AN INTRODUCTION OF SCALAR CONSERVATION LAW

BY LI CHEN

The main reference of this part is from Salsa's book.

Contents

1. Conservation Laws

Usually, we call $u(x,t)$ is the density and $q(u(x,t))$ is the flux function. In 1-dimensional case, for any interval $(x_1,x_2)\in\mathbb{R}$, we know that the change of total mass in (x_1,x_2) with respect to t equals the difference between the flux function at the endpoint, i.e.

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)dx = -q(u(x_2,t)) + q(u(x_1,t)).$$

If furthermore u and q are smooth functions, we will have

$$\int_{x_1}^{x_2}\Big(u_t + [q(u(x,t))]_x\Big)dx = 0.$$

Now because of the arbitrariness of (x_1,x_2), we have the conservation law in the differential form,

$$u_t + [q(u)]_x = 0, \qquad \text{in } \mathbb{R}, \tag{1.1}$$

- If $q(u) = vu$, with v be a given function, then it is just the linear transport equation.
- If $q(u)$ is a given nonlinear function of u, it is called nonlinear conservation law. We will give two examples in this part: Burger's equation and the traffic flow equation.
  - Burger's equation $q(u) = \dfrac{1}{2}u^2$,

– Traffic flow $q(u) = v(u)u$, with $v(u) = v_m(1 - \frac{u}{u_m})$, where v_m is the maximum speed of the cars and u_m is the maximum density.

2. Local existence and smooth solutions

We will take Burger's equation as an example to show the local existence, the reader can discuss the same problem for traffic flow problem in the same way.

$$\begin{aligned} &\rho_t + \rho\rho_x = 0 \quad x \in \mathbb{R}, t > 0 \\ &\rho|_{t=0} = g(x), \quad x \in \mathbb{R} \end{aligned} \tag{2.1}$$

Formally, by using characteristic method, the Cauchy problem of Burger's equation is reduced into

$$\frac{d}{dt}\rho(x(t, x_0), t)) = 0, \quad \rho(x(0), 0) = g(x_0)$$

where the characteristic lines $x(t, x_0)$ satisfies

$$\frac{dx}{dt} = \rho(x, t), \quad x|_{t=0} = x_0. \quad \Leftrightarrow \quad x = x_0 + \rho(x, t)t$$

Then the solution of Burger's equation satisfies an equation

$$\rho(x, t) = g(x - \rho(x, t)t).$$

Theorem 2.1. *If the initial data $g(x) \in C^1(\mathbb{R})$ and*

$$\min_{\mathbb{R}} g'(x) \geq -a > -\infty, \text{ for some } a \geq 0,$$

then problem (2.1) has a unique C^1 solution in domain $\{(x, t) : x \in \mathbb{R}, 0 \leq t < 1/a\}$.

Proof. Let $F(x, t, \rho) = \rho - g(x - \rho t)$, then

$$F_\rho(x, t, \rho) = 1 + g'(x - \rho t)t \geq 1 - at \geq a\varepsilon > 0, \text{ on } \bar{Q}_{a\varepsilon} = \{(x, t) : x \in \mathbb{R}, 0 \leq t \leq 1/a - \varepsilon\}.$$

With the help of implicit function theorem, we have that

$$F(x, t, \rho) = 0$$

has a solution $\rho \in C^1(\bar{Q}_{a\varepsilon})$. □

For general equations, the Cauchy problem is

$$\begin{aligned} &u_t + [q(u)]_x = 0, \quad x \in \mathbb{R}, t > 0 \\ &u|_{t=0} = g(x), \quad x \in \mathbb{R} \end{aligned} \tag{2.2}$$

The global existence of smooth solution exists in the case that the second derivative of q and the first derivative of g have the same sign, which means that the characteristics don't interact each other.

