HEAT EQUATION

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Contents

1. Basic facts of Fourier transform

Fourier transform in multi-D is defined by

$$\hat{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} f(x) e^{-ik\cdot x} dx$$

where $k \cdot x = \sum_{i=1}^n k_i x_i$.

We list here a series properties of Fourier transform without proof,

(1) $(\partial_{x_j} f)^\wedge(k) = ik_j \hat{f}(k)$, and $(\partial^\alpha f)^\wedge(k) = i^{|\alpha|} k^\alpha \hat{f}(k)$
(2) $(xf)^\wedge(k) = i\partial_{k_j} \hat{f}(k)$, and $(x^\alpha f)^\wedge(k) = i^{|\alpha|} \partial^\alpha \hat{f}(k)$
(3) $f(x-a)^\wedge(k) = e^{-ia\cdot k} \hat{f}(k)$
(4) $(f(\lambda x))^\wedge(k) = \dfrac{1}{|\lambda|} \hat{f}(\dfrac{k}{\lambda})$, $\forall \lambda \neq 0$.
(5) $(f * g)^\wedge(k) = (2\pi)^{\frac{n}{2}} \hat{f}(k)\hat{g}(k)$

Example 1. *For $x \in \mathbb{R}$, the Fourier transform of $e^{-a|x|}$ is $\frac{1}{\sqrt{2\pi}}\frac{2a}{k^2+a^2}$.*

$$\begin{aligned}\hat{f}(k) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-a|x|}e^{-ixk}dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-a|x|}(\cos(xk) - i\sin(xk))dx \\ &= \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-ax}\cos(xk)dx = \frac{1}{\sqrt{2\pi}}\frac{2a}{k^2+a^2}.\end{aligned}$$

Example 2. *Fourier transform of Gaussian e^{-x^2} in 1-d is $\frac{1}{\sqrt{2}}e^{-\frac{k^2}{2}}$. More general case in multi dimension is $\forall A > 0$*

$$(e^{-A|x|^2})^\wedge(k) = \prod_1^n (e^{-Ax_i^2})^\wedge(k_i) = \frac{1}{(2A)^{\frac{n}{2}}}e^{-\frac{|k|^2}{4A}}.$$

The inverse Fourier transform can be formally given by

$$\check{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(k)e^{ik\cdot x}dk.$$

Example 3. *The inverse Fourier transform of $\frac{1}{\sqrt{2\pi}}e^{-|k|^2 t}$ can be obtained in the following discussions. Let*

$$I(x) = \check{f}(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}e^{-|k|^2 t}e^{ikx}dk = \frac{2}{2\pi}\int_0^\infty e^{-k^2 t}\cos(xk)dk.$$

We know from $\int_0^\infty e^{-y^2}dy = \frac{\sqrt{\pi}}{2}$ that

$$I(0) = \frac{1}{2\sqrt{\pi t}}$$

On the other hand, differentiate $I(x)$ once and do integral by parts, we have

$$I'(x) + \frac{x}{2t}I(x) = 0.$$

Therefore, by solving this ODE, we have

$$I(x) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}.$$

2. Cauchy Problem

The initial value problem of heat equation is

$$u_t - \triangle u = f(x,t), \qquad x \in \mathbb{R}^n, t > 0, \tag{2.1}$$

$$u|_{t=0} = u_0(x). \tag{2.2}$$

2.1. **Solution formula of the problem.** We will find the formal solution of Cauchy problem by Fourier transform. Take Fourier transform in x for equation (2.1) and its initial variable (2.2),

$$\begin{aligned}\hat{u}_t + |k|^2\hat{u} &= \hat{f}(k,t), \qquad k \in \mathbb{R}^n, t > 0, \\ \hat{u}|_{t=0} &= \hat{u}_0(k).\end{aligned}$$

This ODE problem is easy to solve by Duhamel formula with solution

$$\hat{u}(k,t) = e^{-|k|^2 t}\hat{u}_0(k) + \int_0^t e^{-|k|^2(t-\tau)}\hat{f}(k,\tau)d\tau,$$

Now taking the inverse Fourier transform and using its property for convolutions, we have

$$\begin{aligned} u(x,t) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} * u_0(x) + \int_0^t \frac{1}{(4\pi(t-\tau))^{\frac{n}{2}}} e^{-\frac{|x|^2}{4(t-\tau)}} * f(x,\tau) d\tau \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy + \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-\tau))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t-\tau)}} f(y,\tau) dy d\tau. \end{aligned}$$

One can get formally the solution of (2.1)(2.2) by Fourier transform.

$$\begin{aligned} u(x,t) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} * u_0(x) + \int_0^t \frac{1}{(4\pi(t-\tau))^{\frac{n}{2}}} e^{-\frac{|x|^2}{4(t-\tau)}} * f(x,\tau) d\tau \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy + \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-\tau))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t-\tau)}} f(y,\tau) dy d\tau, \end{aligned}$$

It can be seen from here that the function, so called **heat kernel**

$$K(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \tag{2.3}$$

is very important in getting the solution of heat equation. Actually, the solution can be written in the following form

$$u(x,t) = K(x,t) * u_0 + \int_0^t K(x,t-\tau) * f(x,\tau) d\tau. \tag{2.4}$$

which is called **Poisson formula**.

Now take $f \equiv 0$, let's understand the property of $u(x,t)$ given by Poisson formula.

Theorem 2.1. *$f \equiv 0$, if u_0 is a bounded function in $C(\mathbb{R})$, then $u(x,t)$ given by (2.4) is a bounded classical solution of (2.1)(2.2).*

Proof. It is easy to see that $\forall t > 0$, $u(x,t) = K(x,t) * u_0(x)$ is infinitely differentiable. Another fact is that

$$K_t - \triangle K = 0, \quad \forall t > 0.$$

From these, we can obtain $u_t - \triangle u = 0$ in $\mathbb{R}^n \times (0, \infty)$. Now we are left to prove $\forall x_0 \in R^n$,

$$\lim_{x \to x_0, t \to 0+} u(x,t) = u_0(x_0).$$

By changing of variables, we have

$$\begin{aligned} u(x,t) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy \\ &= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-|z|^2} u_0(x + 2\sqrt{t} z) dz \end{aligned}$$

Since u_0 is bounded, this integral is uniformly convergence in x and t. Now taking limit inside of the integral implies that

$$\lim_{x \to x_0, t \to 0+} u(x,t) = \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-|z|^2} u_0(x_0) dz = u_0(x_0).$$

□

Remark 2.1. Some basic properties of solution $u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy$ can be obtained directly from this formula.

(1) If u_0 is periodic (or odd, or even) in x, then so is $u(x,t)$.

(2) Infinite speed of propagation. If $u_0(x) \geq 0$ has support in a small domain, say $\Omega_0 \subset \mathbb{R}^n$, $u(x,t)$ is positive everywhere in $\mathbb{R}^n$.
(3) Infinite differentiability of $u(x,t)$ for $t > 0$.

