Course on *Bifurcation Theory*, a.y. 2009/10

PROJECTS

on

Multiple Bifurcations of Sample Dynamical Systems

Students of the ‘Bifurcation Theory’ course *can* carry out an *individual* homework, to be discussed during oral examination. <u>As an alternative, they can chose to take a traditional examination</u>.

Here information is given about the assignment procedure of the homework, and the list of possible themes.

Procedure:

- The student chooses one of projects from the following list; then he sends an email to angelo.luongo@univaq.it, with the number of the project he wants to carry out. After that, no changes are allowed.
- The project requires *analytical* calculations, to be performed by a symbolic program (e.g. Mathematica), and *numerical* results. Manual calculations are allowed, but discouraged.
- The student prepares a report, on a hard support *and* a disk, to be submitted to the examining board the day of the oral examination. The report will contain a detailed illustration of the procedure followed, analytical and numerical results obtained, figures and comments.
- The examination consists in a discussion of the project.
- Each project is classified according to its difficulty level (D.L.) (lowest, low, average, high, highest). The final assessment will account for the difficulty level, the completeness, style and accuracy of the report, the effectiveness of the oral presentation, the smartness of the candidate in the discussion.

Homework #1 **D.L.: *lowest***

Comparison between the Center Manifold Method and the Multiple Scale Method for a two-dimensional system, with quadratic and cubic nonlinearities, affected by imperfections, undergoing a simple divergence (symmetric behaviour)

Consider the following dynamical system, with η the imperfection parameter:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_2 xy + c_3 x^3 + \eta \\ bx^2 \end{pmatrix}$$

Apply both the CM and the MSM to obtain a second-order approximation (the first-order approximation is illustrated in the lecture notes).

Analyze equilibria and stability both for $\eta=0$ and $\eta\neq0$.

Present some numerical applications (e.g. bifurcation diagrams and numerical integrations showing transient motions).

Homework # 2 **D.L.: *low***

Comparison between the Center Manifold Method and the Multiple Scale Method for a two-dimensional system, with quadratic and cubic nonlinearities, affected by imperfections, undergoing a simple divergence (asymmetric behaviour)

Consider the following dynamical system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_1 xy + c_2 x^2 + c_3 x^3 + \eta \\ bx^2 \end{pmatrix}$$

Apply both the CM and the MSM to obtain a second-order approximation.

Analyze equilibria and stability both for perfect (η=0) and imperfect (η≠0) systems.

Present some numerical applications (e.g. bifurcation diagrams and numerical integrations showing transient motions).

Homework # 3a **D.L.: *low***

Three-dimensional system, with quadratic and cubic nonlinearities, undergoing a simple Hopf bifurcation.

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega^2 x + c_1 x^2 + c_2 xy + b_1 x^2 \dot{x} = 0 \\ \dot{y} + ky + b_2 x\dot{x} = 0, \qquad\qquad k > 0 \end{cases}$$

Apply the MSM to obtain approximations for the bifurcation equation. Discuss the role of the passive coordinate.

Obtain some numerical solutions showing periodic motions.

Homework # 3b **D.L.: *average***

Three-dimensional system, with quadratic and cubic nonlinearities, undergoing a simple Hopf bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega^2 x + c_1 x^2 + c_2 xy + b_1 x^2 \dot{x} = 0 \\ \dot{y} + ky + b_2 x\dot{x} = 0, \qquad\qquad k > 0 \end{cases}$$

Apply the Center Manifold and Normal Form Theory to obtain the bifurcation equation. Discuss the role of the passive coordinate.

Obtain some numerical solutions showing periodic motions.

Homework # 4 **D.L.: *average***

Comparison between the Center Manifold Method and the Multiple Scale Method for a three-dimensional system, with quadratic and cubic nonlinearities, affected by imperfections, undergoing double divergence

Consider the following dynamical system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} c_1 xz + c_3 x^3 + \eta \\ c_2 yz + c_4 y^3 + \eta \\ b_1 x^2 + b_2 y^2 \end{pmatrix}$$

Apply the CM and the MSM to obtain a second-order approximation (the first-order approximation is illustrated in the lecture notes).

