 + \mathrm{d}_1\, x_1 + \mathrm{d}_0\, x_2) + \nu x_2 \\ \qquad - b[(\mathrm{d}_2\, x_0 + \mathrm{d}_1\, x_1 + \mathrm{d}_0\, x_2) x_0^2 + 2(\mathrm{d}_1\, x_0 + \mathrm{d}_0\, x_1) x_0 x_1 + (\mathrm{d}_0\, x_0) x_1^2 + 2\,\mathrm{d}_0\, x_0 x_0 x_2] \\ \qquad + 3 c_1 (y_0 - x_0)^2 (y_2 - x_2) + 3 c_1 (y_0 - x_0)(y_1 - x_1)^2 \\ \mathrm{d}_0\, y_4 + k y_4 = -\mathrm{d}_4\, y_0 - \mathrm{d}_3\, y_1 - \mathrm{d}_2\, y_1 - \mathrm{d}_1\, y_0 \\ \qquad - 3 c_2 (y_0 - x_0)^2 (y_2 - x_2) - 3 c_2 (y_0 - x_0)(y_1 - x_1)^2 \end{cases}$$

- Generating solution:

$$x_0 = a, \quad y_0 = 0$$

- $\varepsilon^{1/2}$ -order:

  - equations:

$$\begin{cases} \mathrm{d}_0^2\, x_1 = 0 \\ \mathrm{d}_0\, y_1 + k y_1 = 0 \end{cases}$$

  - secular terms: *absent*

  - solution:

$$x_1 = 0, \quad y_1 = 0$$

- ε -order:

  - equations:

$$\begin{cases} \mathrm{d}_0^2 x_2 = -\mathrm{d}_1^2 a + \nu a - c_1 a^3 \\ \mathrm{d}_0 y_2 + k y_2 = c_2 a^3 \end{cases}$$

  - elimination of secular terms:

$$\mathrm{d}_1^2 a = -c_1 a^3 + \nu a$$

  - solution:

$$x_2 = 0, \quad y_2 = \frac{c_2}{k} a^3$$

- $\varepsilon^{3/2}$ -order:

  - equations:

$$\begin{cases} \mathrm{d}_0^2 x_3 = -2\,\mathrm{d}_1\,\mathrm{d}_2\, a + (\mu - ba^2)\,\mathrm{d}_1\, a \\ \mathrm{d}_0\, y_3 + k y_3 = -3\dfrac{c_2}{k} a^2\,\mathrm{d}_1\, a \end{cases}$$

  - elimination of secular terms:

$$2\,\mathrm{d}_1\,\mathrm{d}_2\, a = (\mu - ba^2)\,\mathrm{d}_1\, a$$

  - solution:

$$x_3 = 0, \quad y_3 = -3\frac{c_2}{k^2} a^2\,\mathrm{d}_1\, a$$

- ε^2 -order equation:

  - ➢equations:

$$\begin{cases} d_0^2 x_4 = -d_2^2 a - 2 d_1 d_3 a + (\mu - ba^2) d_2 a + 3\dfrac{c_1 c_2}{k} a^5 \\ d_0 y_4 + k y_4 = NRT \end{cases}$$

  - ➢elimination of the secular terms:

$$d_2^2 a + 2 d_1 d_3 a = (\mu - ba^2) d_2 a + 3\frac{c_1 c_2}{k} a^5$$

- Reconstitution method and parameter reabsorbing:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}a = [\varepsilon^{1/2}\,\mathrm{d}_1 + \varepsilon\,\mathrm{d}_2 + \varepsilon^{3/2}\,\mathrm{d}_3 + \varepsilon^2\,\mathrm{d}_4 + \cdots]^2$$

$$= [\varepsilon\,\mathrm{d}_1^2 + 2\varepsilon^{3/2}\,\mathrm{d}_1\,\mathrm{d}_2 + \varepsilon^2(\mathrm{d}_2^2 + 2\,\mathrm{d}_1\,\mathrm{d}_3) + \cdots]a$$

$$= \varepsilon(-c_1 a^3 + \nu a) + \varepsilon(\mu - b a^2)(\varepsilon^{1/2}\,\mathrm{d}_1\,a + \varepsilon\,\mathrm{d}_2\,a + \cdots) + 3\varepsilon^2\frac{c_1 c_2}{k}a^5 + \cdots$$

$$== \varepsilon(-c_1 a^3 + \nu a) + \varepsilon(\mu - b a^2)\dot{a} + 3\varepsilon^2\frac{c_1 c_2}{k}a^5 + \cdots$$

where all the approximations are consistent with the order of the analysis. By multiplying by $\varepsilon^{1/2}$ and using $\varepsilon^{1/2}a \to a, \quad \varepsilon(\mu,\nu) \to (\mu,\nu)$, the *bifurcation equation* follows:

$$\ddot{a} - \mu\dot{a} - \nu a + b a^2\dot{a} + c_1 a^3 - 3\frac{c_1 c_2}{k}a^5 = 0$$

- Motion of the original system:

$$x(t) = a(t), \quad y(t) = \frac{c_2}{k} a(t)^3 - 3\frac{c_2}{k^2} a(t)^2 \dot{a}(t)$$

- **Note: T**he MSM filters the fast dynamic. In the double-zero bifurcation, *no fast dynamics occurs*, since the frequency involved, $\sqrt{|\nu|}$, is close to zero.