Now we consider the inhomogeneous equation with homogeneous initial data.

$$\begin{aligned} u_t - \Delta u &= f \quad \text{ in } \mathbb{R}^n \times (0,+\infty) \\ u|_{t=0} &= 0. \end{aligned} \tag{2.5}$$

The solution is

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s) f(y,s)dyds = \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)dyds.$$

Theorem 2.2. *If $f \in C^{2,1}(\mathbb{R}^n \times [0,\infty))$ and has compact support, then $u \in C^{2,1}(\mathbb{R}^n \times [0,\infty))$ is the solution of* (2.5).

Proof. By the regularity of f, we have

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^n} K(y,s)(f_t - \Delta f)(x-y,t-s)dyds + \int_{\mathbb{R}^n} K(y,t) f(x-y,0)dy \\ &= \int_0^\varepsilon \int_{\mathbb{R}^n} + \int_\varepsilon^t \int_{\mathbb{R}^n} + \int_{\mathbb{R}^n} K(y,t)f(x-y,0)dy := J_\varepsilon + I_\varepsilon + L. \end{aligned}$$

We deal with the right hand side term by term,

$$|J_\varepsilon| \leq (\|f_t\|_{L^\infty} + \|D^2 f\|_{L^\infty}) \int_0^\varepsilon \int_{\mathbb{R}^n} K(y,s)dyds \leq \varepsilon C.$$

$$\begin{aligned} I_\varepsilon &= \int_\varepsilon^t \int_{\mathbb{R}^n} K(y,s)(f_t - \Delta f)(x-y,t-s)dyds \\ &= \int_\varepsilon^t \int_{\mathbb{R}^n} (\partial_s - \Delta_y)K(y,s) f(x-y,t-s)dyds + \int_{\mathbb{R}^n} K(y,\varepsilon) f(x-y,t-\varepsilon)dy - \int_{\mathbb{R}^n} K(y,t)f(x-y,0)dy \\ &= \int_{\mathbb{R}^n} K(y,\varepsilon)f(x-y,t-\varepsilon)dy - L. \end{aligned}$$

Thus we have

$$u_t - \Delta u = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} K(y,\varepsilon) f(x-y,t-\varepsilon)dy = f(x,t), \quad \forall t > 0.$$

and

$$|u(x,t)| = \left| \int_0^t \int_{\mathbb{R}^n} K(y,s) f(x-y,t-s)dyds \right| \leq t\|f\|_{L^\infty}, \quad \text{ as } t \to 0.$$

□

Remark 2.2. By superposition principle for linear equations, we have

$$u(x,t) = \int_{\mathbb{R}^n} K(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)f(y,s)dyds$$

is the solution of

$$u_t - \Delta u = f, \quad u|_{t=0} = u_0.$$

2.2. **Fundamental solution.** Before introducing the fundamental solution, let's give a basic understanding of Delta function, which was mathematically defined as a distribution. But we will not give a definition here. For those who are interested, please check the detail in the appendix for distributions. Apart from the mathematical definition, we can understand Delta function as a limit of those functions whose integral is 1 and whose limit is $+\infty$ at $x = 0$, 0 at $x \neq 0$. For example, such functions can be taken as follows.

Example 4. *Heat kernel* $K(x,t) = \dfrac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \to \delta(x)$ *as* $t \to 0+$.
$\int_{\mathbb{R}} K(x,t)dx = 1$ *and* $\forall \phi \in C_0^\infty$,

$$\int_{\mathbb{R}} K(x,t)\phi(x)dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} \phi(2\sqrt{t}y)dy \to \phi(0).$$

by dominated convergence.

Example 5. $Q_n(x) = \begin{cases} \frac{n}{2} & |nx| < 1, \\ 0 & |nx| \geq 1. \end{cases} \to \delta(x)$ *as* $n \to \infty$.
$\int_{\mathbb{R}} Q_n(x)dx = 1$ *and* $\forall \phi \in C_0^\infty$,

$$\int_{\mathbb{R}} Q_n(x)\phi(x)dx = \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{n}{2}\phi(x)dx \to \phi(0).$$

Example 6. *Dirichlet kernel* $D_n(x) = \dfrac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}} = 1 + 2\sum_{k=1}^n \cos kx \to 2\pi\delta(x)$ *as* $n \to \infty$.
$\int_{\mathbb{R}} D_n(x)dx = 2\pi$ *and* $\forall \phi \in C_0^\infty$,

$$\int_{-\pi}^{\pi} D_n(x)\phi(x)dx = \int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}\phi(x)dx \to 2\pi\phi(0),$$

which can be proved by using Riemann's lemma and similar argument to the proof of Fourier inverse transform we did for $L^1 \cap C^1$ *functions in the appendix.*

Next we give some motivations in defining fundamental solutions. Formally, the right hand side function $f(x,t)$, the heat source, can be represented by, $\forall t > 0$

$$f(x,t) = \int_{\mathbb{R}^n} \int_0^\infty \delta(x-\xi, t-\tau) f(\xi,\tau) d\xi d\tau,$$

which means that $f(x,t)$ can be treated as a summation of $\delta(x-\xi, t-\tau)f(\xi,\tau)d\xi d\tau$, the point heat source. Then we can expect that if $K(x,t;\xi,\tau)$ is the solution of $u_t - \triangle u = \delta(x-\xi, t-\tau)$, then $K(x-\xi, t-\tau)f(\xi,\tau)d\xi d\tau$ is the solution with point heat source. Then we can imagine that in the case of heat source $f(x,t)$, the solution is

$$u(x,t) = \int_{\mathbb{R}^n} \int_0^\infty K(x-\xi, t-\tau) f(\xi,\tau) d\xi d\tau.$$

Basically, the fundamental solution of heat equation is to find the temperature distribution with a point heat source at (ξ,τ).

Definition 1. $K(x,t;\xi,\tau) = K(x-\xi, t-\tau)$ is called the **fundamental solution** of heat equation.

Let $Q = \mathbb{R}^n \times (0, \infty)$. $\forall (\xi, \tau) \in Q$. $K(x, t; \xi, \tau)$ is a solution (in the sense of distribution) of the following Cauchy problem

$$\begin{aligned} u_t - \triangle u &= \delta(x - \xi, t - \tau), \\ u|_{t=0} &= 0. \end{aligned}$$

For an introduction of distribution, the readers are referred to appendix.