Theorem 2.2. *If $q \in C^2$, $g \in C^1$ and $q'' \cdot g' \geq 0$, then the C^1 solution is uniquely determined by*

$$u(x, t) = g(x - q'(u)t), \tag{2.3}$$

With the method of characteristics we know that the first appearance of shock happens at time t such that

$$1 + tq''(u)g'(x - q'(u)t) = 0.$$

For example, in the case $q(u) = u - u^2$ and $g(x) = \arctan x$,

$$q'(u) = 1 - 2u, \quad q''(u) = -2 < 0, \quad g(x) = \arctan x, \quad g'(x) = \frac{1}{1+x^2} > 0$$

For any $(x,t) \in \mathbb{R} \times (0, +\infty)$, let x_0 be the initial point that can be connected to (x,t) by characteristics. Since $u(x,t)$ is constant along characteristics, i.e. $u(x,t) = g(x_0)$, we have

$$q''(u(x,t))g'(x - q'(u(x,t))t) = q''(g(x_0))g'(x_0) = \frac{-2}{1+x_0^2}.$$

The first time that the continuity of the solution breaks down appears at time t such that

$$1 + t\Big(\frac{-2}{1+x_0^2}\Big) = 0.$$

This means the possible time is $t_s = \frac{1}{2}$ at point $x_0 = 0$.

At time $0 \le t < \frac{1}{2}$, we know the solution is uniquely determined by

$$u - \arctan(x - (1-2u)t) = 0.$$

At time $t > \frac{1}{2}$, it would appear multi-valued functions near $t_s = \frac{1}{2}$. An idea to determine the solution is equal area method. For those who are interested in equal area method, please check Salsa's book [1] Page.176-177.

This example also tells us that we need more theory on conservation law with discontinuous initial data. In this course, we only introduce the Cauchy problem with simplest discontinuous initial data, which is called Riemann problem. We will explain the theory of Riemann problem for traffic flow equation in the next section. The same theory holds for Burger's equation.

3. Riemann problem for traffic flow and Burger's equation

We will explain the theory of Riemann problem for traffic flow equation in the first subsection in detail. Then using the same method for Burger's equation in the second subsection.

3.1. **Traffic flow problem.** This part is almost a copy of Salsa's book [1] 4.3.3 and 4.3.4, one can check that book for more details and some pictures.

we will use ρ to represent the density of cars. The Cauchy problem is

$$\begin{aligned} \rho_t + v_m(1 - \frac{2\rho}{\rho_m})\rho_x = 0 \qquad & \text{in } \mathbb{R} \times (0, +\infty) \\ \rho|_{t=0} = g(x) \qquad & \end{aligned} \tag{3.1}$$

where ρ_m is the maximum density and v_m is the maximum velocity.

By method of characteristics, formally we know that the density is conserved along characteristics, so that one can rewrite (3.1) into

$$\begin{aligned} &\frac{d}{dt}\rho(x(t,x_0),t) = 0, \qquad \rho(x(0,x_0),0) = g(x_0) \\ &\dot{x}(t) = v_m\Big(1 - \frac{2\rho}{\rho_m}\Big). \qquad x|_{t=0} = x_0, \end{aligned}$$

which means $\rho(x(t),t) = g(x_0)$ and

$$\dot{x}(t) = v_m\Big(1 - \frac{2g(x_0)}{\rho_m}\Big), \quad \Rightarrow x(t) = v_m\Big(1 - \frac{2g(x_0)}{\rho_m}\Big)t + x_0, \quad \Rightarrow x_0 = x - v_m\Big(1 - \frac{2g(x_0)}{\rho_m}\Big)t.$$

Thus the formal solution of traffic flow problem is

$$\rho(x,t) = g\Big(x - v_m\Big(1 - \frac{2g(x_0)}{\rho_m}\Big)t\Big). \tag{3.2}$$

with $v_m\Big(1 - \frac{2g(x_0)}{\rho_m}\Big) = q'(g(x_0))$ is the traveling wave propagation speed.

From here one can see that the existence of solution totally depends on the shape of initial data $g(x)$.

We will study the two typical initial data.