- **Note:** If the contribution of the passive coordinate *y* is neglected, the bifurcation equation reduces to the Van der Pol-Duffing equation:

$$\ddot{a} - \mu\dot{a} - \nu a + b a^2 \dot{a} + c_1 a^3 = 0$$

■ Steady solutions

- Fixed points:

The equilibrium positions of the original system are the fixed points $a = a_s = \text{const}$ of the bifurcation equation (y neglected):

$$(T): \quad a_T = 0, \quad \forall(\mu, \nu)$$

$$(B): \quad \nu = c_1 a_B^2, \quad \forall \mu$$

where:

- (T) is the *trivial* equilibrium, existing on the whole parameter-plane;
- (B) is the *buckled* (non-trivial) equilibrium;
- The two solutions intersect along the μ -axis. The static bifurcation is a *pitchfork*; if $c_1 > 0$ it is super-critical, (i.e. (B) exists when $\nu > 0$); if $c_1 < 0$ it is sub-critical, (i.e. (B) exists when $\nu < 0$).

- Stability of the *T*-solution s governed by:

$$\delta\ddot{a} - \mu\delta\dot{a} - \nu\delta a = 0$$

(already discussed).

- Stability of the *B*-solution is governed by :

$$\delta\ddot{a} + (ba_B^2 - \mu)\delta\dot{a} + (3c_1 a_B^2 - \nu)\delta a = 0$$

i.e.:

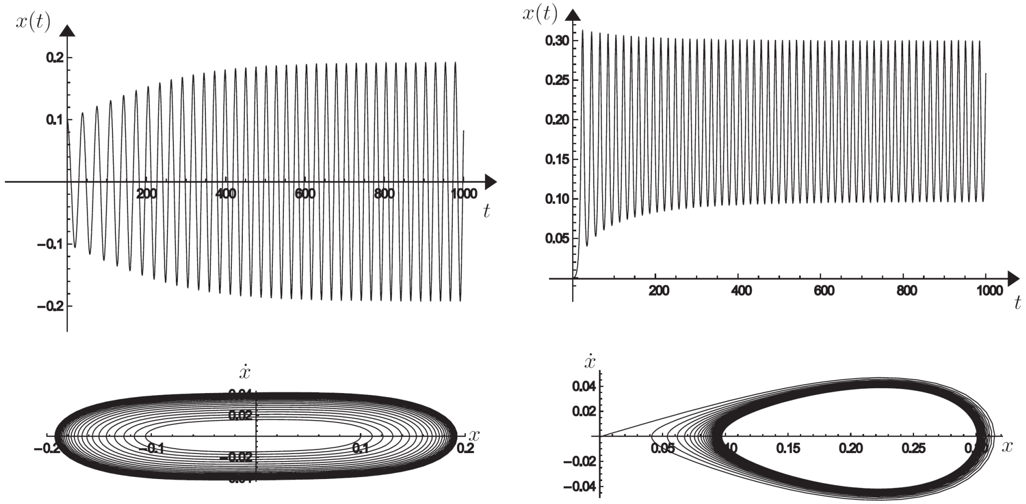
$$\delta\ddot{a} + (\frac{b}{c_1}\nu - \mu)\delta\dot{a} + 2\nu\delta a = 0$$

➢the *B*-solution cannot undergo further static bifurcations.
➢the *B*-solution *suffers a dynamic bifurcation*, when:

$$\nu = \frac{c_1}{b}\mu$$

i.e. along a straight line r_H from the origin..

- Numerical integrations



Motions around: (a) a stable *T*-cycle $(\mu = .01\,, \nu = -.01)$ and (b) un unstable *B*-cycle $(\mu = 0.045\,, \nu = 0.01)$; $b = 1, c_1 = 2$

- For the system considered:
  - The limit cycles bifurcating from the *T*- solutions are *super-critical and stable*; they are symmetric.
  - The limit cycles bifurcating from the *B*- solutions are *sub-critical and unstable*; they are non-symmetric.
  - The unstable *B*-cycles live in a narrow region. At r_h they collide with the trivial equilibrium point and disappear (*homoclinic bifurcation*).