For the first-order approximation, build up the bifurcation chart for perfect (η=0) and imperfect ($\eta\neq$0) systems. Present some numerical applications (e.g. bifurcation diagrams and numerical integrations showing transient motions).

Homework # 5 **D.L.: *lowest***

Single oscillator, with quadratic nonlinearities, affected by imperfections, undergoing a static bifurcation

Consider the following dynamical system, where η is an imperfection:

$$\ddot{x} + \xi \dot{x} - \mu x + c x^2 + \eta = 0, \quad \xi > 0$$

Apply the MSM to obtain a first-order approximation for the bifurcation equation.

Find the equilibrium points, study stability and plot the bifurcation diagram, both for perfect and imperfect systems.

Present some numerical integrations showing transient motions.

Homework # 6 **D.L.: *lowest***

Single oscillator, with cubic nonlinearities, affected by imperfections, undergoing a static bifurcation

Consider the following dynamical system, where η is an imperfection:

$$\ddot{x} + \xi \dot{x} - \mu x + c x^3 + \eta = 0, \quad \xi > 0$$

Apply the MSM to obtain a first-order approximation for the bifurcation equation.

Find the equilibrium points, study stability and plot the bifurcation diagram, both for perfect and imperfect systems.

Present some numerical integrations showing transient motions.

Homework # 7 **D.L.: *low***

Single oscillator, with quadratic and cubic nonlinearities, affected by imperfections, undergoing a static bifurcation

Consider the following dynamical system, where η is an imperfection:

$$\ddot{x} + \xi \dot{x} - \mu x + c_2 x^2 + c_3 x^3 + \eta = 0, \quad \xi > 0$$

Apply the MSM to obtain a second-order approximation for the bifurcation equation.

Find the equilibrium points, study stability and plot the bifurcation diagram, both for perfect and imperfect systems.

Discuss the role of the c_3/c_2 ratio.

Present some numerical integrations showing transient motions.

Homework # 8 **D.L.: *lowest***

Single oscillator, with quadratic and cubic nonlinearities, undergoing a Hopf bifurcation

Consider the following dynamical system:

$$\ddot{x} - \mu \dot{x} + \omega_1^2 x + c_1 x^2 + c_2 x^3 + b_1 x\dot{x} + b_2 x^2 \dot{x} + b_3 \dot{x}^3 = 0$$

Apply the MSM to obtain a second-order approximation for the bifurcation equation.

Present some numerical applications (e.g. bifurcation diagrams and numerical integrations showing transient motions).

Comment the role of the quadratic nonlinearities.

Homework # 9 **D.L.: *low***

Two coupled oscillators, with quadratic and cubic nonlinearities, undergoing a simple Hopf bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu \dot{x} + \omega_1^2 x + c_1 x^2 + c_2 x^3 + b_0 x^2 \dot{x} + b_1 y \dot{x} = 0 \\ \ddot{y} + \xi \dot{y} + \omega_2^2 y + b_2 x^2 = 0 \end{cases}$$

Apply the MSM to obtain a second-order approximation for the bifurcation equation.

Present some numerical applications (e.g. bifurcation diagrams and numerical integrations showing transient motions).

Comment the role of the quadratic nonlinearities and that of the passive variable *y*.

Homework # 10 **D.L.: *low***

Four-dimensional system, with quadratic and cubic nonlinearities, undergoing a simple Hopf bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega^2 x + cx^3 + b_0 x^2 \dot{x} + b_1 y\dot{x} + b_2 z\dot{x} = 0 \\ \dot{y} + k_1 y + b_3 x^2 = 0 \\ \dot{z} + k_2 z + b_4 x^2 = 0 \end{cases}$$

Apply the MSM to obtain a second-order approximation for the bifurcation equation.