3.1.1. *Green light problem.* Initial data

$$g(x) = \begin{cases} \rho_m & x \le 0 \\ 0 & x > 0 \end{cases} \tag{3.3}$$

In this case, the wave speed is

$$q'(g(x)) = \begin{cases} -v_m & x \le 0 \\ v_m & x > 0 \end{cases} \tag{3.4}$$

Now we know in (x,t) plan, the solution can be determined in the domains on the right hand side of $x > v_m t$ and on the left hand side of $x < -v_m t$, i.e.

$$\rho(x,t) = \begin{cases} \rho_m & x < -v_m t \\ 0 & x > v_m t \end{cases}$$

The above analysis shows that by the method of characteristics, we don't know how to determine the value of solution in the domain $-v_m t < x < v_m t$. One way to find the reasonable representation of solution inside of this domain is to use approximation. More precisely, we will use function $g_\varepsilon(x)$ instead of the initial data $g(x)$,

$$g_\varepsilon(x) = \begin{cases} \rho_m & x \le 0 \\ \rho_m(1 - \frac{x}{\varepsilon}) & 0 < x < \varepsilon \\ 0 & x > \varepsilon \end{cases} \tag{3.5}$$

It is easy to calculate the characteristics of this problem, i.e.

$$x = \begin{cases} v_m t + x_0 & x_0 < 0 \\ -v_m(1 - 2\frac{x_0}{\varepsilon})t + x_0 & 0 \le x_0 < \varepsilon \\ v_m t + x_0 & x_0 \ge \varepsilon \end{cases}$$

The characteristics inside of the region $0 \le x_0 < \varepsilon$ looks like a rarefaction fan.

Now the solution of traffic flow problem with initial data (3.5) by characteristic method is

$$\rho_\varepsilon(x,t) = \begin{cases} \rho_m & x < -v_m t \\ \rho_m(1 - \frac{x + v_m t}{2v_m t + \varepsilon}) & -v_m t < x < v_m t + \varepsilon \\ 0 & x > v_m t + \varepsilon \end{cases} \tag{3.6}$$

Letting $\varepsilon \to 0$, the solution is

$$\rho(x,t) = \begin{cases} \rho_m & x < -v_m t \\ \frac{\rho_m}{2}(1 - \frac{x}{v_m t}) & -v_m t < x < v_m t \\ 0 & x > v_m t \end{cases} \tag{3.7}$$

We can see that the solution in the fan is a self-similar solution of the equation. More precisely, if we try to find the self-similar solution of a 1-d conservation law

$$u_t + [q(u)]_x = 0, \text{ find the solution with form } u(\xi) = u(x/t),$$

we will have

$$-\frac{x}{t^2}u_\xi + (q(u))_\xi \frac{1}{t} = 0, \quad \Rightarrow u_\xi(q'(u)\xi - \xi^2) = 0, \quad \Rightarrow q'(u(\xi)) = \xi.$$

Now in traffic flow problem, we have $q(u) = v_m(1 - \frac{u}{\rho_m})u$ and

$$q'(u) = v_m(1 - \frac{2u}{\rho_m}), \quad \Rightarrow v_m(1 - \frac{2u(\xi)}{\rho_m}) = \xi,$$

which exactly gives the rarefaction wave solution

$$u(\xi) = \frac{1}{2}\rho_m(1 - \frac{\xi}{v_m}) = \frac{\rho_m}{2}(1 - \frac{x}{v_m t}).$$

3.1.2. *Red light problem (or traffic jam).* Initial data

$$g(x) = \begin{cases} \frac{1}{8}\rho_m & x < 0 \\ \rho_m & x > 0 \end{cases} \tag{3.8}$$

In this case, the wave speed is

$$q'(g(x)) = \begin{cases} \frac{3}{4}v_m & x < 0 \\ -v_m & x > 0 \end{cases} \tag{3.9}$$

It is obvious to see that some characteristic lines will hit together as $t > 0$. Then the main problem is how to define the solution with jump discontinuity. Remember that our conservation law comes from the integral version,

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx = -q(\rho(x_2,t)) + q(\rho(x_1,t)).$$

If ρ is smooth except a line $x = s(t)$ for $t \in [t_1, t_2]$, then

$$\frac{d}{dt}\Big(\int_{x_1}^{s(t)} \rho(x,t)dx + \int_{s(t)}^{x_2} \rho(x,t)dx\Big) + q(\rho(x_2,t)) - q(\rho(x_1,t)) = 0.$$

By taking the derivative with respect to t,

$$\frac{d}{dt}\int_{x_1}^{s(t)} \rho(x,t)dx = \int_{x_1}^{s(t)} \rho_t(x,t)dx + \rho^-(s(t),t)\frac{ds}{dt}$$
$$\frac{d}{dt}\int_{s(t)}^{x_2} \rho(x,t)dx = \int_{s(t)}^{x_2} \rho_t(x,t)dx - \rho^+(s(t),t)\frac{ds}{dt}$$

where $\rho^\pm(s(t),t) = \lim_{y \to s(t)\pm} \rho(y,t)$.