Remark 2.3. We also know that $K(x, t; \xi, \tau)$ is a solution of

$$\begin{aligned} u_t - \triangle u = 0, \qquad \forall x \in \mathbb{R}^n, t \geq \tau, \\ u|_{t=\tau} = \delta(x - \xi). \end{aligned}$$

Some properties of fundamental solution

(1) $K(x, t; \xi, \tau) > 0$ for $t > \tau$.
(2) $K(x, t; \xi, \tau) = K(\xi, t; x, \tau)$.
(3) $\forall x \in \mathbb{R}^n$, $t > \tau$,

$$\int_{\mathbb{R}^n} K(x, t; \xi, \tau) d\xi = 1.$$

(4) $\forall x, \xi \in \mathbb{R}^n$, $t > \tau$,

$$\begin{aligned} (\partial_t - \triangle_x) K(x, t; \xi, \tau) = 0 \\ (\partial_\tau + \triangle_\xi) K(x, t; \xi, \tau) = 0 \end{aligned}$$

(5) If $\varphi(x)$ is a bounded continuous function in $\mathbb{R}^n$, then

$$\lim_{t \to 0+} \int_{\mathbb{R}^n} K(x, t; \xi, 0) \varphi(\xi) d\xi = \varphi(x).$$

(6) $K(x, t; \xi, \tau)$ is infinitely differentiable and $\exists M > 0$ s.t. in the case of $t > \tau$,

$$|K(x, t; \xi, \tau)| \leq \frac{M}{(t - \tau)^{\frac{n}{2}}}.$$

Remark 2.4. There is another derivation of fundamental solution instead of using Fourier transform. Once can check this method in Evan's book.

2.3. **viscous Burger's equation-Cole Hopf transformation in 1950's.** In 1950's Cole and Hopf found a transformation independently to reduce the viscous Burger's equation into a heat equation. This transformation is now called Cole-Hopf transformation. Then by using the fundamental solution of heat equation, an exact solution of viscous Burger's equation can be obtained.

Viscous Burger's equation is

$$u_t + u u_x = \varepsilon u_{xx}, \qquad x \in \mathbb{R}, t > 0.$$

which can be rewritten into

$$u_t + \left(\frac{1}{2} u^2 - \varepsilon u_x \right)_x = 0.$$

This formula means that the $2 - D$ vector valued function $(-u, \frac{1}{2}u^2 - \varepsilon u_x)$ is curl free. Therefore, there exists a potential $\psi(x, t)$ such that

$$\psi_x = -u, \qquad \psi_t = \frac{1}{2} u^2 - \varepsilon u_x.$$

So ψ solves the equation

$$\psi_t = \frac{1}{2}\psi_x^2 + \varepsilon\psi_{xx}.$$

Now a way to avoid the quadratic term is using a new function φ such that $\psi = g(\varphi)$ with relation g to be determined later.

$$\psi_t = g'(\varphi)\varphi_t, \quad \psi_x = g'(\varphi)\varphi_x, \quad \psi_{xx} = g''(\varphi)(\varphi_x)^2 + g'(\varphi)\varphi_{xx}$$

Then the equation for φ is

$$g'(\varphi)[\varphi_t - \varepsilon\varphi_{xx}] = [\frac{1}{2}(g'(\varphi))^2 + \varepsilon g''(\varphi)](\varphi_x)^2.$$

Now we choose g such that the right hand side vanish. $g(s) = 2\varepsilon \log s$, then the equation for φ reduce to heat equation,

$$\varphi_t - \varepsilon\varphi_{xx} = 0.$$

Then the relation between u and φ is

$$u = -\psi_x = -2\varepsilon\frac{\varphi_x}{\varphi}$$

which is called the **Cole-Hopf transformation**.

The initial data for $u(x,0) = u_0(x)$ is transformed into

$$\varphi_0(x) = \exp\Big\{-\int_a^x \frac{u_0(z)}{2\varepsilon}dz\Big\}, \quad a \in \mathbb{R}.$$

If

$$\frac{1}{x^2}\int_a^x u_0(z)dz \to 0, \qquad \text{as } |x| \to \infty,$$

Then the Cauchy problem for φ has a unique smooth solution,

$$\varphi(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}}\int_{-\infty}^{+\infty}\varphi_0(y)\exp\Big\{-\frac{(x-y)^2}{4\varepsilon t}\Big\}dy.$$

Changing back to the original variables, we know that the Cauchy problem for viscous Burger's equation has solution

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{+\infty}\frac{x-y}{t}\varphi_0(y)\exp\Big\{-\frac{(x-y)^2}{4\varepsilon t}\Big\}dy}{\displaystyle\int_{-\infty}^{+\infty}\varphi_0(y)\exp\Big\{-\frac{(x-y)^2}{4\varepsilon t}\Big\}dy}.$$

3. Half space problem and its Green's function

The main purpose of this section is to give a first insight on how to build up a Green's function on general problem.

Consider the problem

$$\begin{aligned} u_t - u_{xx} &= 0, \quad x \in (0,+\infty), t > 0 \\ u|_{t=0} &= \varphi, \quad x \in (0,+\infty) \\ u|_{x=0} &= 0, \quad t > 0. \end{aligned} \tag{3.1}$$

We want to find a function $G(x,t,\xi,0)$ such that the solution of (3.1) can be represented by

$$u(x,t) = \int_0^\infty G(x,t;\xi,0)\varphi(\xi)d\xi.$$

The important thing here is that we must make sure the solution satisfies boundary condition $u|_{x=0} = 0$. $\forall \xi \in (0, +\infty)$, if the initial data is $\delta(x-\xi)$, we need to find the odd extension of it, i.e. $-\delta(x+\xi)$ to balance the boundary condition. Now we can choose the initial data is

$$\delta(x-\xi) - \delta(x+\xi),$$

and solve the Cauchy problem with this initial data. Since the problem is linear, our the solution should be

$$K(x,t;\xi,0) - K(x,t;-\xi,0).$$

Thus the Green's function for half space problem (3.1) can be chosen to be

$$G(x,t;\xi,0) = K(x,t;\xi,0) - K(x,t;-\xi,0),$$

and the solution of (3.1) is expected to be $u(x,t) = \int_0^\infty G(x,t;\xi,0)\varphi(\xi)d\xi$.

Theorem 3.1. *φ is a bounded smooth function on $(0,+\infty)$ and $\varphi(0) = 0$, $u(x,t) = \int_0^\infty G(x,t;\xi,0)\varphi(\xi)d\xi$ is the solution of* (3.1).

The proof of this theorem is easy...

Remark 3.1. For inhomogeneous problem

$$\begin{aligned} u_t - u_{xx} &= f, \quad x \in (0,+\infty), t > 0 \\ u|_{t=0} &= \varphi. \quad x \in (0,+\infty) \\ u|_{x=0} &= 0. \quad t > 0 \end{aligned}$$

The formal solution is

$$u(x,t) = \int_0^\infty G(x,t;\xi,0)\varphi(\xi)d\xi + \int_0^t d\tau \int_0^\infty G(x,t;\xi,\tau)f(\xi,\tau)d\xi.$$

Remark 3.2. Similarly, one can find the Green's function for half space problem with homogeneous Neumann boundary condition.

4. Initial boundary value problem

Heat equation with initial boundary value problem in 1-d space variable is

$$\begin{aligned} u_t - u_{xx} &= f, \quad x \in (0,1), t > 0, \\ u|_{t=0} &= \varphi. \quad x \in (0,1), \\ u|_{x=0} = u|_{x=1} &= 0. \quad t > 0. \end{aligned} \tag{4.1}$$

The method of separation of variable is easy to be applied here. It was Fourier who first used this method to solve heat equation, which was the beginning of Fourier analysis.

