POISSON EQUATION

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Contents

1. Fundamental Solution

The Poisson's equation in $\mathbb{R}^n$ reads

$$-\Delta u = f \text{ in } \mathbb{R}^n. \tag{1.1}$$

We will first try to find some special solution formally. Since Laplace operator is radially symmetric, it is natural to find radially symmetric solutions. Assume $u(x) = v(|x|) = v(r)$, where $r = |x|$, then

$$u_{x_i} = v_r \frac{\partial r}{\partial x_i} = v_r \frac{x_i}{r}, \quad u_{x_i x_j} = v_{rr} \frac{x_i^2}{r^2} + v_r(\frac{1}{r} - \frac{x_i^2}{r^3}),$$

thus

$$\Delta u = v_{rr} + \frac{n-1}{r} v_r = 0, \quad \Rightarrow \quad (\log v_r)_r = \frac{1-n}{r}, \text{ in che case of } v_r \neq 0.$$

Consequently, there exist constants C and C' such that $v_r = Cr^{1-n}$ and

$$v(r) = \begin{cases} C \log r + C' & n = 2 \\ \dfrac{C}{r^{n-2}} + C' & n \geq 3 \end{cases}$$

Date: 2011 Nov.

Definition 1. Let

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$$

where $\alpha(n)$ is the volumn of n dimension ball. $\Phi(x)$ is called the **fundamental solution** of Poisson equation.

Properties

(1) $|\nabla\Phi| \leq \frac{C}{|x|^{n-1}}$, $|D^2\Phi| \leq \frac{C}{|x|^n}$ for $x \neq 0$.
(2) $\Delta\Phi = 0$ for $x \neq 0$ and $\Delta\Phi(x-y) = 0$ for $x \neq y$, $\forall y \in \mathbb{R}^n$

Then we are able to represent the solution of Poisson equation by using fundamental solution. More precisely we have the following theorem.

Theorem 1.1. *If $f \in C_0^2(\mathbb{R}^n)$, then $u = \Phi * f$ is a solution of problem* (1.1)

Proof. First we prove that $u \in C^2(\mathbb{R}^n)$. In fact,

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y)\frac{f(x+he_i-y) - f(x-y)}{h}dy.$$

Since we know that f has compact support and $\frac{\partial f(x-y)}{\partial x_i} = \lim_{h\to 0}\frac{f(x+he_i-y)-f(x-y)}{h}$, combined with the fact that Φ is locally integrable, we have that, by letting $h \to 0$,

$$\frac{\partial u}{\partial x_i} = \int_{\mathbb{R}^n}\Phi(y)\frac{\partial f}{\partial x_i}(x-y)dy.$$

By similar discussions, we have that u is twice differential and

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \int_{\mathbb{R}^n}\Phi(y)\frac{\partial^2 f}{\partial x_i\partial x_j}(x-y)dy.$$

Next we will prove $-\Delta u = f$. $\forall\varepsilon > 0$ small enough,

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^n}\Phi(y)\Delta_x f(x-y)dy \\ &= \int_{B_\varepsilon(0)}\Phi(y)\Delta_x f(x-y)dy + \int_{\mathbb{R}^n\setminus B_\varepsilon(0)}\Phi(y)\Delta_x f(x-y)dy \\ &:= I_\varepsilon + J_\varepsilon. \end{aligned}$$

where

$$|I_\varepsilon| \leq C\|D^2 f\|_{L^\infty}\int_{B_\varepsilon(0)}|\Phi(y)|dy \leq \begin{cases} C\varepsilon^2|\log\varepsilon| & n = 2 \\ C\varepsilon^2 & n \geq 3 \end{cases}$$

Integral by parts for J_ε,

$$J_\varepsilon = -\int_{\mathbb{R}^n\setminus B_\varepsilon(0)}\nabla_y\Phi(y)\nabla_y f(x-y)dy - \int_{\partial B_\varepsilon(0)}\Phi(y)\nabla_y f(x-y)\cdot\gamma dS_y := K_\varepsilon + L_\varepsilon,$$

L_ε can be estimated by

$$|L_\varepsilon| \leq \|Df\|_{L^\infty}\int_{\partial B_\varepsilon(0)}|\Phi(y)|dS_y \leq \begin{cases} C\varepsilon|\log\varepsilon| & n = 2 \\ C\varepsilon & n \geq 3 \end{cases}$$

K_ε contributes the main part of the calculation. When ε goes to 0, this term practiced like a Delta function applied on f. Due to the fact that $\Delta\Phi(y) = 0$ for $y \neq 0$, we have

$$K_\varepsilon = \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Delta\Phi(y) f(x-y)dy + \int_{\partial B_\varepsilon(0)} \nabla\Phi \cdot \gamma f(x-y)dS_y = \int_{\partial B_\varepsilon(0)} \nabla\Phi \cdot \gamma f(x-y)dS_y$$

Now we can calculate that on $\partial B_\varepsilon(0)$,

$$\nabla_y \Phi(y) \cdot \gamma = -\frac{1}{n\alpha(n)} \frac{y}{|y|^n} \frac{y}{|y|} = -\frac{1}{n\alpha(n)\varepsilon^{n-1}}.$$

Thus we have

$$K_\varepsilon = -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B_\varepsilon(0)} f(x-y)dS_y = -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B_\varepsilon(x)} f(y)dS_y.$$

Taking $\varepsilon \to 0$, we know that

$$K_\varepsilon \to f(x).$$

□

Remark 1.1. From the above proof, we understand the constants appeared in definition of fundamental solution.