We arrive at

$$\int_{x_1}^{x_2} \rho_t(x,t)dx + \Big(\rho^-(s(t),t) - \rho^+(s(t),t)\Big)\frac{ds}{dt} = q(\rho(x_1,t)) - q(\rho(x_2,t)).$$

Letting $x_2 \to s(t)+$ and $x_1 \to s(t)-$, formally we have

$$\Big(\rho^-(s(t),t) - \rho^+(s(t),t)\Big)\frac{ds}{dt} = q(\rho^-(s(t),t)) - q(\rho^+(s(t),t)).$$

Then

$$\frac{ds}{dt} = \frac{q(\rho^+(s(t),t)) - q(\rho^-(s(t),t))}{\rho^+(s(t),t) - \rho^-(s(t),t)}. \tag{3.10}$$

which is called the **Rankine-Hugoniot condition**.

Usually, we call the discontinuity propogating of solution **Shock wave**.

Now in the traffic flow problem with red light initial data (3.8), $\rho^+ = \rho_m$, $\rho^- = \frac{\rho_m}{8}$, $q(\rho^+) = 0$, $q(\rho^-) = \frac{7}{64}v_m\rho_m$,

$$\frac{ds}{dt} = \frac{q(\rho^+) - q(\rho^-)}{\rho^+ - \rho^-} = -\frac{1}{8}v_m,$$

So in this case the solution is

$$\rho(x,t) = \begin{cases} \frac{1}{8}\rho_m & x < -\frac{1}{8}v_m t \\ \rho_m & x > -\frac{1}{8}v_m t \end{cases} \tag{3.11}$$

3.2. **Burger's equation.** Riemann problem of Burger's equation,

$$u_t + uu_x = 0, \quad u|_{t=0} = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$$

The difference between traffic flow and Burger's equation is the flux function $q(u)$, which is concave in traffic flow and convex in Burger's equation.

Then the situations to have rarefaction wave and shock wave in Burger's equation are just opposite to those in traffic flow problem. More precisely, we will have rarefaction wave when $u_l < u_r$ and shock wave when $u_l > u_r$.

In the case of $u_l < u_r$, the rarefaction wave solution is

$$u(x,t) = \begin{cases} u_l & x < u_l t \\ x/t & u_l t \le x \le u_r t \\ u_r & x > u_r \end{cases}$$

In the case of $u_l > u_r$, the shock wave solution is

$$u(x,t) = \begin{cases} u_l & x < \frac{u_l + u_r}{2}t \\ u_r & x > \frac{u_l + u_r}{2}t \end{cases}$$

4. Weak Entropy Solution

We know that except the characteristics don't interact, we couldn't expect smooth solution for conservation laws no matter how smooth the initial data is. As the analysis for traffic flow problem we did in previous section, we need to find new definition of solution within which class the solution exist in some sense.

Now we give the definition of weak solution in the sense of distribution

Definition 1. If u is a bounded function defined on $\mathbb{R} \times [0, +\infty)$, $\forall v \in C_0^\infty(\mathbb{R} \times [0, +\infty))$, the following is true

$$\int_0^{+\infty}\int_{\mathbb{R}} (uv_t + q(u)v_x)dxdt + \int_{\mathbb{R}} g(x)v(x,0)dx = 0, \tag{4.1}$$

then we call u is a weak solution of (2.2).