4.1. **Separation of variable.** First by solving the eigenvalue problem

$$\begin{aligned} X'' + \lambda X &= 0, \qquad x \in (0,1) \\ X(0) = X(1) &= 0, \end{aligned}$$

We have that

$$\lambda_n = (n\pi)^2, \quad X_n = \sin n\pi x.$$

Then if the solution $u(x,t)$ has form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin n\pi x,$$

$T_n(t)$ will solve the initial value problem of an ODE

$$\begin{aligned} T_n' + (n\pi)^2 T_n &= f_n(t), \\ T_n(0) &= \varphi_n, \end{aligned}$$

where

$$f_n(t) = 2\int_0^1 f(x,t) \sin n\pi x dx, \qquad \varphi_n = 2\int_0^1 \varphi(x) \sin n\pi x dx.$$

This ODE problem has a solution

$$T_n(t) = e^{-(n\pi)^2 t}\varphi_n + \int_0^t e^{-(n\pi)^2(t-\tau)} f_n(\tau) d\tau, \quad n = 1, 2, \cdots$$

Thus our formal solution for problem (4.1) can be written as

$$u(x,t) = \sum_{n=1}^{\infty} \sin n\pi x \Big(e^{-(n\pi)^2 t}\varphi_n + \int_0^t e^{-(n\pi)^2(t-\tau)} f_n(\tau) d\tau \Big). \tag{4.2}$$

A natural question is to ask under which condition is (4.2) a $C^{2,1}((0,1) \times (0,\infty))$ solution. Left to reader...

Basic properties of the solution of heat (or more generally, parabolic) equation, "Infinitely differentiable inside of the domain". It is mainly due to the exponential decay in time $t > 0$. More precisely, $\forall (x,t) \in (0,1) \times (0,\infty)$, for any nonnegative integer k, l, the solution given by (4.2) is $(k+l)$-differentiable at (x,t). For example, in the case of $f = 0$, we know that

$$\frac{\partial^{k+l} u(x,t)}{\partial x^k \partial t^l} = \sum_{n=1}^{\infty} (-1)^l (n\pi)^{k+2l} \varphi_n e^{-(n\pi)^2 t} \sin(n\pi x + \frac{k\pi}{2}).$$

The discussion for the case $f \neq 0$ is the same, but the formula is a bit mass, we omit it here.

4.2. **Energy estimates.** We will give the energy estimate for initial boundary value problem of heat equation in multi-dimension. Ω is a bounded open subset of $\mathbb{R}^n$. Let $Q_T = \Omega \times (0,T)$.

$$\begin{aligned} u_t - \triangle u &= f, \quad (x,t) \in Q_T, \\ u|_{t=0} &= \varphi. \quad x \in \Omega, \\ u|_{\partial\Omega} &= 0. \quad t > 0. \end{aligned} \tag{4.3}$$

Theorem 4.1. *If $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ is a solution of problem (4.3), then $\exists M > 0$ depends only on T, s.t.*

$$\sup_{0 \le t \le T} \|u(\cdot, t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)} \le M(\|\varphi\|_{L^2(\Omega)} + \|f\|_{L^2(Q_T)}). \tag{4.4}$$

Proof. Multiplying the equation by u and integrating it in Q_t, doing integral by parts, we have

$$\frac{1}{2}\int_\Omega u^2 dx - \frac{1}{2}\int_\Omega \varphi^2 dx + \int_0^t \int_\Omega |\nabla u|^2 dxdt = \int_0^t \int_\Omega f u dxdt.$$

Young's inequality gives

$$\int_\Omega u^2 dx + 2\int_0^t \int_\Omega |\nabla u|^2 dxdt \le \int_0^t \int_\Omega u^2 dxdt + \int_\Omega \varphi^2 dx + \int_0^t \int_\Omega f^2 dxdt$$

Then (4.4) can be obtained directly from Gronwall's inequality.

□

Remark 4.1. The discuss on uniqueness and stability of solution by energy estimates is similar to what we have done for wave equation.

Remark 4.2. For homogeneous Neumann boundary condition $u \cdot \gamma|_{\partial\Omega} = 0$, where γ be the unit outer normal vector of $\partial\Omega$, the energy estimate is similar.

Remark 4.3. For nonhomogeneous boundary condition, i.e. $u|_{\partial\Omega} = \psi_D$, one can try to homogenize it or just use $u - \psi_D$ as test function.

5. Maximum Principle

Ω is a bounded open subset of $\mathbb{R}^n$. Let $Q_T = \Omega \times (0, T]$ and the parabolic boundary of Q_T be $\partial_p Q_T = \Omega \times \{t = 0\} \cup \partial\Omega \times (0, T]$. $Lu = u_t - \triangle u$.

5.1. **Weak maximum principle.**

Theorem 5.1. *If $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ and $Lu \le 0$ in Q_T, then the maximum of u in $\bar{Q}_T$ must be achieved on $\partial_p Q_T$, i.e.*

$$\max_{\bar{Q}_T} u(x, t) = \max_{\partial_p Q_T} u. \tag{5.1}$$

Proof. We first assume $Lu < 0$ in Q_T. If (5.1) is not true, which means $\exists (x_0, t_0) \in Q_T$ s.t.

$$u(x_0, t_0) = \max_{\bar{Q}_T} u(x, t),$$

then we know that $\nabla u(x_0, t_0) = 0$, $\triangle u(x_0, t_0) \le 0$ and $u_t(x_0, t_0) \ge 0$. Thus,

$$f(x_0, t_0) = Lu(x_0, t_0) \ge 0,$$

which is a contradiction with the assumption $Lu < 0$.

If Lu is non-positive. $\forall \varepsilon > 0$, we will use auxiliary function $v(x, t) = u(x, t) - \varepsilon t$. Now

$$Lv = Lu - \varepsilon = f - \varepsilon < 0.$$

By the conclusion we obtained above, we have

$$\max_{\bar{Q}_T} v = \max_{\partial_p Q_T} v.$$

Going back to the variable u, it gives

$$\begin{aligned}\max_{\bar{Q}_T} u(x,t) &= \max_{\bar{Q}_T}(v+\varepsilon t) \le \max_{\bar{Q}_T} v + \varepsilon T\\ &\le \max_{\partial_p Q_T} v + \varepsilon T = \max_{\partial_p Q_T}(u-\varepsilon t) + \varepsilon T\\ &\le \max_{\partial_p Q_T} u + \varepsilon T.\end{aligned}$$

Let $\varepsilon \to 0$, we have (5.1). □

By the same discussion or just let $v = -u$, we will have

Corollary 5.1. *If $Lu \ge 0$, then*

$$\min_{\bar{Q}_T} u(x,t) = \min_{\partial_p Q_T} u.$$

Furthermore,

Corollary 5.2. *If $Lu = 0$, then both maximum and minimum of u are achieved on the parabolic boundary.*

Now we will have the very useful tool, the **comparison principle**, as a corollary

Corollary 5.3. *If $u, v \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$, $Lu \le Lv$ and $u|_{\partial_p Q_T} \le v|_{\partial_p Q_T}$, then*

$$u(x,t) \le v(x,t) \quad \text{in } \bar{Q}_T.$$

5.2. **Dirichlet Boundary Condition.** The initial boundary value problem of heat equation with Dirichlet BC

$$\begin{aligned} u_t - \triangle u &= f(x,t) \quad \text{in } Q_T\\ u|_{t=0} &= \varphi(x) \\ u|_{\partial\Omega} &= g(x,t).\end{aligned} \tag{5.2}$$

Theorem 5.2. *If $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ is a solution of (5.2), then*

$$\max_{\bar{Q}_T} |u| \le FT + B, \tag{5.3}$$

where $F = \max_{\bar{Q}_T} |f|$, $B = \max\{\max_\Omega |\varphi|, \max_{\partial\Omega\times[0,T]} |g|\}$.