Present some numerical applications (e.g. bifurcation diagrams and numerical integrations showing transient motions).

Comment the role of the quadratic nonlinearities and of the passive variable *y*.

Homework # 11 **D.L.: *lowest***

Two coupled Van der Pol-Duffing oscillators, undergoing a nonresonant double Hopf bifurcation

Consider the following dynamical system, with $\omega_2 / \omega_1 \notin \mathbb{Q}$:

$$\begin{cases} \ddot{x} - \mu \dot{x} + \omega_1^2 x + b_1 \dot{x}^3 + c x^3 - b_0 (y - x)^2 (\dot{y} - \dot{x}) = 0 \\ \ddot{y} - \nu \dot{y} + \omega_2^2 y + b_2 \dot{y}^3 + c y^3 + b_0 (y - x)^2 (\dot{y} - \dot{x}) = 0 \end{cases}$$

Apply the MSM to obtain a first-order approximation for the bifurcation equations.

Repeat the whole analysis carried out in the lecture notes for the Raileigh-Duffing system, but choose different numerical values for the auxiliary parameters *b* and *c*.

Present, in addition, some numerical integrations showing transient motions.

Homework # 12 **D.L.: *low***

Two oscillators, with quadratic and cubic coupling , undergoing a nonresonant double Hopf bifurcation

Consider the following dynamical system, with $\omega_2 / \omega_1 \notin \mathbb{Q}$:

$$\begin{cases} \ddot{x} - \mu \dot{x} + \omega_1^2 x + c_1 x^3 - c_0 (y - x)^2 - b_0 (y - x)^2 (\dot{y} - \dot{x}) = 0 \\ \ddot{y} - \nu \dot{y} + \omega_2^2 y + c_1 y^3 + c_0 (y - x)^2 + b_0 (y - x)^2 (\dot{y} - \dot{x}) = 0 \end{cases}$$

Apply the MSM to obtain a second-order approximation for the bifurcation equations.

Repeat the whole analysis carried out in the lecture notes for the case of no quadratic nonlinearities.

Investigate and comment the role of the quadratic nonlinearities.

Homework # 13 **D.L.: *low***

Two oscillators, with cubic coupling, undergoing a nonresonant double Hopf bifurcation

Consider the following dynamical system, with $\omega_2 / \omega_1 \notin \mathbb{Q}$:

$$\begin{cases} \ddot{x} - \mu \dot{x} + \omega_1^2 x + c_1 x^2 - c_0 (y - x)^3 - b_0 (\dot{y} - \dot{x})^3 = 0 \\ \ddot{y} - \nu \dot{y} + \omega_2^2 y + c_1 y^2 + c_0 (y - x)^3 + b_0 (\dot{y} - \dot{x})^3 = 0 \end{cases}$$

Apply the MSM to obtain a second-order approximation for the bifurcation equations.

Repeat the whole analysis carried out in the lecture notes for a similar problem.

Investigate and comment the role of the quadratic nonlinearities.

Homework # 14 **D.L.: *average***

A five-dimensional system, undergoing a nonresonant double Hopf bifurcation

Consider the following dynamical system, with $\omega_2 / \omega_1 \notin \mathbb{Q}$:

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega_1^2 x + cx^3 - b_0(\dot{y} - \dot{x})^3 + b_1 z\dot{x} = 0 \\ \ddot{y} - \nu\dot{y} + \omega_2^2 y + cy^3 + b_0(\dot{y} - \dot{x})^3 + b_2 z\dot{y} = 0 \\ \dot{z} + kz + b_3(\dot{x}^2 + \dot{y}^2) = 0, \qquad k > 0 \end{cases}$$

Apply the MSM to obtain a second-order approximation for the bifurcation equations.

Build up the bifurcation chart and one or more bifurcation diagrams.

Investigate the role of the passive variable z.