With this definition, a series of questions should be asked

- If u has some discontinuity, is it determined by in the weak formula? (Answer: R-H condition)
- Is the weak solution unique? (Answer: No.)
- If it is not unique, how to choose the physically correct solution? (Answer: Entropy condition)

4.1. **Rankine-Hugoniot condition.** We answer the first question. Let V be an open subset in $\mathbb{R} \times [0, +\infty)$. If u is piecewise smooth in $\bar{V} = \bar{V}^+ \cup \bar{V}^-$ and $u \in C^1(\bar{V}^+)$ and $u \in C^1(\bar{V}^-)$. Let $\Gamma = \bar{V}^+ \cap \bar{V}^-$,

$$u^{\pm}(x_0, t_0) = \lim_{(y,t)\in V^{\pm} \to (x_0,t_0)} u(y), \qquad \forall (x_0, t_0) \in \Gamma.$$

We assume that $u^+ \neq u^-$. The by the definition of weak solution, $\forall v \in C_0^\infty(K)$ where K is a compact subset in V, we have

$$\begin{aligned} 0 &= \int_0^{+\infty} \int_{\mathbb{R}} [uv_t + q(u)v_x] dxdt \\ &= \int_{V^+} [uv_t + q(u)v_x] dxdt + \int_{V^-} [uv_t + q(u)v_x] dxdt \\ &= -\int_{V^+} [u_t + (q(u))_x] v dxdt - \int_{V^-} [u_t + (q(u))_x] v dxdt \\ &\quad + \int_\Gamma v(u^+, q(u^+))^T \cdot \gamma dl - \int_\Gamma v(u^-, q(u^-))^T \cdot \gamma dl \end{aligned}$$

where γ is the unit outer normal vector of V^+ on Γ. Then by the arbitrariness of v, we have

$$(u^+ - u^-, q(u^+) - q(u^-))^T \cdot \gamma = 0 \quad \text{ along } \Gamma,$$

which means that the discontinuity of u, Γ is a line with

$$\dot{s}(t) = \frac{q(u^+(s,t) - q(u^-(s,t))}{u^+(s,t) - u^-(x,t)}.$$

This is called the Rankine-Hugoniot condition for shock curve Γ. Usually, we abbreviate the notation to be

$$\dot{s}(t) = \frac{[q(u)]}{[u]}. \tag{4.2}$$

4.2. **Nonuniqueness of weak solution.** We will use Burger's equation to give an example,

$$u_t + uu_x = 0, \quad u|_{t=0} = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

We know already that this problem has a rarefaction wave solution

$$u(x,t) = \begin{cases} 0 & x \leq 0 \\ x/t & 0 < x < t \\ 1 & x \geq t \end{cases}$$

However this problem has another weak solution (shock solution) which also satisfies the R-H condition (4.2).

$$w(x,t) = \begin{cases} 0 & x < t/2 \\ 1 & x > t/2 \end{cases}$$

In fact the situation is even worse, for Burger's equation with initial data

$$u|_{t=0} = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases} \qquad u_l < u_r.$$

There is a family of infinite number of weak solutions, i.e., $\forall u_m \in [u_l, u_r]$, $s_m = \frac{u_l + u_m}{2}$,

$$u(x,t) = \begin{cases} u_l & x \le s_m t \\ u_m & s_m t \le x \le u_m t \\ x/t & u_m t \le x \le u_r t \\ u_r & x > u_r t \end{cases}$$

4.3. **Entropy condition.** A natural question to ask is that which is the physically correct solution? We must have some method to select the physically relevant solution, an apriori condition on solutions which distinguishes the correct solution from the others. In the case of $q'' > 0$, we will give a motivation on how to give a reasonable condition to have uniquely determined solution. For the concave case, i.e. $q'' < 0$, one can get similar conditions.

From method of characteristics, the solution is uniquely determined by

$$G(x,t,u) = u - g(x - q'(u)t) = 0.$$

Then implicit function theorem implies if $g' > 0$ and $q'' \ge c > 0$ then

$$u_x = -\frac{G_x}{G_u} = \frac{g'}{1 + tg'q''} \le \frac{g'}{1 + tcg'} \le \frac{E}{t}, \quad E = \frac{1}{c}.$$

We thus have from the mean value theorem,

$$u(x+z,t) - u(x,t) \le \frac{E}{t} z, \quad \forall x, z \in \mathbb{R}, z > 0, t > 0, \tag{4.3}$$

A weak solution satisfies entropy condition is called entropy solution.