Proof. We will use comparison principle and introduce auxiliary function $w(x,t) = Ft + B \pm u(x,t)$. It is easy to check that

$$\begin{aligned} Lw = F \pm f \ge 0,\\ w|_{\partial_p Q_T} \ge Ft + B \pm g|_{\partial_p Q_T} \ge 0.\end{aligned}$$

By comparison principle, corollary 5.3, we have $w(x,t) \ge 0$ in Q_T, which implies

$$|u| \le FT + B, \quad \text{in } Q_T.$$

□

This maximum estimate can be used to prove the uniqueness and stability of classical solutions.

Corollary 5.4. *$C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ solution of (5.2) is unique.*

Corollary 5.5. *$C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ solution of (5.2) is stable in the following sense. If $u_1, u_2 \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ are solutions separately with data φ_1, f_1, g_1 and φ_2, f_2, g_2, then*

$$\max_{\bar{Q}_T} |u_1 - u_2| \leq \|f_1 - f_2\|_\infty T + \|\varphi_1 - \varphi_2\|_\infty + \|g_1 - g_2\|_\infty.$$

6. Problems

(1) Find the formal solution of the following problem by Fourier transform

$$\begin{aligned} i\partial_t u + \triangle u = 0, &\quad (x,t) \in \mathbb{R}^n \times (0, +\infty), \\ u|_{t=0} = g(x), &\quad x \in \mathbb{R}^n. \end{aligned}$$

(2) (a) Find the formal solution of the following problems

$$\begin{aligned} u_t - \triangle u + 2u = f(x,t), &\quad (x,t) \in \mathbb{R}^n \times (0, \infty), \\ u|_{t=0} = \varphi(x), &\quad x \in \mathbb{R}^n. \end{aligned}$$

(b)

$$\begin{aligned} u_t - u_{xx} + xu = 0, &\quad (x,t) \in \mathbb{R} \times (0, \infty), \\ u|_{t=0} = \varphi(x), &\quad x \in \mathbb{R}. \end{aligned}$$

(c)

$$\begin{aligned} u_t = a^2 u_{xx}, &\quad (x,t) \in (0, +\infty) \times (0, \infty), \\ u|_{t=0} = 0, &\quad x \in (0, +\infty), \\ u_x|_{x=0} = -1, &\quad t > 0. \end{aligned}$$

(3) Find the Green's function of half line problem

$$\begin{aligned} u_t - u_{xx} &= f, \quad x \in (0, +\infty), t > 0, \\ u|_{t=0} &= \varphi, \quad x \in (0, +\infty), \\ u_x|_{x=0} &= 0, \quad t > 0. \end{aligned}$$

And give the formal solution formula of this problem.

(4) Ω is a bounded open subset of $\mathbb{R}^n$, $Q = \Omega \times (0, T]$. If $u \in C^{2,1}(Q) \cap C(\bar{Q})$ is a solution of the following initial boundary value problem,

$$\begin{aligned} u_t - \triangle u = f(x,t), &\quad (x,t) \in Q, \\ u|_{t=0} = \varphi(x), &\quad x \in \Omega, \\ u|_{\partial\Omega} = 1. & \end{aligned}$$

Try to prove there exists a constant C (depends on T and $|\Omega| = \int_\Omega dx$) such that the following inequality holds

$$\sup_{0 \leq t \leq T} \int_\Omega u^2(x,t) dx + \int_0^T \int_\Omega |\nabla u|^2 dx dt \leq C\Big(\int_\Omega \varphi^2 dx + \int_0^T \int_\Omega f^2 dx dt + 1 \Big).$$

(5) Find the formal solution of the following problem by using separation of variable

$$\begin{aligned} u_t = a^2 u_{xx}, &\quad (x,t) \in (0,1) \times (0, \infty), \\ u|_{t=0} = x^2(1-x), &\quad x \in (0,1), \\ u_x|_{x=0} = u|_{x=1} = 0, &\quad t > 0. \end{aligned}$$

(6) Ω is a bounded open subset of $\mathbb{R}^n$, $Q_T = \Omega \times [0, T)$. $c(x, t) \geq -c_0$ with constant $c_0 > 0$. If $u \in C(\bar{Q}_T) \cap C^{2,1}(Q_T)$ satisfies

$$u_t - a^2 \triangle u + c(x, t)u \leq 0, \quad (x, t) \in Q_T.$$

$$u|_{\partial_p Q} \leq 0.$$

Try to prove that $u \leq 0$ in Q_T. (Hint: try to use auxiliary function $e^{-ct}u$)

7. *Appendix: Short review of Fourier transform and distribution

The contents in appendix is not required in this course. I list it here for those who are interested.

7.1. Fourier transform. Let's remind first the Fourier series, $\forall f \in L^1(-l,l)$, which is defined by

$$f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}).$$

where

$$A_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx, \quad n = 0, 1, 2, \cdots$$
$$B_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx, \quad n = 1, 2, \cdots$$

Let

$$S_N(x) = \frac{A_0}{2} + \sum_{n=1}^{N}(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}).$$

Theorem 7.1. *(Convergence in L^2 norm)*

$$\lim_{N\to\infty}\|f - S_N\|_{L^2} = 0, \quad \textit{for } L^2(-l,l).$$

Theorem 7.2. *(Bessel inequality) For $f \in L^2(-l,l)$, it holds*

$$\frac{A_0^2}{2} + \sum_{n=1}^{\infty}(A_n^2 + B_n^2) \le \frac{1}{l}\int_{-l}^{l} f^2 dx.$$

Theorem 7.3. *(Parseval's equality) For $f \in L^2(-l,l)$, it holds*

$$\frac{A_0^2}{2} + \sum_{n=1}^{\infty}(A_n^2 + B_n^2) = \frac{1}{l}\int_{-l}^{l} f^2 dx.$$

By Euler formula, we can change the items in summation into

$$A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l} = A_n' e^{i\frac{n\pi x}{l}} + B_n' e^{-i\frac{n\pi x}{l}}.$$

Thus the Fourier series can be rewritten into

$$f(x) \sim \sum_{-\infty}^{\infty} a_n e^{i\frac{n\pi x}{l}}, \qquad a_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-i\frac{n\pi x}{l}}dx$$

i.e.

$$f(x) \sim \frac{1}{2l}\sum_{-\infty}^{\infty}\int_{-l}^{l} f(y)e^{-i\frac{n\pi y}{l}}dy e^{i\frac{n\pi x}{l}}.$$

Now let $k = \frac{n\pi}{l}$, the formula is

$$f(x) \sim \frac{1}{2\pi}\sum_{-\infty}^{\infty}\int_{-l}^{l} f(y)e^{-iky}dy e^{ikx}\frac{\pi}{l}.$$

As $l \to \infty$, one could expect that

$$f(x) \sim \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)e^{-iky}dy e^{ikx}dk.$$

These formal computations will give the motivation of Fourier transform on $\mathbb{R}$.

