7. DEFECTIVE BIFURCATIONS: THE DOUBLE-HOPF CASE

- A double-Hopf bifurcation occurs when, at an equilibrium point, the Jacobian matrix admits two pairs of complex conjugate, purely imaginary eigenvalues $\lambda^{(1,\bar{1})} = \pm i\omega_1, \lambda^{(2,\bar{2})} = \pm i\omega_2$.
- If these pairs coincide, i.e. if $\omega_1 = \omega_2$, a 1:1 *resonant* double-Hopf bifurcation takes place.
- This kind of bifurcation has already been analyzed for a system admitting *two* distinct eigenvectors associated with $\lambda^{(1)} = \lambda^{(2)}$ (i.e. to a system with a *diagonalizable* Jacobian matrix). This is a *non-generic case*.
- In the generic case, *just one* eigenvector is associated with the double eigenvalue $\lambda^{(1)} = \lambda^{(2)}$, so that the system is defective (i.e. it has a *non-diagonalizable* Jacobian matrix).
- Here, the MSM is applied to tackle *defective double-Hopf bifurcations*.

■ **A SELF-EXCITED NONLINEAR SYSTEM WITH NON-SYMMETRIC STIFFNESS AND DAMPING**

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} - \begin{pmatrix} \mu & 0 \\ \nu & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{pmatrix} \omega^2 & 1 \\ \sigma & \omega^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$+ \begin{pmatrix} b_1 \dot{x} x^2 - b_0 (\dot{y} - \dot{x})(y - x)^2 - c(y - x)^3 \\ b_2 \dot{y} y^2 + b_0 (\dot{y} - \dot{x})(y - x)^2 + c(y - x)^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

■ **Linear stability of the trivial equilibrium: *exact analysis***

- Variational equation:

$$\begin{pmatrix} \delta\ddot{x} \\ \delta\ddot{y} \end{pmatrix} - \begin{pmatrix} \mu & 0 \\ \nu & 0 \end{pmatrix} \begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \end{pmatrix} + \begin{pmatrix} \omega^2 & 1 \\ \sigma & \omega^2 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Eigenvalue problem:

By letting $\delta x(t) = \delta\hat{x}\exp(\lambda t), \delta y(t) = \delta\hat{y}\exp(\lambda t)$, a *quadratic* eigenvalue problem follows:

$$\begin{pmatrix} \lambda^2 - \mu\lambda + \omega^2 & 1 \\ \sigma - \nu\lambda & \lambda^2 + \omega^2 \end{pmatrix} \begin{pmatrix} \delta\hat{x} \\ \delta\hat{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

➢When, in particular, $\mu = \nu = \sigma = 0$:

$$\begin{pmatrix} \lambda_0^2 + \omega^2 & 1 \\ 0 & \lambda_0^2 + \omega^2 \end{pmatrix} \begin{pmatrix} \delta\hat{x} \\ \delta\hat{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which $\lambda_0^{(1,2)} = i\omega$ and $\lambda_0^{(\bar{1},\bar{2})} = -i\omega$, with one eigenvector, $(\delta\hat{x}, \delta\hat{y}) = (1,0)$ (two eigenvectors $(1, \pm i\omega, 0, 0)$ in the state-space).

➢ A defective double-Hopf bifurcation occurs at the origin *O* of the (μ, ν, σ)-parameter-space (codimension-3)

- Characteristic equations for $\lambda = \lambda(\mu, \nu, \sigma)$:

$$(\lambda^2 - \mu\lambda + \omega^2)(\lambda^2 + \omega^2) + \nu\lambda - \sigma = 0$$

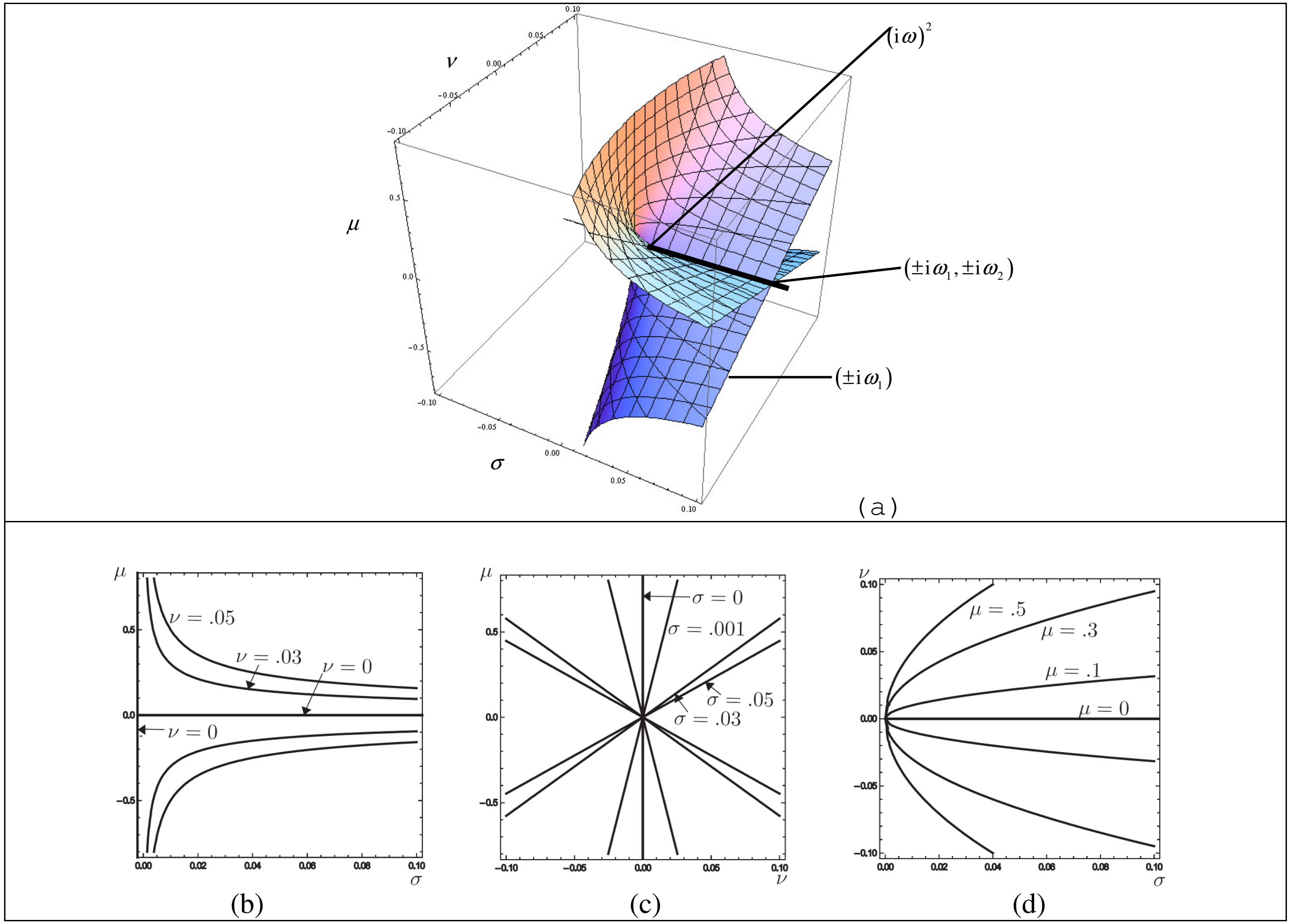
- Boundaries of the stability region:
  We look for the locus of the (μ, ν, σ)-points at which $\mathrm{Re}(\lambda) = 0$. By requiring $\lambda = i\beta,\ \beta \in \mathbb{R}$, two conditions follow:

$$(\beta^2 - \omega^2)^2 = \sigma, \quad \beta[\mu(\beta^2 - \omega^2) + \nu] = 0$$

  from which, eliminating β:

$$\nu = \pm\mu\sqrt{\sigma}$$

  This is a codimension-1 manifold in the three-dimensional parameter space.