Generally speaking, entropy condition (4.3) says that if we fix time $t > 0$, and let x move from $-\infty$ to $+\infty$, then our solution must jump down. It is a one dimensional irreversible property, which is the reason why it is called entropy condition.

From this entropy condition we obtain

- $x \to u(x,t) - \dfrac{E}{t} x$ is decreasing;
- If x is a discontinuity point of $u(x,t)$, then

$$u_+(x,t) < u_-(x,t).$$

In the case that q is strictly convex,

$$q'(u_+) < \frac{q(u_+) - q(u_-)}{u_+ - u_-} < q'(u_-)$$

Then R-H condition implies that if $x = \dot{s}(t)$ is a shock curve, the entropy condition has another representation

$$q'(u_+(x,t)) < \dot{s}(t) < q'(u_-(x,t)). \tag{4.4}$$

Theorem 4.1. *If $q \in C^2(\mathbb{R})$ is convex (or concave), g is bounded, there exists a unique entropy solution of*

$$\begin{cases} u_t + (q(u))_x = 0, & x \in \mathbb{R}, t > 0 \\ u|_{t=0} = g(x), & x \in \mathbb{R} \end{cases}$$

We refer the proof of this theorem to J. Smoller's book.

5. Riemann problem for general scalar conservation law

In this part, we will give the entropy solution of scalar conservation law

$$u_t + (q(u))_x = 0, \quad x \in \mathbb{R}, t > 0 \tag{5.1}$$

$$u|_{t=0} = \begin{cases} u_+, & x > 0 \\ u_-, & x < 0 \end{cases} \tag{5.2}$$

where $u_+ \neq u_-$ are constants.

Theorem 5.1. *If $q \in C^2(\mathbb{R})$ is strictly convex and $q'' \geq h > 0$ we have the following result*

- *If $u_+ < u_-$, then the unique entropy solution is shock wave, i.e.*

$$u(x,t) = \begin{cases} u_+, & x/t > \dot{s}(t) \\ u_-, & x/t < \dot{s}(t) \end{cases}$$

where

$$\dot{s}(t) = \frac{q(u_+) - q(u_-)}{u_+ - u_-}.$$

- *If $u_+ > u_-$, then the unique entropy solution is the rarefaction wave, i.e.*

$$u(x,t) = \begin{cases} u_+, & x/t > q'(u_+) \\ r(x/t), & q'(u_-) < x/t < q'(u_+) \\ u_-, & x/t < q'(u_-) \end{cases}$$

where $r = (q')^{-1}$.

Proof. The shock wave obtained directly from RH condition. Since $u_+ < u_-$, it automatically satisfy the entropy condition.

For the rarefaction wave solution, we try to search the self similar solution $u(x,t) = r(x/t)$, then the equation changes into

$$u_t + (q(u))_x = r'(\frac{x}{t})\frac{1}{t}[q'(r) - \frac{x}{t}] = 0, \quad \Rightarrow \quad q'(r) = \frac{x}{t}.$$

which implies directly that it is a weak solution. It remains to check entropy condition, we only check it in the region

$$q'(u_-)t \leq x \leq q'(u_+)t.$$

$\forall x, z$ such that $q'(u_-)t \leq x < x + z \leq q'(u_+)t$, $\exists 0 < z^* < z$ such that

$$u(x+z,t) - u(x,t) = r(\frac{x+z}{t}) - r(\frac{x}{t}) = r'(\frac{x+z^*}{t})\frac{z}{t} \leq \frac{z}{ht},$$

where we have used $r'(s) = \dfrac{1}{q''(r)} \leq \dfrac{1}{h}$. □

Remark 5.1. The result is similar in the concave flux case, $q'' \leq h < 0$

6. Vanishing Viscosity method

Vanishing viscosity method is natural from physical point of view to study discontinuous solutions for hyperbolic equations. In fluid dynamics, Navier-Stokes system is a viscous version of Euler system.

Roughly speaking, vanishing viscosity is to use $u_t + q(u) = \varepsilon u_{xx}$ as an approximation of $u_t + q(u) = 0$, then take $\varepsilon \to 0$, one could get desired properties of solution. Usually, parabolic equations are easier, and the solutions have better properties.