Definition 2. $\forall f \in L^1(\mathbb{R})$, its Fourier transform is defined by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx,$$

A result which can be obtained directly from the definition is $\hat{f}(k) \in L^\infty(\mathbb{R})$, i.e.

$$|\hat{f}(k)| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \right| \leq \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}.$$

So by definition, Fourier transform is a continuous linear mapping from L^1 to L^∞. Moreover, if $f \geq 0$, then $\|\hat{f}\|_{L^\infty} = \|f\|_{L^1}$.

Theorem 7.4. *If $f \in L^1(\mathbb{R})$, then $\hat{f}(k)$ is uniformly continuous in $\mathbb{R}$.*

Proof. (For those who are interested) $\forall \varepsilon > 0$, $\exists A > 0$, such that

$$\frac{1}{\sqrt{2\pi}} \int_{|x|>A} 2|f| dx \leq \frac{\varepsilon}{2}.$$

$\forall 0 < h < \dfrac{\sqrt{2\pi}\varepsilon}{4A\|f\|_{L^1}}$, we have

$$\begin{aligned} |\hat{f}(k+h) - \hat{f}(k)| &= \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} f(x) e^{-ixk} [e^{-ixh} - 1] dx \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{|x|>A} 2|f| dx + \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} |x| \cdot |h| \cdot |f| dx \\ &\leq \frac{1}{2\varepsilon} + \frac{1}{2\varepsilon} = \varepsilon. \end{aligned}$$

□

Remark 7.1. Similarly, one can define Fourier transform in multi-D case, $\forall f \in L^2(\mathbb{R}^n)$,

$$\hat{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} f(x) e^{-ik\cdot x} dx$$

where $k \cdot x = \sum_{i=1}^n k_i x_i$. It is also a continuous linear mapping from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

We list here a series properties of Fourier transform here without proof,

(1) $(\partial_{x_j} f)^\wedge(k) = ik_j \hat{f}(k)$, and $(\partial^\alpha f)^\wedge(k) = i^{|\alpha|} k^\alpha \hat{f}(k)$
(2) $(xf)^\wedge(k) = i\partial_{k_j} \hat{f}(k)$, and $(x^\alpha f)^\wedge(k) = i^{|\alpha|} \partial^\alpha \hat{f}(k)$
(3) $f(x-a)^\wedge(k) = e^{-ia\cdot k} \hat{f}(k)$
(4) $(f(\lambda x))^\wedge(k) = \dfrac{1}{|\lambda|} \hat{f}(\dfrac{k}{\lambda})$, $\forall \lambda \neq 0$.
(5) $(f * g)^\wedge(k) = (2\pi)^{\frac{n}{2}} \hat{f}(k) \hat{g}(k)$

Example 7. *Fourier transform of Gaussian e^{-x^2} in 1-d is $\dfrac{1}{\sqrt{2}} e^{-\frac{k^2}{2}}$. More general case in multi dimension is $\forall A > 0$*

$$(e^{-A|x|^2})^\wedge(k) = \prod_1^n (e^{-Ax_i^2})^\wedge(k_i) = \frac{1}{(2A)^{\frac{n}{2}}} e^{-\frac{|k|^2}{4A}}.$$

The **inverse Fourier transform** can be formally given by

$$\check{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(k) e^{ik\cdot x} dk.$$

Theorem 7.5. *If $f \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$, then*

$$\lim_{N\to\infty} \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-N}^{N} \hat{f}(k) e^{ikx} dk = f(x).$$

Proof. (For those who are interested) We know that $\hat{f}(k)$ is uniformly bounded and continuous in $k \in \mathbb{R}$, by the definition of Fourier transform, we have

$$\begin{aligned} & \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-N}^{N} \hat{f}(k) e^{ikx} dk \\ = \ & \frac{1}{2\pi} \int_{-N}^{N} \int_{-\infty}^{\infty} f(y) e^{-iky} dy e^{ikx} dk \\ = \ & \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big(\int_{-N}^{N} e^{ik(x-y)} dk \Big) f(y) dy \end{aligned}$$

where

$$\int_{-N}^{N} e^{ik(x-y)} dk = 2\frac{\sin N(x-y)}{x-y}.$$

This is similar to the Dirichlet kernel, one can expect that the whole integral will converge to $f(x)$ as $N \to \infty$. Next we will prove it in detail.

Change variable $x = y - x$ gives

$$\begin{aligned} & \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-N}^{N} \hat{f}(k) e^{ikx} dk \\ = \ & \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin N(x-y)}{x-y} f(y) dy = \frac{1}{\pi} \int_{-\infty}^{\infty} f(z+x) \frac{\sin Nz}{z} dz \end{aligned}$$

Now we should separate the integral on $\mathbb{R}$ into two parts $I_1 = \displaystyle\int_{|z|\le M}$ and $I_2 = \displaystyle\int_{|z|\ge M}$, where M to be determined later. In the next, we will estimate I_1 by Riemann' lemma, and estimate I_2 by $1/M$.

$\forall \varepsilon > 0$, choose $M = \dfrac{2\|f\|_{L^1}}{\pi\varepsilon}$, we have

$$I_2 = \frac{1}{\pi} \int_{|z|\ge M} f(z+x) \frac{\sin Nz}{z} dz \le \frac{1}{\pi M} \|f\|_{L^1} = \frac{\varepsilon}{2}.$$

The way to estimate I_1 is by using

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi,$$

from which we know that $\exists N \ge 0$ s.t.

$$\Big| \frac{f(x)}{\pi} \int_{-MN}^{MN} \frac{\sin z}{z} dz - f(x) \Big| \le \frac{\varepsilon}{4}.$$

Now I_1 is

$$\begin{aligned} I_1 &= \frac{1}{\pi}\int_{|z|\le M} f(z+x)\frac{\sin Nz}{z}dz \\ &= \frac{1}{\pi}\int_{|z|\le M}\frac{f(z+x)-f(x)}{z}\sin Nz dz + \frac{f(x)}{\pi}\int_{|z|\le M}\frac{\sin Nz}{z}dz \\ &= \frac{1}{\pi}\int_{|z|\le M}\int_0^1 f'(x+\tau z)d\tau \sin Nz dz + \frac{f(x)}{\pi}\int_{|z|\le M}\frac{\sin Nz}{z}dz \\ &\le \frac{\|f'\|_{L^\infty}}{\pi}\int_{|z|\le M}\sin Nz dz + \frac{f(x)}{\pi}\int_{|z|\le M}\frac{\sin Nz}{z}dz \end{aligned}$$

By Riemann's lemma, we know $\exists N_1 > 0$ s.t. when $N \ge N_1$, we have

$$\frac{\|f'\|_{L^\infty}}{\pi}\int_{|z|\le M}\sin Nz dz \le \frac{\varepsilon}{4}.$$

□

Fourier transform for L^2 functions

Theorem 7.6. *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\hat{f} \in L^2(\mathbb{R})$ and*

$$\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$$

Furthermore, $f \to \hat{f}$ has a unique extension to a continuous, linear map from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ which is isometry.