Homework # 15 **D.L.: *average***

Two coupled oscillators, affected by imperfections, undergoing a Hopf-Divergence bifurcation

Consider the following dynamical system, with η being the imperfection:

$$\begin{cases} \ddot{x} - \mu \dot{x} + \omega^2 x + c x^3 - b_0 (y-x)^2 (\dot{y} - \dot{x}) - c_0 (y-x)^3 = 0 \\ \ddot{y} + \xi \dot{y} - \nu y + c y^3 + b_0 (y-x)^2 (\dot{y} - \dot{x}) + c_0 (y-x)^3 + \eta = 0, \qquad \xi > 0 \end{cases}$$

Apply the MSM to obtain a first-order approximation for the bifurcation equations.

Build up the bifurcation chart, and one or more bifurcation diagrams, for η=0 as well as $\eta \neq 0$.

Show some numerical integrations comparing exact and asymptotic solutions.

Homework # 16 **D.L.:** ***average***

A three-dimensional system, affected by imperfections, undergoing a Hopf-Divergence bifurcation

Consider the following dynamical system, with η being the imperfection:

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega^2 x - b_1(y-x)^2(\dot{y}-\dot{x}) = 0 \\ \dot{y} - \nu y + b_2(y-x)^2(\dot{y}-\dot{x}) + \eta = 0 \end{cases}$$

Apply the MSM to obtain a first-order approximation for the bifurcation equations.

Build up the bifurcation chart, and one or more bifurcation diagrams, for η=0 as well as $\eta\neq 0$.

Show some numerical integrations comparing exact and asymptotic solutions

Homework # 17 **D.L.: *average***

A five-dimensional system, undergoing a Hopf-Divergence bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu \dot{x} + \omega^2 x + c_1 x^3 - b_0 (y-x)^2 (\dot{y} - \dot{x}) - c_0 (y-x)^3 + b_1 \dot{x} z = 0 \\ \ddot{y} + \xi \dot{y} - \nu y + c_1 y^3 + b_0 (y-x)^2 (\dot{y} - \dot{x}) + c_0 (y-x)^3 + b_2 \dot{y} z = 0, \qquad \xi > 0 \\ \dot{z} + kz + c_2 (x^2 + y^2) = 0, \qquad k > 0 \end{cases}$$

Apply the MSM to obtain a second-order approximation for the bifurcation equations.

Build up the bifurcation chart, and one or more bifurcation diagrams.

Investigate and comment the role of the passive variable.

Homework # 18 **D.L.:** ***average***

Two coupled oscillators, undergoing a non-defective, 1:1 resonant, double Hopf bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu \dot{x} + \omega^2 x + c x^3 - b_0 (y - x)^2 (\dot{y} - \dot{x}) = 0 \\ \ddot{y} - \nu \dot{y} + (\omega + \sigma)^2 y + c y^3 + b_0 (y - x)^2 (\dot{y} - \dot{x}) = 0 \end{cases}$$

Apply the MSM to obtain a first-order approximation for the bifurcation equations.

Fix two parameters (e.g. σ and ν) and find, if necessary by numerical methods, the fixed points of the reduced system as function of the third parameter (e.g. μ). Then analyze stability of fixed points.

Illustrate by graphs the results found by the MSM, and compare them with the exact, numerical, results.

Homework # 19 **D.L.: *high***

Two coupled oscillators, with quadratic nonlinearities, undergoing a non-defective, 1:1 resonant, double Hopf bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega^2 x + c_1 x^3 + c_2 x^2 - b_0 (y-x)^2 (\dot{y} - \dot{x}) = 0 \\ \ddot{y} - \nu\dot{y} + (\omega+\sigma)^2 y + c_1 y^3 + c_2 y^2 + b_0 (y-x)^2 (\dot{y} - \dot{x}) = 0 \end{cases}$$

Apply the MSM to obtain a second-order approximation for the bifurcation equations.

