INTRODUCTION

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The main references of this part is Evan's book.

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1. General formulation of PDE

Some Notations: A multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ and

$$D^\alpha u = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \cdots D_{x_n}^{\alpha_n} u.$$

Ω is an open subset of $\mathbb{R}^n$,
F is a mapping in the following sense

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}.$$

Symbolically a k-th order partial differential equation can be written in the form of

$$F(D^k u(x), D^{k-1} u(x), \cdot, Du(x), u(x), x) = 0, \quad \forall x \in \Omega \tag{1.1}$$

The unknown in the equation is $u(x) : \Omega \to \mathbb{R}$.

Moreover, if F is a vector valued mapping such that

$$\mathbf{F} : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \cdots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m.$$

Then the corresponding equation is a k-th order partial differential system

$$\mathbf{F}(D^k \mathbf{u}(x), D^{k-1} \mathbf{u}(x), \cdot, D\mathbf{u}(x), \mathbf{u}(x), x) = \mathbf{0}, \quad \forall x \in \Omega \tag{1.2}$$

and $\mathbf{u} : \Omega \to \mathbb{R}^m$ is unknown.

If F has the form such that (1.1) is

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u = f(x) \tag{1.3}$$

with $a_\alpha(x)$ and $f(x)$ are given functions, we call it **linear**.

If F has the form

$$\sum_{|\alpha|=k} a_\alpha(x)D^\alpha u + a_0(D^{\alpha-1}u, \cdots, Du, u, x) = 0, \tag{1.4}$$

with $a_\alpha(x)$ a given function and a_0 a given mapping, we call it **semilinear**.

If F has the form

$$\sum_{|\alpha|=k} a_\alpha(D^{\alpha-1}u, \cdots, Du, u, x)D^\alpha u + a_0(D^{\alpha-1}u, \cdots, Du, u, x) = 0, \tag{1.5}$$

with a_α and a_0 are given mappings, we call it **quasilinear**.

A PDE is called **fully nonlinear** if the highest order derivative is nonlinear.

Theorem 1.1. *(Superposition principle for homogeneous linear equations). Let $f \equiv 0$ in* (1.3)*. If u_1 and u_2 are both solutions of* (1.3)*, then any linear combination of u_1 and u_2 are still solution of* (1.3)*.*

There are also more detailed classifications for second order PDE, which will not be covered in this course. For those who are interested, please check Pan's notes, introduction, on the mathmods' web page.

We list here some examples of PDE.

Linear PDEs

(1) Linear transport equation $u_t + \mathbf{b} \cdot \nabla u = f$
(2) Wave equation $u_{tt} - \Delta u = f$
(3) Heat equation $u_t - \Delta u = f$
(4) Poisson equation $-\Delta u = f$
(5) Schrödinger equation $iu_t = -\Delta u$
(6) ......

there are also some other higher order linear equations, mostly are from physics. Check Evan's book.

Nonlinear PDEs There are plenty of nonlinear equations in the literature. But we couldn't touch them in this course except one, the 1-D scalar first order hyperbolic conservation laws.

(1) Scalar conservation law $u_t + \mathrm{div}\mathbf{F}(u) = 0$
(2) Hamilton-Jacobi equation $u_t + H(Du, x) = 0$
(3) Nonlinear Poisson equation $-\Delta u = f(u)$
(4) Monge-Ampère equation $\det D^2 u = f$
(5) Porous medium equation $u_t - \Delta u^m = 0$
(6) Nonlinear Schrödinger equation $iu_t = -\Delta u + |u|^2 u$
(7) ......

For PDE systems, we will not give any detailed equations here since we don't touch them here. Some typical PDE systems are fluid dynamic type systems like Euler equations, Navier-Stokes equations, Maxwell's equations, etc.

2. Set up of the problems

To have a complete PDE problem, we need to take into account of the related boundary conditions.

For time evolutionary PDE, some initial data of the problem is deserved. For example, for heat equation $u_t - \Delta u = f$, we need initial conditions like $u|_{t=0} = u_0(x)$ where $u_0(x)$ is a given function. For wave equation $u_{tt} - \Delta u = f$, we need $u|_{t=0} = g$ and $u_t|_{t=0} = h$ since we have double derivatives in time t.

If we study the PDE in Ω, a subset of $\mathbb{R}^n$, and $\partial\Omega \neq \emptyset$, we need to give boundary conditions. There are three kinds of boundary conditions, each of them has its physical backgrounds.