Proof. $\forall f \in L^1 \cap L^2$, $\forall \varepsilon > 0$, consider

$$\int_{\mathbb{R}} |\hat{f}(k)|^2 e^{-\varepsilon|k|^2} dk.$$

By the definition of Fourier transform, we have

$$\int_{\mathbb{R}} |\hat{f}(k)|^2 e^{-\varepsilon|k|^2} dk = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{f(x)} f(y) e^{ik(x-y)} e^{-\varepsilon|k|^2} dxdydk$$

We know that $\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\varepsilon k^2} e^{ik(x-y)} dk = (e^{-\varepsilon k^2})^\vee(x-y)$, by Fubini theorem, the above integral is

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\sqrt{2}\varepsilon} e^{-\frac{(x-y)^2}{4\varepsilon}} \overline{f(x)} f(y) dxdy$$

Since

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-(\frac{x}{2\sqrt{\varepsilon}})^2} d(\frac{x}{2\sqrt{\varepsilon}}) = 1,$$

by theorem **??**, we have for $f \in L^2$,

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{1}{\sqrt{2}\varepsilon} e^{-\frac{(x-y)^2}{4\varepsilon}} f(y) dy \to f(x) \text{ strongly in } L^2(\mathbb{R}).$$

Thus,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} |\hat{f}(k)|^2 e^{-\varepsilon|k|^2} dk = \int_{\mathbb{R}} |f|^2 dx.$$

Then monotone convergence shows $\hat{f} \in L^2$ and

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2}.$$

If $f \in L^2$ but not in $L^1 \cap L^2$, since $L^1 \cap L^2$ is dense in L^2, there exists $\{f_j\} \subset L^1 \cap L^2$ such that

$$\|f_j - f\|_{L^2} \to 0.$$

On the other hand, since Fourier transform is linear, we have

$$\|\hat{f}_j - \hat{f}_m\|_{L^2} = \|f_j - f_m\|_{L^2} \to 0, \text{ as } j, m \to \infty$$

Hence, $\{\hat{f}_j\}$ is a Cauchy sequence in L^2. L^2 is complete, so $\exists g \in L^2$ such that $\hat{f}_j \to g$ strongly in L^2.

Now we define $\hat{f} = g$ then we have

$$\|\hat{f}\|_{L^2} = \lim_{j\to\infty} \|\hat{f}_j\|_{L^2} = \lim_{j\to\infty} \|f_j\|_{L^2} = \|f\|_{L^2}.$$

Continuity and linearity are left for reader. □

Remark 7.2. Fourier transform can be extended for L^p in similar way, and

$$\|\hat{f}\|_{L^q} \le C(p,q)\|f\|_{L^p}, \ \frac{1}{p} + \frac{1}{q} = 1.$$

7.2. **Distribution and weak derivative.** Ω is an open subset of $\mathbb{R}^n$.

7.2.1. *Distribution.*

Definition 3. Test function space $\mathcal{D}(\Omega)$ consists of all the functions in $C_0^\infty(\Omega)$ supplemented by the following convergence: $\phi_m \to \phi \in C_0^\infty(\Omega)$ iff

(1) $\exists$ a compact set $K \subset \Omega$ such that supp$\phi_m \subset K$, $\forall m$.
(2) $\forall \alpha$ multi-index,

$$\sup_k |\partial^\alpha \phi_m - \partial^\alpha \phi| \to 0. \quad (m \to \infty).$$

Remark 7.3. $\mathcal{D}(\Omega)$ is a linear space.

Definition 4. Distribution is the dual space of $\mathcal{D}(\Omega)$. i.e. is a continuous linear functional on $\mathcal{D}(\Omega)$. we denoted it by $\mathcal{D}'(\Omega)$. Namely, $T : \mathcal{D}(\Omega) \to \mathbb{C}$ s.t.

(1) $\langle T, \alpha\phi_1 + \beta\phi_2\rangle = \alpha\langle T, \phi_1\rangle + \beta\langle T, \phi_2\rangle$
(2) If $\phi_m \to \phi$ in $\mathcal{D}(\Omega)$, then $\langle T, \phi_m\rangle \to \langle T, \phi\rangle$.

Remark 7.4. It is usually nonsense to multiply two distributions, since it is not well defined.

Remark 7.5. A distribution multiplied by a smooth function can be defined by the following, $T \in \mathcal{D}'$, $f \in C^\infty$, then

$$\langle Tf, \phi\rangle = \langle T, f\phi\rangle, \qquad \forall \phi \in \mathcal{D}.$$

Remark 7.6. The support of a distribution and convolution of two distributions can be also defined with the help of test functions, since we will not use these in our course, we omit the detail here.

Example 8. $L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$.
$\forall f \in L^1_{loc}(\Omega)$, $T_f \in \mathcal{D}'(\Omega)$ is defined by

$$\langle T_f, \phi\rangle = \int_\Omega f(x)\phi(x)dx, \qquad \forall \phi \in \mathcal{D}(\Omega).$$

Remark 7.7. Similarly, $L^p_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$. And $L^p_{loc}(\Omega) \subset L^q_{loc}(\Omega)$, $\forall q < p$.

Theorem 7.7. *L^1_{loc} functions are uniquely determined by distributions. More precisely, $\forall f, g \in L^1_{loc}(\Omega)$ and*

$$\int_\Omega f\phi dx = \int_\Omega g\phi dx, \qquad \forall \phi \in \mathcal{D}(\Omega),$$

Then $f = g$ a.e. in Ω.

The proof is left to the readers.

Example 9. *Probability distribution on $\mathbb{R}$ is a subset of $\mathcal{D}'(\mathbb{R})$.*

For any probability distribution P, $T_P \in \mathcal{D}'(\mathbb{R})$ is defined by

$$\langle T_P, \phi \rangle = \int_{\mathbb{R}} \phi(x) dP, \qquad \forall \phi \in \mathcal{D}(\mathbb{R}).$$

Example 10. *δ function.*

Delta function $\delta(x)$ is defined by

$$\langle \delta, \phi \rangle = \phi(0), \qquad \forall \phi \in \mathcal{D}(\Omega).$$

Remark 7.8. $\delta \notin L^1_{loc}$.