Linear stability diagram for defective double-Hopf bifurcations: (a) 3D-view of the critical manifold; (b)-(d) sections ; $\omega = 1$.

■ Linear stability of the trivial equilibrium: *perturbation analysis*

The stability analysis of the trivial equilibrium is repeated, as an example, via evaluation of the *eigenvalue sensitivities.*

- Rescaling:

$$(\mu,\nu,\sigma)\rightarrow\varepsilon(\mu,\nu,\sigma)$$

- Characteristic equation:

$$(\lambda^2+\omega^2)^2-\varepsilon[\mu\lambda(\lambda^2+\omega^2)+\sigma-\nu\lambda)]=0$$

- Series expansion
  When $\varepsilon\rightarrow 0$, then $\lambda\rightarrow\pm i\omega,\pm i\omega$. A *fractional power series expansion* must be used:

$$\lambda=\lambda_0+\varepsilon^{1/2}\lambda_1+\varepsilon\lambda_2+\varepsilon^{3/2}\lambda_3+\cdots$$

- Perturbation equations:

$$
\begin{aligned}
\varepsilon^{0} \quad &: (\lambda_0^2 + \omega^2) = 0 \\
\varepsilon^{1/2} &: (\lambda_0^2 + \omega^2) 4\lambda_0 \lambda_1 = 0 \\
\varepsilon^{1} \quad &: (\lambda_0^2 + \omega^2) 4\lambda_0 \lambda_2 = -2\lambda_1^2 (3\lambda_0^2 + \omega^2) + (\lambda_0^2 + \omega^2)\lambda_0 \mu + \sigma - \lambda_0 \nu \\
\varepsilon^{3/2} &: (\lambda_0^2 + \omega^2) 4\lambda_0 \lambda_3 = 4\lambda_0 \lambda_1^3 - 4\lambda_1 \lambda_2 (3\lambda_0^2 + \omega^2) + (3\lambda_0^2 + \omega^2)\lambda_1 \mu - \lambda_1 \nu \\
&\quad \cdots
\end{aligned}
$$

- Generating solution:

$$\lambda_0 = i\omega$$

(the solutions generated by $\lambda_0 = -i\omega$ is obtained by complex conjugation).

- $\varepsilon^{1/2}$ -order:

trivially satisfied

- solvability at ε -order:

$$\lambda_1^2 = \frac{1}{4\omega^2}(-\sigma + i\omega\nu)$$

- solvability at $\varepsilon^{3/2}$ -order:

$$\lambda_1\lambda_2 = \lambda_1(-i\frac{\sigma}{8\omega^3} + \frac{1}{4}\mu)$$

Two cases arise:

(a) *generic perturbation*, in which σ and ν do not vanish simultaneously $(\lambda_1 \neq 0)$;

(b) *singular perturbation* in which σ and ν vanish $(\lambda_1 = 0)$. $\varepsilon^{3/2}$ -solvability is trivially satisfied!

- generic perturbation:

$$\lambda_1^{(1,2)} = \pm\frac{1}{2\omega}\sqrt{-\sigma + i\omega\nu}, \quad \lambda_2^{(1,2)} = -i\frac{\sigma}{8\omega^3} + \frac{1}{4}\mu, \quad \cdots$$

from which, after reabsorbing ε:

$$\lambda^{(1,2)} = i\omega \pm \frac{1}{2\omega}\sqrt{-\sigma + i\omega\nu} + \frac{1}{4}\mu - i\frac{\sigma}{8\omega^3}$$

not valid close to $\sigma = 0, \nu = 0$.

- singular perturbation:

An ordering violation occurs, since the leading term vanishes. An integer power expansion would be necessary. Not an efficient procedure!

- Reconstitution method:

A uniformly valid expression is built up, recombining *in a whole* all the solvability conditions:

$$\Delta\lambda = \lambda - \lambda_0 = \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + \varepsilon^{3/2}\lambda_3 + \cdots$$

$$\Delta\lambda^2 = \varepsilon\lambda_1^2 + 2\varepsilon^{3/2}\lambda_1\lambda_2 + \cdots$$

$$= \varepsilon\frac{1}{4\omega^2}(-\sigma + i\omega\nu) + 2\varepsilon^{3/2}\lambda_1(-i\frac{\sigma}{8\omega^3} + \frac{1}{4}\mu) =$$

$$\varepsilon[\frac{1}{4\omega^2}(-\sigma + i\omega\nu) + 2\Delta\lambda(-i\frac{\sigma}{8\omega^3} + \frac{1}{4}\mu)]$$

After reabsorbing ε, a *reconstituted sensitivity equation* is obtained:

$$\Delta\lambda^2 + (\frac{1}{2}\mu - i\frac{\sigma}{4\omega^3})\Delta\lambda + \frac{1}{4\omega^2}(\sigma - i\omega\nu) = 0$$

- Asymptotic expression for the critical manifold

On the critical manifold $\mathrm{Re}(\lambda) \equiv \mathrm{Re}(\Delta\lambda) = 0$. In order that $\Delta\lambda = i\beta$:

$$\beta^2 + \frac{\sigma}{4\omega^3}\beta - \frac{\sigma}{4\omega^2} = 0, \quad \nu = -2\omega\mu\beta$$

By eliminating β :

$$\nu = \mu(\pm\sqrt{\sigma}\sqrt{1+\frac{\sigma}{16\omega^4}} + \sigma) = \pm\mu\sqrt{\sigma} + \mathrm{O}(\mu\sigma)$$

which recovers the exact result to within an error of order $\mathrm{O}(\varepsilon^2)$, not accounted for in the analysis.

■ Nonlinear, multiple-scale bifurcation analysis

We investigate the dynamics of the nonlinear system around the bifurcation point.