Fix two parameters (e.g. σ and ν) and find, if necessary by numerical methods, the fixed points of the reduced system as function of the third parameter (e.g. μ). Then analyze stability of fixed points.

Illustrate by graphs the results found by the MSM, and compare them with the exact, numerical, results.

Homework # 20 **D.L.: *high***

A five-dimensional system, undergoing a non-defective, 1:1 resonant, double Hopf bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu \dot{x} + \omega^2 x + c_1 x^3 - b_0 (y-x)^2 (\dot{y} - \dot{x}) + b_3 z \dot{x} = 0 \\ \ddot{y} - \nu \dot{y} + (\omega + \sigma)^2 y + c_1 y^3 + b_0 (y-x)^2 (\dot{y} - \dot{x}) + b_3 z \dot{y} = 0 \\ \dot{z} + kz + c_2 (x^2 + y^2) = 0, \qquad k > 0 \end{cases}$$

Apply the MSM to obtain a second-order approximation for the bifurcation equations.

Fix two parameters (e.g. σ and ν) and find, if necessary by numerical methods, the fixed points of the reduced system as function of the third parameter (e.g. μ). Then analyze stability of fixed points.

Investigate and comments the role of the passive variable z.

Homework # 21 **D.L.: *average***

Two coupled oscillators, undergoing a 1:3 resonant, double Hopf bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega^2 x + cx^3 - b_0(y-x)^2(\dot{y}-\dot{x}) = 0 \\ \ddot{y} - \nu\dot{y} + (3\omega+\sigma)^2 y + cy^3 + b_0(y-x)^2(\dot{y}-\dot{x}) = 0 \end{cases}$$

Apply the MSM to obtain a first-order approximation for the bifurcation equations.

Fix two parameters (e.g. σ and ν) and find, if necessary by numerical methods, the fixed points of the reduced system as function of the third parameter (e.g. μ). Then analyze stability of fixed points.

Illustrate by graphs the results found by the MSM, and compare them with the exact, numerical, results.

Homework # 22 **D.L.: *high***

Two coupled oscillators, with quadratic nonlinearities added, undergoing a 1:3 resonant, double Hopf bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega^2 x + c_1 x^3 + c_2 x^2 - b_0 (y - x)^2 (\dot{y} - \dot{x}) = 0 \\ \ddot{y} - \nu\dot{y} + (3\omega + \sigma)^2 y + c_1 y^3 + c_2 y^2 + b_0 (y - x)^2 (\dot{y} - \dot{x}) = 0 \end{cases}$$

Apply the MSM to obtain a second-order approximation for the bifurcation equations.

Fix two parameters (e.g. σ and ν) and find, if necessary by numerical methods, the fixed points of the reduced system as function of the third parameter (e.g. μ). Then analyze stability of fixed points.

Illustrate by graphs the results found by the MSM, and compare them with the exact, numerical, results.

Homework # 23 **D.L.: *high***

A five-dimensional system, undergoing a 1:3 resonant, double Hopf bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x}-\mu\dot{x}+\omega^2 x+c_1 x^3-b_0(y-x)^2(\dot{y}-\dot{x})+b_3 z\dot{x}=0 \\ \ddot{y}-\nu\dot{y}+(3\omega+\sigma)^2 y+c_1 y^3+b_0(y-x)^2(\dot{y}-\dot{x})+b_3 z\dot{y}=0 \\ \dot{z}+kz+c_2(x^2+y^2)=0, \qquad k>0 \end{cases}$$

Apply the MSM to obtain a second-order approximation for the bifurcation equations.

Fix two parameters (e.g. σ and ν) and find, if necessary by numerical methods, the fixed points of the reduced system as function of the third parameter (e.g. μ). Then analyze stability of fixed points.

Investigate and comments the role of the passive variable z.