Proof. If not, there exists $f \in L^1_{loc}$ s.t. $\forall \phi \in C_0^\infty$

$$\langle \delta, \phi \rangle = \int_{\mathbb{R}} f\phi dx.$$

Now we choose $\varphi_n(x) = \begin{cases} e^{\frac{1}{|nx|^2-1}} & |nx| < 1, \\ 0 & |nx| \geq 1. \end{cases}$ Then on the one hand, we have

$$\langle \delta, \phi_n \rangle = \phi_n(0) = e^{-1},$$

on the other hand, since $f \in L^1_{loc}$,

$$\int_{\mathbb{R}} f\varphi_n dx = \int_{|x| \leq \frac{1}{n}} f(x) e^{\frac{1}{|nx|^2-1}} dx \to 0, \quad n \to \infty.$$

Contradiction. □

In the following we will show some sequences which converge to δ-function in the sense of distribution, to have more understanding of δ-function.

Example 11. *Heat kernel $f_t(x) = \dfrac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$.*

$\int_{\mathbb{R}} f_t(x) dx = 1$ and $\forall \phi \in C_0^\infty$,

$$\int_{\mathbb{R}} f_t(x)\phi(x) dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} \phi(2\sqrt{t}y) dy \to \phi(0).$$

by dominated convergence.

Example 12. $Q_n(x) = \begin{cases} \frac{n}{2} & |nx| < 1, \\ 0 & |nx| \geq 1. \end{cases}$.

$\int_{\mathbb{R}} Q_n(x) dx = 1$ and $\forall \phi \in C_0^\infty$,

$$\int_{\mathbb{R}} Q_n(x)\phi(x) dx = \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{n}{2} \phi(x) dx \to \phi(0).$$

Example 13. *Dirichlet kernel* $D_n(x) = \dfrac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}} = 1 + 2\sum_{k=1}^{n} \cos kx$.

$\int_{\mathbb{R}} D_n(x)dx = 2\pi$ *and* $\forall \phi \in C_0^\infty$,

$$\int_{-\pi}^{\pi} D_n(x)\phi(x)dx = \int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}\phi(x)dx \to 2\pi\phi(0),$$

which can be proved by using Riemann's lemma and similar argument to the proof of Fourier inverse transform we did for $L^1 \cap C^1$ *functions.*

7.2.2. *weak derivative of Distributions.* The definition of weak derivative is enlightened by integral by parts, if $f \in C^1$, $\forall \phi \in C_0^\infty$, we have

$$\int_\Omega \partial_i f \phi dx = -\int_\Omega f \partial_i \phi dx.$$

Definition 5. $\forall T \in \mathcal{D}'(\Omega)$, $\partial_i T$ is defined by

$$\langle \partial_i T, \phi \rangle = -\langle T, \partial_i \phi \rangle, \quad \phi \in \mathcal{D}(\Omega).$$

Since $-\partial_i \phi \in \mathcal{D}'(\Omega)$, we know that $\partial_i T$ is well defined. One can define the higher order derivative in the same way, α is a multi-index,

$$\langle \partial^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \phi \rangle, \quad \phi \in \mathcal{D}(\Omega).$$

Remark 7.9. According to this definition, we know that all distributions are infinitely weakly differentiable.

Example 14. *The derivatives of* δ*-function.* $\forall \phi \in \mathcal{D}$

$$\langle \delta', \phi \rangle = -\langle \delta, \phi' \rangle = -\phi'(0),$$
$$\langle \delta^{(k)}, \phi \rangle = (-1)^k \langle \delta, \phi^{(k)} \rangle = (-1)^k \phi^{(k)}(0),$$

Example 15. *The derivatives of Heaviside function* $H = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$.

$\forall \phi \in \mathcal{D}$

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle = -\int_0^\infty \phi'(x)dx = \phi(0) = \langle \delta, \phi \rangle$$

Also enlightened by integral (changing variables), we can give the translation of distributions.

Definition 6.

$$\langle T(x-a), \phi(x) \rangle = \langle T(x), \phi(x+a) \rangle.$$

For example, $\delta_a(x) = \delta(x-a)$ is defined by

$$\langle \delta, \phi(x+a) \rangle = \phi(a)$$

7.3. Tempered distribution and its Fourier transform.

Definition 7. Schwartz class function $\mathcal{S}(\mathbb{R}^n)$, (Rapidly decreasing function)

$$\mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) | \sup_x |x^\gamma \partial^\alpha \phi| < +\infty, \forall \text{ multi-index } \alpha, \gamma\}$$

We call a sequence $\{\phi_j\} \subset \mathcal{S}(\mathbb{R}^n)$ convergent to $\phi \in \mathcal{S}(\mathbb{R}^n)$ if

$$\sup_x |x^\gamma \partial^\alpha (\phi_j - \phi)| \to 0, \forall \text{ multi-index } \alpha, \gamma$$

We will use the notation

$$D_j = \frac{1}{i}\partial_j.$$

Then the properties of Fourier transform are

$$(D_j\phi)^\wedge = k_j\hat{\phi}(k), \quad (x_j\phi)^\wedge = -D_j\hat{\phi}.$$

Theorem 7.8. *If $\phi \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{\phi} \in \mathcal{S}(\mathbb{R}^n)$*

Proof. $\forall \alpha, \gamma$ multi-index, we know from Fourier transform that

$$k^\alpha D^\gamma \hat{\phi}(k) = \Big(D^\alpha\big((-x)^\gamma \phi(x)\big)\Big)^\wedge = \int_{\mathbb{R}^n} e^{-ix\cdot k} D^\alpha\big((-x)^\gamma\phi(x)\big)dx.$$

By taking sup in k,

$$\begin{aligned} \sup_k |k^\alpha D^\gamma \hat{\phi}(k)| &\leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |D^\alpha\big((-x)^\gamma\phi(x)\big)|dx \\ &\leq \frac{1}{(2\pi)^{\frac{n}{2}}} C \sup_x (1+|x|)^{n+1} |D^\alpha\big((-x)^\gamma \phi(x)\big)| < +\infty, \end{aligned}$$

where $C = \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+1}}dx$.

□

Definition 8. Dual space of $\mathcal{S}(\mathbb{R}^n)$ is called tempered distribution $\mathcal{S}'(\mathbb{R}^n)$.

Remark 7.10.

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n), \qquad \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$$

Definition 9. $\forall T \in \mathcal{S}'(\mathbb{R}^n)$, its Fourier transform is defined by $\forall \phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \hat{T}, \phi\rangle = \langle T, \hat{\phi}\rangle$$

Example 16. $\hat{\delta} = \frac{1}{(2\pi)^{\frac{n}{2}}}$.

$\forall \phi \in \mathcal{S}(\mathbb{R}^n)$, by definition,

$$\langle \hat{\delta}, \phi\rangle = \langle \delta, \hat{\phi}\rangle = \hat{\phi}(0) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi(x) e^{-i0\cdot x} dx = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi(x)dx = \langle \frac{1}{(2\pi)^{\frac{n}{2}}}, \phi\rangle$$

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