- Rescaling:

By introducing the rescaling:

$$(\mu,\nu,\sigma) \rightarrow \varepsilon(\mu,\nu,\sigma)\,, \qquad (x,y) \rightarrow \varepsilon^{1/2}(x,y)$$

the equations read:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} + \begin{pmatrix} \omega^2 & 1 \\ 0 & \omega^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \varepsilon\begin{pmatrix} -\mu\dot{x} - b(\dot{y}-\dot{x})(y-x)^2 - c(y-x)^3 \\ \sigma x - \nu\dot{x} + b(\dot{y}-\dot{x})(y-x)^2 + c(y-x)^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Fractional series expansions:

$$\begin{pmatrix} x(t;\varepsilon) \\ y(t;\varepsilon) \end{pmatrix} = \begin{pmatrix} x_0(t_0,t_1,\cdots) \\ y_0(t_0,t_1,\cdots) \end{pmatrix} + \varepsilon^{1/2}\begin{pmatrix} x_1(t_0,t_1,\cdots) \\ y_1(t_0,t_1,\cdots) \end{pmatrix} + \varepsilon\begin{pmatrix} x_2(t_0,t_1,\cdots) \\ y_2(t_0,t_1,\cdots) \end{pmatrix} + \varepsilon^{3/2}\begin{pmatrix} x_3(t_0,t_1,\cdots) \\ y_3(t_0,t_1,\cdots) \end{pmatrix} + \cdots$$

where $t_k = \varepsilon^{k/2}\ (k = 0,1,\cdots)$. By applying the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{d}_0 + \varepsilon^{1/2}\,\mathrm{d}_1 + \varepsilon\,\mathrm{d}_2 + \varepsilon^{3/2}\,\mathrm{d}_3 + \cdots$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \mathrm{d}_0^2 + 2\varepsilon^{1/2}\,\mathrm{d}_0\,\mathrm{d}_1 + \varepsilon(\mathrm{d}_1^2 + 2\,\mathrm{d}_0\,\mathrm{d}_2) + 2\varepsilon^{3/2}(\mathrm{d}_0\,\mathrm{d}_3 + \mathrm{d}_1\,\mathrm{d}_2) + \cdots$$

- Perturbation equations:

$$\varepsilon^0 \quad : \begin{cases} \mathrm{d}_0^2\, x_0 + \omega^2 x_0 + y_0 = 0 \\ \qquad \mathrm{d}_0^2\, y_0 + \omega^2 y_0 = 0 \end{cases}$$

$$\varepsilon^{1/2} : \begin{cases} \mathrm{d}_0^2\, x_1 + \omega^2 x_1 + y_1 = -2\,\mathrm{d}_0\,\mathrm{d}_1\, x_0 \\ \qquad \mathrm{d}_0^2\, y_1 + \omega^2 y_1 = -2\,\mathrm{d}_0\,\mathrm{d}_1\, y_0 \end{cases}$$

$$\varepsilon \quad : \begin{cases} \mathrm{d}_0^2\, x_2 + \omega^2 x_2 + y_2 = -(\mathrm{d}_1^2 + 2\,\mathrm{d}_0\,\mathrm{d}_2) x_0 - 2\,\mathrm{d}_0\,\mathrm{d}_1\, x_1 + \mu\,\mathrm{d}_0\, x_0 \\ \qquad\qquad + b_1 x_0^2\,\mathrm{d}_0\, x_0 + b_0 (y_0 - x_0)^2 (\mathrm{d}_0\, y_0 - \mathrm{d}_0\, x_0) + c(y_0 - x_0)^3 \\ \qquad \mathrm{d}_0^2\, y_2 + \omega^2 y_2 = -(\mathrm{d}_1^2 + 2\,\mathrm{d}_0\,\mathrm{d}_2) y_0 - 2\,\mathrm{d}_0\,\mathrm{d}_1\, y_1 - \sigma x_0 + \nu\,\mathrm{d}_0\, x_0 \\ \qquad\qquad + b_2 y_0^2\,\mathrm{d}_0\, y_0 + -b_0 (y_0 - x_0)^2 (\mathrm{d}_0\, y_0 - \mathrm{d}_0\, x_0) - c(y_0 - x_0)^3 \end{cases}$$

(continue)

$$\varepsilon^{3/2}:\begin{cases} d_0^2 x_3+\omega^2 x_3+y_3=-2(d_0 d_3+d_1 d_2)x_0-(d_1^2+2d_0 d_2)x_1-2d_0 d_1 x_2 \\ \qquad +\mu(d_1 x_0+d_0 x_1) \\ \qquad +b_1[x_0^2(d_1 x_0+d_0 x_1)+2x_0 x_1 d_0 x_0] \\ \qquad +b_0\{(y_0-x_0)^2[(d_1(y_0-x_0)+d_0(y_1-x_1)] \\ \qquad\quad +2(y_0-x_0)(y_1-x_1)d_0(y_0-x_0)] \\ \qquad +3c(y_0-x_0)^2(y_1-x_1) \\ d_0^2 y_3+\omega^2 y_3=-2(d_0 d_3+d_1 d_2)y_0-(d_1^2+2d_0 d_2)y_1-2d_0 d_1 y_2 \\ \qquad +\nu(d_1 x_0+d_0 x_1) \\ \qquad +b_2[y_0^2(d_1 y_0+d_0 y_1)+2y_0 y_1 d_0 y_0] \\ \qquad -b_0\{(y_0-x_0)^2[(d_1(y_0-x_0)+d_0(y_1-x_1)] \\ \qquad\quad +2(y_0-x_0)(y_1-x_1)d_0(y_0-x_0)] \\ \qquad -3c(y_0-x_0)^2(y_1-x_1) \end{cases}$$

……

- Generating solution:

The generating equation admits the general solution:

$$\begin{cases} x_0 = A(t_1, t_2, t_3, \cdots)\,\mathrm{e}^{i\omega t_0} + \dfrac{i}{2\omega} B(t_1, t_2, t_3, \cdots) t_0\, \mathrm{e}^{i\omega t_{0.}} + c.c \\ y_0 = B(t_1, t_2, t_3, \cdots)\,\mathrm{e}^{i\omega t_0} + c.c. \end{cases}$$

with $(A, B) \in \mathbb{C}$. To eliminate secular terms, $B = 0$ must be taken; therefore:

$$\begin{cases} x_0 = A(t_1, t_2, t_3, \cdots)\,\mathrm{e}^{i\omega t_0} \\ y_0 = 0 \end{cases}$$

- Higher-order equations:

  They are, at any order, of the following type:

$$\begin{cases} \mathrm{d}_0^2\, x_j + \omega^2 x_j + y_j = \sum_{k=1,3,\cdots} f_{jk}\, \mathrm{e}^{ik\omega t_0} + c.c. \\ \mathrm{d}_0^2\, y_j + \omega^2 y_j = \sum_{k=1,3,\cdots} g_{jk}\, \mathrm{e}^{ik\omega t_0} + c.c. \end{cases}$$

  with $(f_{jk}, g_{jk}) \in \mathbb{C}$ constant on the t_0 -scale.