Homework # 24 **D.L.:** ***average***

Two coupled oscillators, undergoing a 1:2 resonant, double Hopf bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu\dot{x} + \omega^2 x + bx^2\dot{x} + cx^3 - b_0(\dot{y} - \dot{x})^2 = 0 \\ \ddot{y} - \nu\dot{y} + (2\omega + \sigma)^2 y + by^2\dot{y} + cy^3 + b_0(\dot{y} - \dot{x})^2 = 0 \end{cases}$$

Apply the MSM to obtain a second-order approximation for the bifurcation equations.

Fix two parameters (e.g. σ and ν) and find, if necessary by numerical methods, the fixed points of the reduced system as function of the third parameter (e.g. μ). Then analyze stability of fixed points.

Illustrate by graphs the results found by the MSM, and compare them with the exact, numerical, results.

Homework # 25 **D.L.: *high***

A five-dimensional system, undergoing a 1:2 resonant, double Hopf bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x}-\mu\dot{x}+\omega^2 x+b_1 x^2\dot{x}+cx^3-b_0(\dot{y}-\dot{x})^2+b_3 z\dot{x}=0 \\ \ddot{y}-\nu\dot{y}+(2\omega+\sigma)^2 y+b_1 y^2\dot{y}+cy^3+b_0(\dot{y}-\dot{x})^2+b_3 z\dot{y}=0 \\ \dot{z}+kz+c_2(x^2+y^2)=0, \qquad k>0 \end{cases}$$

Apply the MSM to obtain a second-order approximation for the bifurcation equations.

Fix two parameters (e.g. σ and ν) and find, if necessary by numerical methods, the fixed points of the reduced system as function of the third parameter (e.g. μ). Then analyze stability of fixed points.

Investigate and comments the role of the passive variable z.

Homework # 26 **D.L.:** ***average***

Two Van der Pol-Duffing coupled oscillators undergoing a double-zero bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu\dot{x} - \nu x + bx^2\dot{x} - c(y-x)^3 = 0 \\ \qquad \ddot{y} + \xi\dot{y} + \omega^2 y + c(y-x)^3 = 0 \qquad \xi > 0 \end{cases}$$

Apply the MSM, up to the highest order terms appear in the bifurcation equation.

Investigate and comments the role of the passive variable *y*.

Perform some numerical integrations showing different qualitative behaviors in the parameter space.

Homework # 27a **D.L.: *average***

A three-dimensional system, with cubic nonlinearities, undergoing a double-zero bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu\dot{x} - \nu x + bx^2\dot{x} - c_1(y-x)^3 = 0 \\ \dot{y} + ky + c_2(y-x)^3 = 0 \qquad k > 0 \end{cases}$$

already analyzed in the lecture notes. Put the equation in the state-form (three first-order equations) and apply the MSM to this form.

Compare and comment the two approaches.

Perform some numerical integrations showing different qualitative behaviors in the parameter space.

Homework # 27b **D.L.: *average***

A three-dimensional system, with cubic nonlinearities, undergoing a double-zero bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu\dot{x} - \nu x + bx^2\dot{x} - c_1(y-x)^3 = 0 \\ \dot{y} + ky + c_2(y-x)^3 = 0 \qquad k>0 \end{cases}$$

already analyzed in the lecture notes by the MSM. Apply, in contrast, the Center Manifold Method and the Normal Form Theory, and re-obtain the same bifurcation equation.

Perform some numerical integrations showing different qualitative behaviors in the parameter space.

Homework # 28 **D.L.: *average***

A three-dimensional system, with cubic nonlinearities, affected by imperfections, undergoing a double-zero bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu \dot{x} - \nu x + b x^2 \dot{x} - c_1 (y - x)^3 + \eta = 0 \\ \dot{y} + k y + c_2 (y - x)^3 = 0 \qquad k > 0 \end{cases}$$

Apply the MSM, up to the highest order terms appear in the bifurcation equation.

Perform some numerical integrations highlighting the effects of the imperfections on the perfect system.