(1) **Dirichlet boundary condition**, the unknown itself is given on the boundary

$$u|_{\partial\Omega} = u_D$$

where $u_D(x)$ is a given function defined on $\partial\Omega$.

(2) **Neumann boundary condition**, the normal derivative of the unknown is given on the boundary

$$\nabla u \cdot \gamma|_{\partial\Omega} = u_N(x)$$

where γ is the pointwise outer normal vector of $\partial\Omega$.

(3) **Robin boundary condition**, the nontrivial linear combination of the unknown and its normal derivative is given on the boundary

$$\alpha u + \beta \nabla u \cdot \gamma|_{\partial\Omega} = u_R(x)$$

where $\alpha(x)$ and $\beta(x)$ are nonnegative functions given on the boundary.

3. Basic knowledges for studying PDE problems

After settled down a reasonable PDE problem, the main purpose is to find the solution and study it to have a more clear understanding of the solution behavior. However, it is almost impossible to find an explicit formula for the solution in most problems. This doesn't mean that we couldn't do anything on them. In most of the problems, we can still get some detailed information by analysis of the problems itself without have an solution formula at hand.

Given a PDE problem, we need to study its well-posedness, including

(1) **Existence**. The direct way to get existence of solution is to solve it explicitly. Then it is easy to get existence of classical solution by solution formula. Here classical solution means k-th derivative of the solution exists for k-th order PDE. For those equations which are hopeless to get solution formula, there are several ways to get existence. For example, fixed point theorem in functional analysis, variational methods. The main thing one should keep in mind is in which function class one could expect the existence of solution. Usually the existence should be studied in the sense that the equation is satisfied in some weaker form, instead of its classical one. Then one can try to find out if this existed weak solution has some regularity.

(2) **Uniqueness**. Once we have existence in hand, a natural question is to ask if it is unique, in which class it is unique. For some problems we couldn't always expect uniqueness. The larger the existence class, the less hope to get uniqueness.

(3) **Stability**. The stability of a solution means that if you change a small size of the given data, is it true that your solution also doesn't change much. In other words, it is the continuous dependence of the solution on its given data, including initial and boundary conditions, or any other given functions in the equation.

(4) **Solution behavior** In some sense, solution behavior is the most important thing the PDE analysis should give. Here solution behavior is talked about in the very general sense. It can be the large time behavior, boundary layer behavior or singular behavior if you have a very small constant in the equation.

There are some general methods to study PDE. Although the ways to find solution formula couldn't be directly used in modern PDE study, they still play a very important role in understanding some basic theories of PDE. In some sense, the study of PDE start from finding explicit solutions. The basic tools we will talk about in this course are

(1) Method of characteristics
(2) Fourier transform and Laplace transform
(3) Separation of variables (Fourier series)
(4) Greens function

There are also many other tools to study the solution behavior without using formula,

(1) Energy method
(2) Maximum principle
(3) Asymptotic expansion—*

As explained above, for some physical problems, one couldn't expect that they always have classical solutions. In this course, we will also talk about the ideas on how to define weak solutions. Moreover, a very brief introduction on distributions.

4. Linear Transport Equations

4.1. **Constant speed.** Linear transport equation in 1-D is the simplest partial differential equation,

$$\begin{aligned} \rho_t + a\rho_x &= 0, \qquad \text{in } \mathbb{R}\times\mathbb{R} \\ \rho(x,0) &= \rho_0(x), \end{aligned}$$

where a is a constant, this equation with constant speed is just an ordinary differential system in the sense that

$$\begin{aligned} &\frac{d}{dt}\rho(x(t),t) = 0, \\ &\frac{d}{dt}x(t) = a, \\ &\rho(x(0),0) = \rho_0(x_0). \end{aligned}$$

Obviously, it has solution

$$\rho(x,t) = \rho_0(x - at).$$

This strategy is called the **method of characteristics**. $x(t)$ is called the characteristic line.

Moreover, one can also solve the transport equation in nonhomogeneous case,

$$\begin{aligned} \rho_t + a\rho_x &= f(x,t), \qquad \text{in } \mathbb{R}\times\mathbb{R} \\ \rho(x,0) &= \rho_0(x), \end{aligned}$$

by using the same characteristic line $x(t) = x_0 + at$, the solution is

$$\rho(x,t) = \rho_0(x - at) + \int_0^t f(x - a(t-s), s)ds.$$

Remark 4.1. By using the method of characteristics, one can easily solve the Multi-D equation

$$\rho_t + \mathbf{b}\cdot\nabla\rho = 0, \qquad \text{in } \mathbb{R}\times\mathbb{R}^n$$
$$\rho(x,0) = \rho_0(x).$$

where $\mathbf{b}$ is a constant vector.