- Higher-order solutions:

  Solutions are harmonic and polynomial-harmonic. By ignoring these latter (secular terms), we let:

$$(x_j, y_j) = \sum_k (\hat{x}_{jk}, \hat{y}_{jk})\, \mathrm{e}^{ik\omega t_0} + c.c.$$

  from which an algebraic problem follows:

$$\begin{cases} \omega^2(1-k^2)\hat{x}_{jk} + \hat{y}_{jk} = f_{jk} \\ \omega^2(1-k^2)\hat{y}_{jk} = g_{jk} \end{cases}$$

- if $k \neq 1$ (*non-resonant* forcing terms) the equation are *non-singular* and therefore they admit an unique solution:

$$\begin{cases} \hat{x}_{jk} = \dfrac{f_{jk}}{(1-k^2)\omega^2} - \dfrac{g_{jk}}{(1-k^2)^2\omega^4} \\ \hat{y}_{jk} = \dfrac{g_{jk}}{(1-k^2)\omega^2} \end{cases}$$

- If $k = 1$ (*resonant* forcing terms), the equations are *singular*, and therefore call for a compatibility (or *solvability*) condition:

$$g_{j1} = 0$$

If this holds, they admit infinite solutions:

$$\begin{cases} \hat{x}_{j1} = C \\ \hat{y}_{j1} = f_{j1} \end{cases} \quad \forall C \in \mathbb{C}$$

However, since $C\exp(i\omega t_0) + c.c.$ repeats the generating solution, $C = 0$ is taken:

$$\begin{cases} \hat{x}_{j1} = 0 \\ \hat{y}_{j1} = f_{j1} \end{cases}$$

- $\varepsilon^{1/2}$-order:

  - equations:

$$\begin{cases} \mathrm{d}_0^2\, x_1 + \omega^2 x_1 + y_1 = -2i\omega\, \mathrm{d}_1\, A\, \mathrm{e}^{i\omega t_0} + c.c. \\ \mathrm{d}_0^2\, y_1 + \omega^2 y_1 = 0 \end{cases}$$

  - solvability condition:

automatically satisfied

  - solution:

$$\begin{cases} x_1 = 0 \\ y_1 = -2i\omega\, \mathrm{d}_1\, A\, \mathrm{e}^{i\omega t_0} + c.c. \end{cases}$$

while $\mathrm{d}_1 A$ remains undetermined at this order.

- ε-order:
  - equations:

$$\begin{cases} d_0^2 x_2 + \omega^2 x_2 + y_2 = f_{21}\,e^{i\omega t_0} + f_{23}\,e^{3i\omega t_0} + c.c. \\ d_0^2 y_2 + \omega^2 y_2 = g_{21}\,e^{i\omega t_0} + g_{23}\,e^{3i\omega t_0} + c.c. \end{cases}$$

where:

$$\begin{pmatrix} f_{21} \\ g_{21} \end{pmatrix} = \begin{pmatrix} i\omega\mu \\ -\sigma + i\omega\nu \end{pmatrix} A + \begin{pmatrix} -3c - i\omega(b_0 + b_1) \\ 3c + i\omega b_0 \end{pmatrix} A^2 \overline{A} - \begin{pmatrix} 2i\omega \\ 0 \end{pmatrix} d_2\, A - \begin{pmatrix} 1 \\ 4\omega^2 \end{pmatrix} d_1^2\, A$$

$$\begin{pmatrix} f_{23} \\ g_{23} \end{pmatrix} = \begin{pmatrix} -c - i\omega(b_0 + b_1) \\ c + i\omega b_0 \end{pmatrix} A^3$$

  - solvability condition:

By requiring $g_{21} = 0$, it follows:

$$d_1^2\, A = \frac{1}{4\omega^2}[(-\sigma + i\omega\nu)A + (3c + i\omega b_0)A^2 \overline{A}]$$

➢ solution:

By substituting $\mathrm{d}_1^2 A$, f_{21} is updated as follows:

$$f_{21} = \frac{1}{4\omega^2}\{(\sigma - i\omega\nu + 4i\omega^3\mu)A$$
$$-[3(1+4\omega^2)c + i\omega b_0 + 4i\omega^3(b_0+b_1)]A^2\overline{A} - 8i\omega^3\,\mathrm{d}_2\,A\}$$

and the solution reads:

$$\begin{cases} x_2 = \dfrac{1}{64\omega^4}[c(8\omega^2-1) - ib_0\omega + 8i\omega^3(b_0+b_1)]A^3\,\mathrm{e}^{3i\omega t_0} + c.c. \\ y_2 = \dfrac{1}{4\omega^2}\{(\sigma - i\omega\nu + 4i\omega^3\mu)A - [3(1+4\omega^2)c + i\omega b_0 + 4i\omega^3(b_0+b_1)]A^2\overline{A} \\ \qquad -8i\omega^3\,\mathrm{d}_2\,A\}\,\mathrm{e}^{i\omega t_0} - \dfrac{c+ib_0\omega}{8\omega^2}A^3\,\mathrm{e}^{3i\omega t_0} + c.c. \end{cases}$$

- $\varepsilon^{3/2}$ -order:
  - equations:

$$
\begin{cases}
d_0^2\, x_3 + \omega^2 x_3 + y_3 = NRT \\
\qquad d_0^2\, y_3 + \omega^2 y_3 = \dfrac{1}{2\omega}\{[4\omega^3\mu + \omega\nu - i\sigma + 2\omega b_0 \\
\qquad\qquad -8\omega^3(2b_0 + b_1) + 3ic(1+4\omega^2)]A\bar{A}\, d_1\, A \\
\qquad\qquad +[b_0\omega - 4\omega^3(2b_0 + b_1) + 3ic(1+4\omega^2)]A^2\, d_1\, \bar{A} \\
\qquad\qquad -16\omega^3\, d_1\, d_2\, A + 4i\omega\;\; d_1^3\, A\}\, e^{i\omega t_0} + c.c. + NRT
\end{cases}
$$

- solvability condition:

The resonant terms contain $\mathrm{d}_1^3 A$. By expressing it as $\mathrm{d}_1(\mathrm{d}_1^2 A)$ and using the ε-order compatibility, it is expressed as:

$$\mathrm{d}_1^3 A = \frac{1}{4\omega^2}[(-\sigma + i\omega\nu)\,\mathrm{d}_1 A + 2(3c + i\omega b_0)A\bar{A}\,\mathrm{d}_1 A + (3c + i\omega b_0)A^2\,\mathrm{d}_1 \bar{A}]$$

The $\varepsilon^{3/2}$ -order compatibility then supplies:

$$\mathrm{d}_1\,\mathrm{d}_2\,A = \frac{1}{8\omega^3}\{(2\omega^3\mu - i\sigma)\,\mathrm{d}_1 A + [6ic(1 + 2\omega^2) - 4\omega^3(2b_0 + b_1)]A\bar{A}\,\mathrm{d}_1 A$$
$$+[3ic(1 + 2\omega^2) - 2\omega^3(2b_0 + b_1)]A^2\,\mathrm{d}_1 \bar{A}\}$$

- Reconstitution:

$$\frac{\mathrm{d}^2 A}{\mathrm{d}t^2} = (\varepsilon\, \mathrm{d}_1^2 + 2\varepsilon^{3/2}\, \mathrm{d}_1\, \mathrm{d}_2 + \cdots)A$$

By reabsorbing the perturbation parameter, a second-order complex bifurcation equation follows:

$$\begin{aligned}\ddot{A} = {} & \frac{1}{4\omega^2}[(-\sigma + i\omega\nu)A + (3c + i\omega b_0)A^2\bar{A}] \\ & + \frac{1}{4\omega^3}\{(2\omega^3\mu - i\sigma)\dot{A} \\ & +[6ic(1+2\omega^2) - 4\omega^3(2b_0 + b_1)]A\bar{A}\dot{A} + [3ic(1+2\omega^2) - 2\omega^3(2b_0 + b_1)]A^2\dot{\bar{A}}\}\end{aligned}$$

which is equivalent to a four-dimensional system in real variables.