The viscous Burger's equation is

$$u_t + [q(u)]_x = \varepsilon u_{xx}. \tag{6.1}$$

Here we are interested to know how the solution could connect two constant states by studying

$$\lim_{x\to-\infty} u(x,t) = u_L, \qquad \lim_{x\to+\infty} u(x,t) = u_R, \qquad u_L \neq u_R.$$

The method is to study the traveling wave solution of (6.1). That is to say we will study a typical solution $u(x,t) = U(x-vt)$ with undetermined wave speed v. Let $\xi = x - vt$, then the ODE for U is

$$(q'(U) - v)U' = \varepsilon U'', \quad U(-\infty) = u_L, \quad U(+\infty) = u_R.$$

By $U(-\infty) = u_L$ and $U(+\infty) = u_R$, we have $U' \to 0$ as $\xi \to \pm\infty$. Then integrate the equation once, we have

$$q(U) - vU + A = \varepsilon U', \quad q(u_L) - vu_L + A = 0, \quad q(u_R) - vu_R + A = 0.$$

Thus a consequence for wave speed v is that

$$v = \frac{q(u_R) - q(u_L)}{u_R - u_L}, \qquad A = \frac{-q(u_R)u_L + q(u_L)u_R}{u_R - u_L}.$$

So if a traveling wave solution exists, the wave speeding is exactly the shock speed in Rankine-Hugoniot condition.

Now we are going to check whether traveling wave solution exists. Back to the equation for

$$\varepsilon U' = q(U) - vU + A.$$

We know that this equation has two equilibria $U = u_R$ and $U = u_L$. If $q'' < 0$, from ODE theory, we know that only the case $u_L < u_R$ can be connected. If $q'' > 0$, then $u_L > u_R$.

7. Problems

(1) Solve Burger's equation $u_t + uu_x = 0$ with different initial data

$$g_1(x) = \begin{cases} 1 & x \leq 0 \\ 1-x & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

and

$$g_2(x) = \begin{cases} 0 & x \leq 0 \\ 1 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

(2) If $g \in C^1(\mathbb{R})$ has compact support, prove that

$$\begin{cases} \rho_t + v_m(1 - \dfrac{2\rho}{\rho_m})\rho_x = 0 & (x,t) \in \mathbb{R} \times (0,+\infty) \\ \rho(x,0) = g(x) \end{cases}$$

has a locally existed C^1 solution.

(3) Show that, for every $\alpha \geq 1$, the function

$$u_\alpha(x,t) = \begin{cases} 1 & 2x < (1-\alpha)t \\ -\alpha & (1-\alpha)t < 2x < 0 \\ \alpha & 0 < 2x < (\alpha-1)t \\ -1 & (\alpha-1)t < 2x. \end{cases}$$

is a weak solution of the problem

$$\begin{cases} u_t + uu_x = 0 & t > 0, x \in \mathbb{R} \\ u(x,0) = \begin{cases} 1 & x < 0 \\ -1 & x > 0. \end{cases} \end{cases}$$

Is it also an entropy solution, at least for some α?

(4) (Traffic flow in tunnel found by Greenberg in 1959) Study the traffic flow problem with flux

$$q(\rho) = v_m \rho \log \frac{\rho_m}{\rho},$$

and initial data

$$\rho(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0. \end{cases}$$

where v_m is the maximum velocity, ρ_m is the maximum density and $u_l = \frac{1}{2}u_m$, $u_r = \frac{1}{3}u_m$. Give the solution, and draw a picture for partial path of car trajectories.

(5) Find the solutions of

$$\begin{cases} u_t \pm uu_x = 0 & t > 0, x \in \mathbb{R} \\ u(x,0) = x, & x \in \mathbb{R}. \end{cases}$$

(6) Draw the characteristics and describe the evolution for $t \to +\infty$ of the solution of the problem

$$\begin{cases} u_t + uu_x = 0 & t > 0, x \in \mathbb{R} \\ u(x,0) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & x \leq 0 \text{ or } x \geq \pi. \end{cases} \end{cases}$$

References

[1] S. Salsa, Partial differential equations in Action.
[2] L. C. Evans, Partial differential equations.

Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, People's Republic of China
E-mail address: lchen@math.tsinghua.edu.cn