Homework # 29 **D.L.: *high***

A three-dimensional system, with quadratic and cubic nonlinearities, undergoing a double-zero bifurcation

Consider the following dynamical system:

$$\begin{cases} \ddot{x} - \mu\dot{x} - \nu x + b_1 x^2 + b_2 x\dot{x} + b_3 x^2 \dot{x} - c_1 (y - x)^3 = 0 \\ \dot{y} + ky + c_2 (y - x)^3 = 0 \qquad k > 0 \end{cases}$$

Apply the MSM, up to the highest order terms appear in the bifurcation equation.

Perform some numerical integrations showing different qualitative behaviors in the parameter space.

Homework # 30 **D.L.: *high***

A three-dimensional system, with quadratic and cubic nonlinearities, affected by imperfections, undergoing a double-zero bifurcation

Consider the following dynamical system, with η being the imperfection:

$$\begin{cases} \ddot{x} - \mu\dot{x} - \nu x + b_1 x^2 + c_1 xy + b_2 x\dot{x} + b_3 x^2\dot{x} + \eta = 0 \\ \dot{y} + ky + c_2 x^2 = 0 \qquad k > 0 \end{cases}$$

Apply the MSM, up to the highest order terms appear in the bifurcation equation.

Perform some numerical integrations highlighting the effects of the imperfections on the perfect system.

Homework # 31 **D.L.: *highest***

A four-dimensional system, with quadratic and cubic nonlinearities, undergoing a triple-zero bifurcation

Consider the following dynamical system:

$$\begin{cases} \dddot{x}-\rho\ddot{x}-\mu\dot{x}-\nu x+c_1x^2+c_2xy+b_1x\dot{x}+b_2y\dot{x}+b_3x\ddot{x}+b_4\dot{x}^2+b_5\ddot{x}^2=0 \\ \dot{y}+ky+c_3x^2=0 \qquad k>0 \end{cases}$$

By using the MSM, analyze the bifurcation around $\boldsymbol{\mu}:=(\mu,\nu,\rho)=\mathbf{0}$. Show that the bifurcation equation is of the following type:

$$\dddot{a}=\mathrm{L}(\underbrace{a\boldsymbol{\mu},\ a^2}_{\varepsilon^{3/3}};\ \underbrace{\dot{a}\boldsymbol{\mu},\ a\dot{a}}_{\varepsilon^{4/3}};\ \underbrace{\ddot{a}\boldsymbol{\mu},\ a\ddot{a},\ \dot{a}^2}_{\varepsilon^{5/3}};\ \underbrace{\boldsymbol{\mu}^3,\ a^2\boldsymbol{\mu},\ a^3,\ \dot{a}\ddot{a}}_{\varepsilon^{6/3}};\ \underbrace{\dot{a}\boldsymbol{\mu}^2,\ a^2\dot{a},\ a\dot{a}\boldsymbol{\mu},\ \ddot{a}^2}_{\varepsilon^{7/3}})$$

with L a linear operator. Find the equilibria and analyze their stability. Compare some asymptotic and numerical solutions.

Hint: rescale the variables as $(x,y)\to(\varepsilon x,\varepsilon y)$, expand them in series of $\varepsilon^{1/3}$, and introduces times $t_k=\varepsilon^{k/3}t,\ k=0,1,2,\cdots$.

Homework # 32 **D.L.: *highest***

A four-dimensional system, with quadratic and cubic nonlinearities, undergoing a defective double-Hopf bifurcation

Consider the following dynamical system:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} - \begin{pmatrix} \mu & 0 \\ \nu & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{pmatrix} \omega^2 & 1 \\ \sigma & \omega^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -c_0(y-x)^2 - b_0(\dot{y}-\dot{x})(y-x)^2 \\ c_0(y-x)^2 + b_0(\dot{y}-\dot{x})(y-x)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Apply the MSM, up to the highest order terms appear in the bifurcation equation.

Study the fixed points of the bifurcation equations. Plot bifurcation diagrams in one or two parameters, the other being fixed.