- Real form of the bifurcation equation:

Using the polar form $A(t) := a(t)\,\mathrm{e}^{i\theta}/2$, it follows:

➢ three RAME:

$$\begin{cases} \dot{a} = u \\ \dot{u} = -\dfrac{\sigma}{4\omega^2}a + \dfrac{1}{2}\mu a u + \dfrac{\sigma}{4\omega^3}a\psi + a\psi^2 + \dfrac{3c}{16\omega^2}a^3 - \dfrac{3}{8}(2b_0 + b_1)a^2 u - \dfrac{3c}{16\omega^3}(1+2\omega^2)a^3\psi \\ a\dot{\psi} = \dfrac{\nu}{4\omega}a - \dfrac{\sigma}{4\omega^3}au - 2u\psi + \dfrac{1}{2}\mu a\psi + \dfrac{b_0}{16\omega}a^3 + \dfrac{9c}{16\omega^3}(1+2\omega^2)a^2 u - \dfrac{1}{8}(2b_0 + b_1)a^3\psi \end{cases}$$

➢ one phase equation:

$$\dot{\theta} = \psi$$

with a the real amplitude, u its velocity and ψ the instantaneous frequency correction, i.e. $\Omega(t) = \omega + \psi(t)$.

- Response:

$$\begin{cases} x = a(t)\cos[\Phi(t)] + \dfrac{8\omega^2 - 1}{256\omega^4} c a^3(t)\cos[3\Phi(t)] \\ \quad + \dfrac{1}{256\omega^3}[b_0 - 8\omega^2(b_0 + b_1)]a^3(t)\sin[3\Phi(t))] + \cdots \\ y = [2\omega\dot{a}(t) + (\dfrac{\nu}{4\omega} - \mu\omega)a(t) \\ \quad + \{2\omega a\psi(t) + \dfrac{1}{\omega}[\dfrac{3}{32}b_0 + \dfrac{\omega^2}{4}(b_0 + b_1)]a^3(t)]\}\sin[\Phi(t)] \\ \quad + [\dfrac{\sigma}{4\omega^2}a(t) - \dfrac{3}{16}c(1 + 4\omega^2)a^3(t)]\cos[\Phi(t)] \\ \quad - \dfrac{c}{32\omega^2}a^3(t)\cos[3\Phi(t)] + \dfrac{b_0}{16\omega}a^3(t)\cos[2\Phi(t)]\sin[\Phi(t)] + \cdots \end{cases}$$

where $\Phi(t) := \omega t + \theta(t)$ is the total phase.

■ **Steady solutions and their stability**

The steady motions are the fixed points $(a,u,\psi)=(a_s,0,\psi_s)$ of the bifurcation equations. They are solutions of:

$$\begin{cases} a_s[-\dfrac{\sigma}{4\omega^2}+\dfrac{\sigma}{4\omega^3}\psi_s+\psi_s^2+\dfrac{3c}{16\omega^2}a_s^2-\dfrac{3c}{16\omega^3}(1+2\omega^2)a_s^2\psi_s]=0 \\ a_s[\dfrac{\nu}{4\omega}+\dfrac{1}{2}\mu\psi_s+\dfrac{b_0}{16\omega}a_s^2-\dfrac{1}{8}(2b_0+b_1)a_s^2\psi_s]=0 \end{cases}$$

- *Trivial solution* $a_T=0, \forall\psi_T, \forall(\mu,\nu,\sigma)$: equilibrium of the original system.
- *Non-trivial solutions* (a_P,ψ_P): periodic motions of amplitude a_P and frequency $\Omega_P=\omega+\psi_P$. By eliminating ψ_P, a cubic equation in a_P^2 is obtained, so that *from zero to three* (non-trivial) real solutions exist in each point of the parameter space.

- Periodic motion:

$$\begin{cases} x = a_P \cos[\Phi_P(t)] + \dfrac{8\omega^2 - 1}{256\omega^4} c a_P^3 \cos[3\Phi_P(t)] \\ \quad + \dfrac{1}{256\omega^3}[b_0 - 8\omega^2(b_0 + b_1)] a_P^3 \sin[3\Phi_P(t))] + \cdots \\ y = (\dfrac{\nu}{4\omega} - \mu\omega) a_P + \{2\omega a_P \psi_P + \dfrac{1}{\omega}[\dfrac{3}{32} b_0 + \dfrac{\omega^2}{4}(b_0 + b_1)] a_P^3\} \sin[\Phi_P(t)] \\ \quad + [\dfrac{\sigma}{4\omega^2} a_P - \dfrac{3}{16} c(1 + 4\omega^2) a_P^3] \cos[\Phi_P(t)] \\ \quad - \dfrac{c}{32\omega^2} a_P^3 \cos[3\Phi_P(t)] + \dfrac{b_0}{16\omega} a_P^3 c\,\mathrm{os}[2\Phi_P(t)] \sin[\Phi_P(t)] + \cdots \end{cases}$$

where $\Phi_P(t) := (\omega + \psi_P)t + \theta_0$.

- Stability of the steady solutions:
  - Trivial solution:

    Since ψ_T is undetermined, it is convenient to resort to the variation of the *complex* bifurcation equation:

$$\delta\ddot{A}-\frac{1}{4\omega^3}(2\omega^3\mu-i\sigma)\delta\dot{A}+\frac{1}{4\omega^2}(\sigma-i\omega\nu)\delta A=0$$

By letting $\delta A(t)=\delta\hat{A}\exp(\Lambda\, t)$, its associated eigenvalue problem reads:

$$\Lambda^2-\frac{1}{4\omega^3}(2\omega^3\mu-i\sigma)\Lambda+\frac{1}{4\omega^2}(\sigma-i\omega\nu)=0$$

Since this *coincides with the reconstituted sensitivity equation*, $\Lambda\equiv\Delta\lambda$. Hence, the trivial solution loses stability on the critical manifold, where $\mathrm{Re}(\Delta\lambda)=0$. Here, one ore more *P*-solutions bifurcate.

❑ **Note:** Multiple Scale analysis *includes* sensitivity analysis.

➢ Non-trivial solutions.