4.2. **Nonconstant speed.** The Cauchy problem we consider in this part is

$$\begin{aligned} \rho_t + (v(x)\rho)_x &= 0, \qquad \text{in } \mathbb{R}\times\mathbb{R} \\ \rho(x,0) &= \rho_0(x). \end{aligned} \tag{4.1}$$

where $v(x)$ is a given Lipschitz continuous function.

A reformulation of the equation is

$$\rho_t + v(x)\rho_x + v'(x)\rho = 0.$$

If $v(x)$ is Lipchitz continuous, then the characteristic line $x(t)$ satisfies that

$$\begin{aligned} \frac{dx}{dt} &= v(x), \\ x(0) &= x_0. \end{aligned} \tag{4.2}$$

With the help of this line the equation is

$$\begin{aligned} \frac{d\rho(x(t,x_0),t)}{dt} &= -v'(x(t,x_0))\rho, \\ \rho(x(t,x_0),t)|_{t=0} &= \rho_0(x_0). \end{aligned}$$

By separation of variable in solving ODE,

$$\begin{aligned} \ln\rho(x(t,x_0),t) &= \ln\rho_0(x_0) + \int_0^t -v'(x(\tau,x_0))d\tau, \\ &= \ln\rho_0(x_0) + \int_{x_0}^{x(t,x_0)} -\frac{v'(x)}{v(x)}dx \\ &= \ln\rho_0(x_0) - \ln v(x(t,x_0)) + \ln v(x_0). \end{aligned}$$

So the problem has solution

$$\rho(x(t,x_0),t) = \rho_0(x_0)\frac{v(x_0)}{v(x(t,x_0))}.$$

Thus the exact solution of the nonhomogeneous problem (4.1) is reduced to the solvability of the characteristic line (4.2).

5. Half line problem

We will only study the half line problem for transport equation with constant speed. One should be careful in giving boundary conditions because of the "directions" of characteristic lines. For example, one couldn't arbitrarily give boundary condition at $x=0$ if the characteristic lines starting from $t=0$ will meet $x=0$ at some time t.

$$\begin{aligned} u_t + u_x &= 0 \qquad x \in (0,+\infty)\times(0,+\infty) \\ u|_{t=0} &= u_0(x), \qquad u|_{x=0} = 0 \end{aligned}$$

For compatibility, we need $u_0(0) = 0$. It is easily seen that the solution is

$$u(x,t) = \begin{cases} 0 & x \le t \\ u_0(x-t) & x > t \end{cases}$$

If the boundary condition is nonhomogeneous, $u|_{x=0} = g(t)$, the compatibility condition is $u(0,0) = g(0)$. In this case, we can use $v(x,t) = u(x,t) - g(t)$ which is zero on boundary $x = 0$. And the problem v satisfies is

$$\begin{aligned} v_t + v_x &= -g_t(t) \qquad x \in (0,+\infty) \times (0,+\infty) \\ v|_{t=0} &= u_0(x) - g(0), \qquad u|_{x=0} = 0 \end{aligned}$$

6. Problems

(1) Point out the type of these equations (linear, semilinear, quasilinear, fully nonlinear)
  - (a) $u_t - u_x u_{xxx} = x^2$
  - (b) $-\triangle u + u^2 = 1$
  - (c) $u_{tt} - \mathrm{div}((x^2+t)\nabla u) = f(x)$
  - (d) $u_t + \mathrm{div}\Big[u\nabla\Big(\dfrac{\triangle\sqrt{u}}{\sqrt{u}}\Big)\Big] = 0$
  - (e) $\triangle(u^2) = f(x)$
  - (f) $|\nabla u| = 1$

(2) Find the solution formula by using characteristic method
  - (a) $\begin{cases} u_t + (1+x^2)u_x - u = 0, & t > 0, -\infty < x < \infty, \\ u|_{t=0} = \arctan x, & -\infty < x < \infty \end{cases}$
  - (b) $\begin{cases} u_t + b\cdot\nabla u = f, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u|_{t=0} = g, & x \in \mathbb{R}^n \end{cases}$

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