The variation of the *real* bifurcation equations reads:

$$\begin{pmatrix} \delta\dot{a} \\ \delta\dot{u} \\ \delta\dot{\psi} \end{pmatrix} = \mathbf{J}_P \begin{pmatrix} \delta a \\ \delta u \\ \delta\psi \end{pmatrix}$$

where:

$$J_{11} = 0, \quad J_{12} = 1, \quad J_{13} = 0,$$

$$J_{21} = \frac{1}{16\omega^3}\{9c[\omega - \psi_p(1+2\omega^2)]a_P^2 + 4[\sigma(\psi_p - \omega) + 4\psi_P^2\omega^2],$$

$$J_{22} = \frac{1}{8}[4\mu - 3(2b_0 + b_1)a_P^2], \; J_{23} = \frac{1}{16\omega^3}[-3c(1+2\omega^2)a_P^3 + 4(\sigma + 8\omega^3\psi_p)a_P],$$

$$J_{31} = \frac{1}{16\omega a_p}\{4(\nu + 2\omega\mu\psi_p) + 3[b_0 - (2b_0 + b_1)\omega\psi_p]a_P^2,$$

$$J_{32} = \frac{1}{16\omega^3 a_p}[9c(1+2\omega^2)a_P^2 - 4(\sigma + 8\omega^3\psi_p)], \quad J_{33} = \frac{1}{8}[4\mu - (2b_0 + b_1)a_P^2]$$

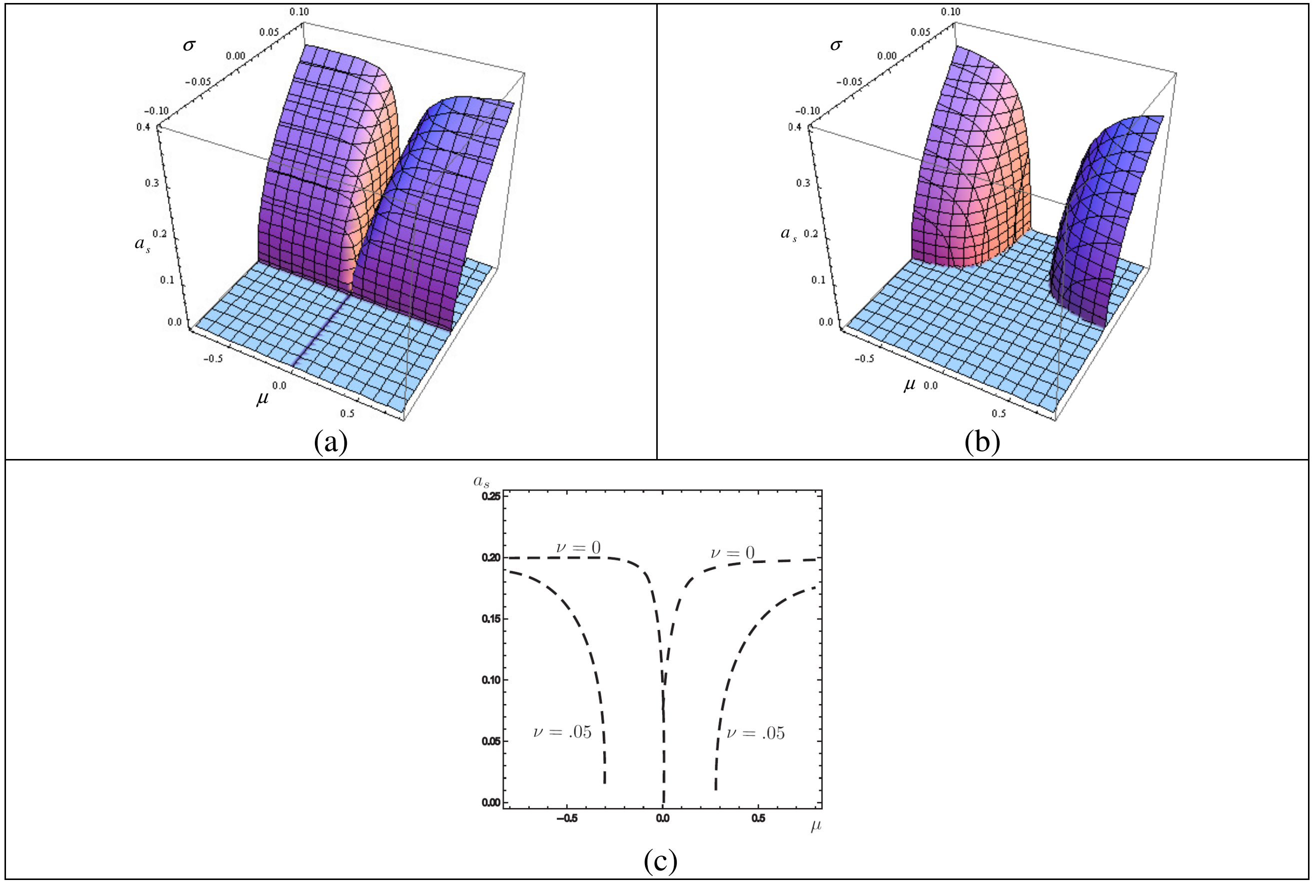
Stability is governed by the eigenvalues of $\mathbf{J}_P$.

■ **Parametric analysis**

The bifurcation diagram would require plotting a_P and/or ψ_P, versus the bifurcation parameters (μ, ν, σ). Three- or bi-dimensional sections are built-up. Three systems analyzed:

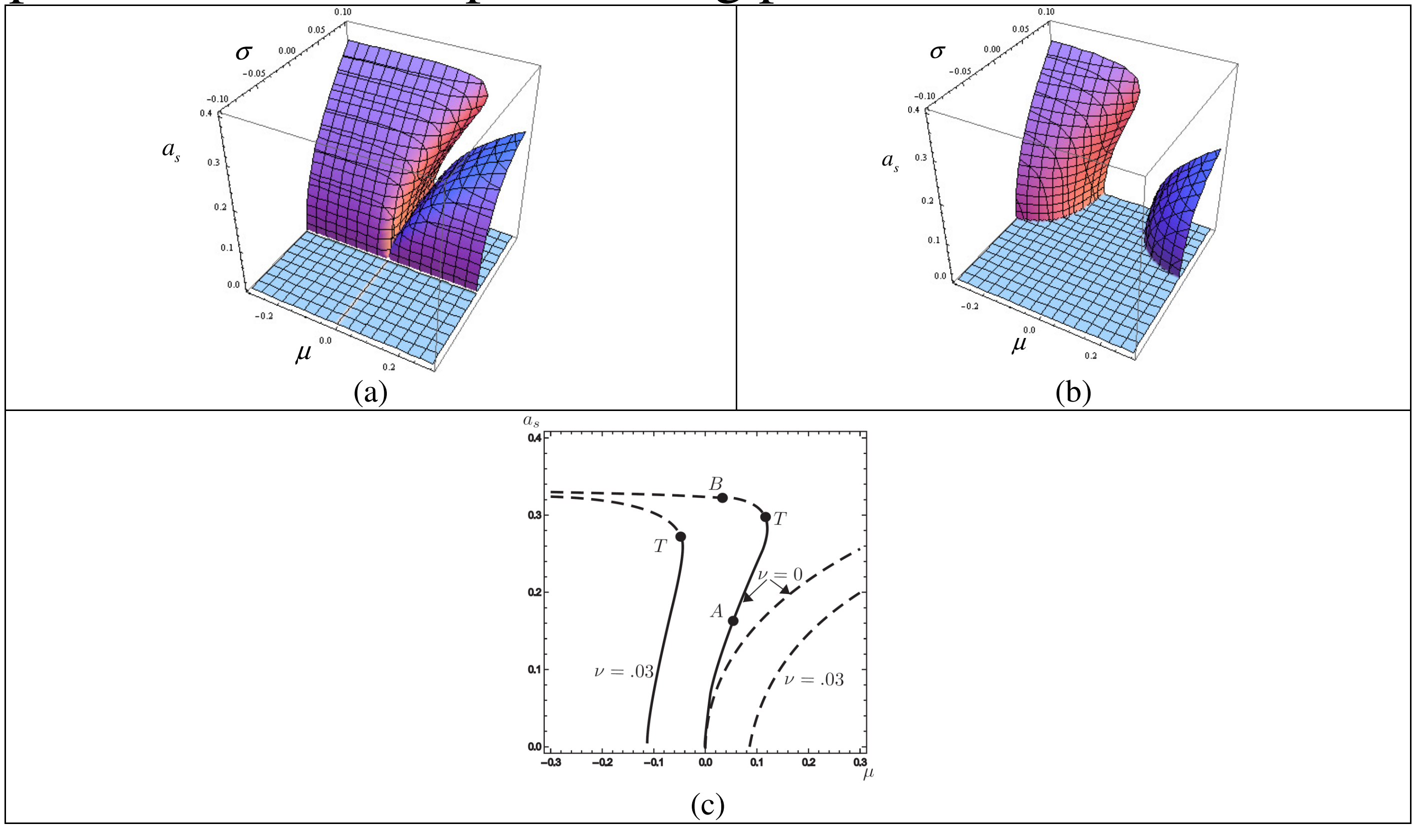
- (S1) system: no damping and hardening elastic coupling,
  $b_0 = 1, b_1 = 0, c = 1, \omega = 1$;
- (S2) system: large damping and hardening elastic coupling,
  $b_0 = 1, b_1 = 10, c = 1, \omega = 1$;
- (S3) system: large damping and softening elastic coupling,
  $b_0 = 1, b_1 = 10, c = -5, \omega = 1$.

- System (S1):
  - Sub-critical bifurcation
  - All the surface branches are unstable.



Bifurcation diagrams for defective double-Hopf bifurcations, $b_0 = 1, b_1 = 0, c = 1, \omega = 1$; (a) $\nu = 0$, (b) $\nu = 0.05$, (c) $\sigma = 0.03$ (— stable, --- unstable)

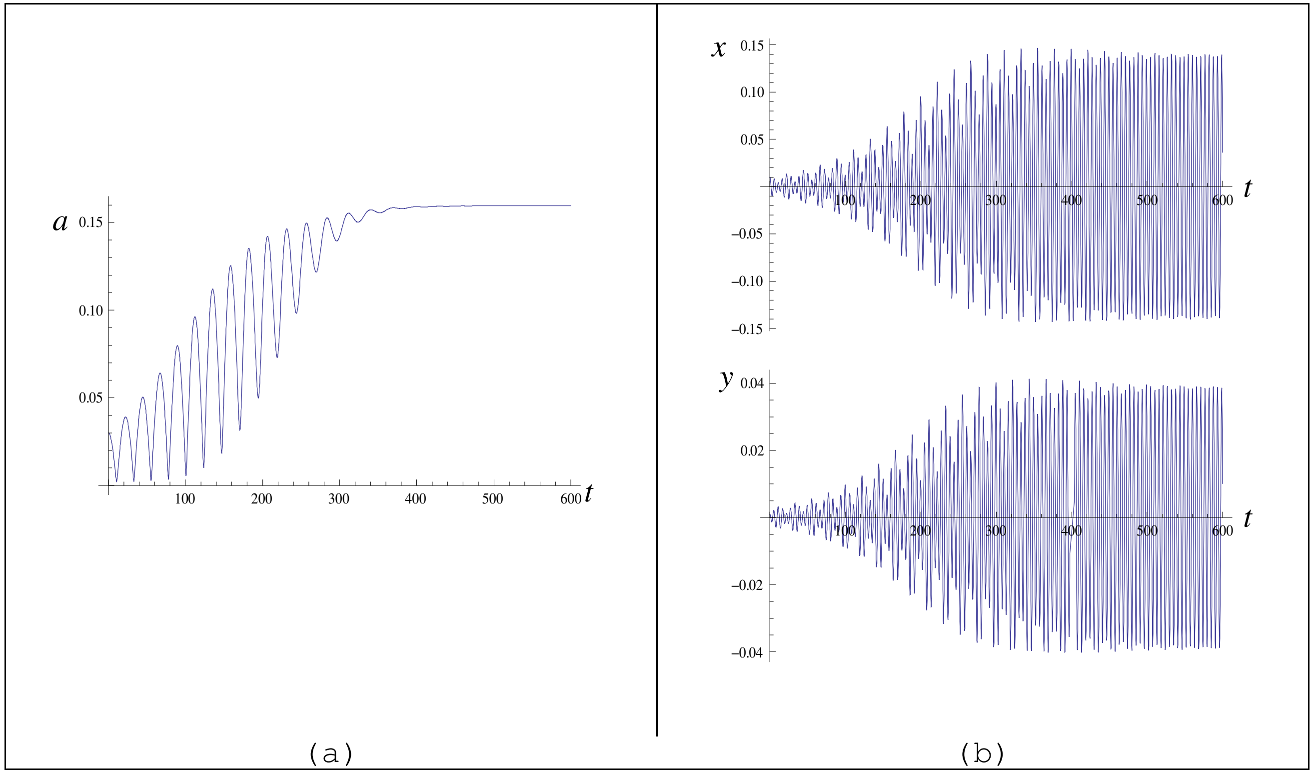
- System (S2):
  - ➢ Folding of the surfaces; multivalued responses.
  - ➢ Most of the surface branches are unstable. Lower, multivalued responses are stable, up-to turning points *T*.



Bifurcation diagrams for defective double-Hopf bifurcations, $b_0 = 1, b_1 = 10, c = 1, \omega = 1$; (a) $\nu = 0$, (b) $\nu = 0.03$, (c) $\sigma = 0.08$ (— stable, --- unstable)

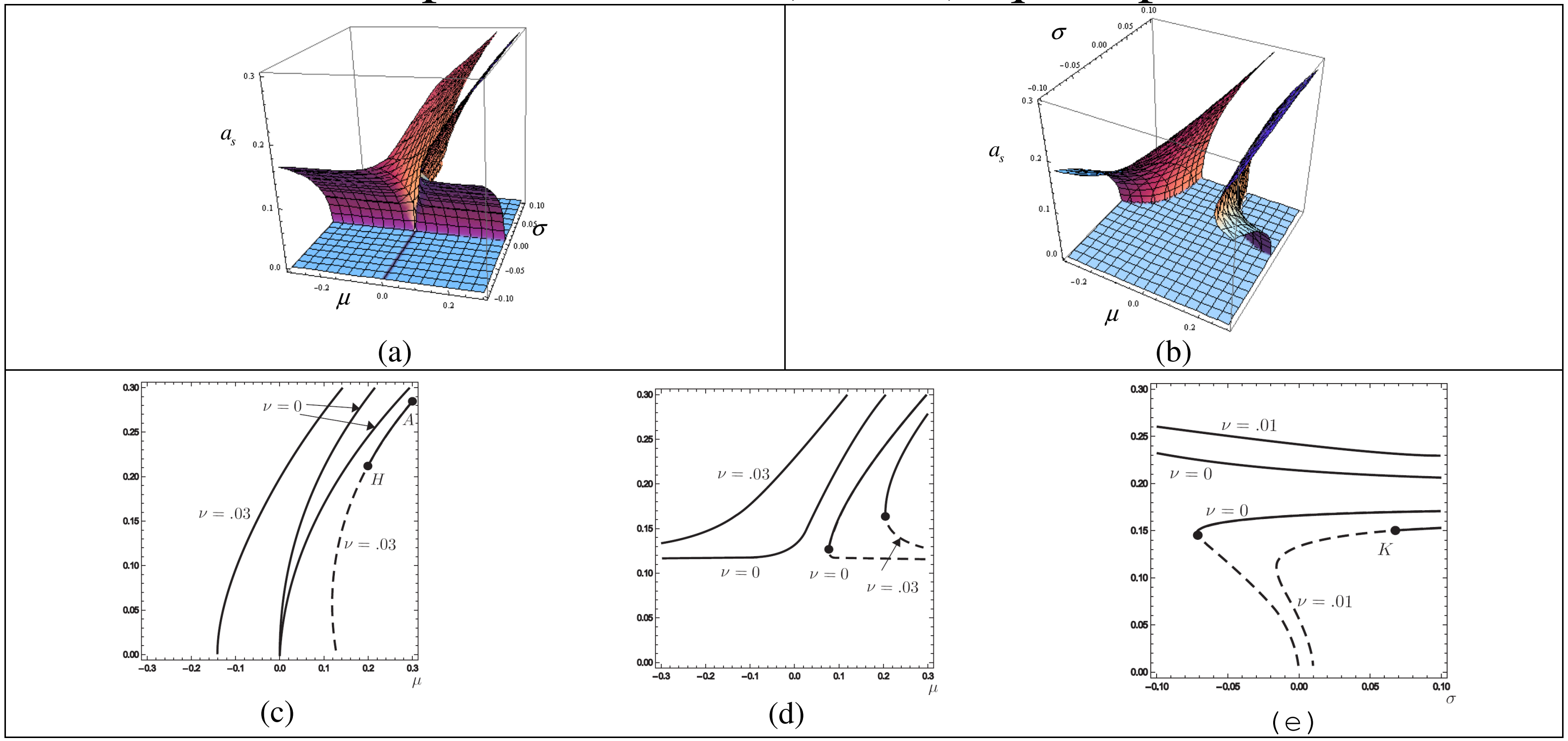
- Numerical integrations for system (S2)

  Response at point A of the previous figure.



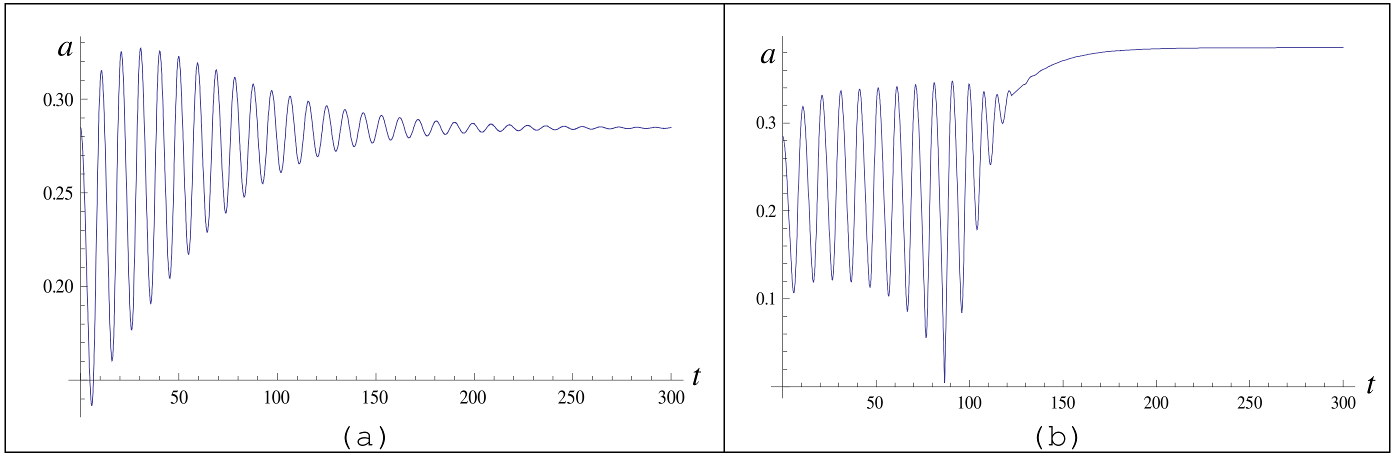
Numerical time-histories in the over-critical bifurcation region: (a) amplitude, (b) original configuration variables; $b_0 = 1, b_1 = 10, c = 1, \omega = 1$; $\nu = 0, \sigma = 0.08, \mu = 0.05$.

- System (S3):
  - Super-critical bifurcation; multivalued responses.
  - Branches originating from the lower (upper) part of the critical manifold are stable (unstable).
  - Occurence of Hopf bifurcations (at *H*,*K*): quasi-periodic motions arise.



Bifurcation diagrams for defective double-Hopf bifurcations, $b_0 = 1, b_1 = 10, c = -5, \omega = 1$; (a) $\nu = 0$, (b) $\nu = 0.03$, (c) $\sigma = 0.05$, (d) $\sigma = -0.05$, (e) $\mu = 0.10$ (— stable, --- unstable)

- Numerical integrations for system (S3)



Numerical time-histories close to point *A* of previous figure; initial conditions: $a(0) = 0.2846, u(0) = 0$ and (a) $\psi(0) = -0.27$,(b) $\psi(0) = -0.25$; $b_0 = 1, b_1 = 10, c = -5, \omega = 1$; $\nu = 0.03, \sigma = 0.05, \mu = 0.30$.