By using the same method, we can prove that $-\Delta\Phi = \delta(x)$ in the sense of distribution.

Theorem 1.2.

$$\Phi(x,y) = \Phi(x-y) = \begin{cases} -\frac{1}{2\pi} \log|x-y| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x-y|^{n-2}} & n \geq 3 \end{cases} \tag{1.2}$$

is a solution of

$$-\Delta\Phi = \delta(x-y)$$

in the sense of distribution. More precisely, $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$, *it holds*

$$\langle -\Delta\Phi(x-y), \varphi(x)\rangle = -\int_{\mathbb{R}^n} \Phi(x-y)\Delta\varphi(x)dy = \varphi(y) = \langle \delta(x-y), \varphi(x)\rangle.$$

2. Properties of Harmonic Function

Let Ω be an open subset of $\mathbb{R}^n$.

Definition 2. If $\Delta u = 0$ in Ω with $u \in C^2(\Omega)$, then u is called a **harmonic function**.

2.1. **Mean Value theorem.**

Theorem 2.1. *If* $u \in C^2(\Omega)$ *is harmonic, then* $\forall$ *ball* $B(x,r) \in \Omega$, *it holds that*

$$u(x) = \fint_{\partial B(x,r)} u dS_y = \fint_{B(x,r)} u dy. \tag{2.1}$$

Proof. Let

$$w(r) = \fint_{\partial B(x,r)} u(y)dS_y = \fint_{\partial B(0,1)} u(x+rz)dS_z$$

Then by taking derivative with respect to r, we have

$$\begin{aligned} w'(r) &= \fint_{\partial B(0,1)} \nabla u(x+rz)\cdot z dS_z \\ &= \fint_{\partial B(x,r)} \nabla u(y)\cdot \frac{y-x}{r} dS_y = \frac{r}{n|B(x,r)|}\int_{B(x,r)} \Delta u(y)dy = 0, \end{aligned}$$

which implies that $w(r)$ is a constant. Thus we have

$$w(r) = \lim_{s\to 0} w(s) = \lim_{s\to 0}\fint_{\partial B(x,s)} u(y)dS_y = u(x).$$

For the mean value on $B(x,r)$, we know that

$$\begin{aligned} \int_{B(x,r)} u(y)dy &= \int_0^r \Big(\int_{\partial B(x,s)} u(y)dS_y\Big)ds \\ &= u(x)\int_0^r n\alpha(n)s^{n-1}ds = \alpha(n)r^n u(x), \end{aligned}$$

which is exactly

$$u(x) = \fint_{B(x,r)} u(y)dy.$$

□

Theorem 2.2. *(Converse to the mean value property) If $u\in C^2(\Omega)$ such that*

$$u(x) = \fint_{\partial B(x,r)} u(y)dS_y, \quad \forall B(x,r)\subset\Omega,$$

Then u is harmonic in Ω i.e. $\Delta u = 0$ in Ω.

Proof. If $\Delta u \not\equiv 0$, there must exist a ball $B(x,r)\subset\Omega$ such that $\Delta u > 0$ in $B(x,r)$. On the other hand,

$$0 = w'(r) = \frac{r}{n}\fint_{B(x,r)} \Delta u(y)dy > 0,$$

which gives a contradiction. □

2.2. **Strong maximum principle.**

Theorem 2.3. *If $u\in C^2(\Omega)\cap C(\bar\Omega)$ is harmonic in Ω, then*

(1) $\max_\Omega u = \max_{\partial\Omega} u$
(2) *If Ω is connected and $\exists x_0\in\Omega$ such that*

$$u(x_0) = \max_\Omega u(x),$$

then u is constant within Ω.

Proof. The first statement is easy, we only prove that second one here. Suppose that $\exists x_0\in\Omega$ such that $u(x_0) = \max_\Omega u = M$, then $\forall 0<r<dist(x_0,\partial\Omega)$, the mean value property implies that

$$M = u(x_0) = \fint_{B(x,r)} u(y)dy \le M,$$

which means that u is constant within $B(x_0,r)$, i.e. $u\equiv M$ in $B(x_0,r)$. Hence the set

$$U_M = \{x\in\Omega | u(x) = M\}$$

is both open and close in Ω. So if Ω is connected, then $U_M = \Omega$. □

Corollary 2.1. *If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic and $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω.*

Corollary 2.2. *(Uniqueness) Dirichlet boundary value problem $-\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$ has at most one $C^2(\Omega) \cap C(\bar{\Omega})$ solution.*

2.3. **Regularity.**

Theorem 2.4. *If $u \in C(\Omega)$ satisfies mean value property for all ball $B(x,r)$ in Ω, then $u \in C^\infty(\Omega)$*

Remark 2.1. The smoothness up to $\partial\Omega$ usually is not true, which depends on the regularity of the boundary.

Proof. *** The proof of regularity will use mollification, which appeared in the appendix of heat equation. For those who are interested, please read this proof by yourself. $\forall \varepsilon > 0$, let

$$\Omega_\varepsilon = \{x \in \Omega | dist(x, \partial\Omega) > \varepsilon\}.$$

Let's study $u_\varepsilon(x) = j_\varepsilon(x) * u(x)$, by direct calculation and mean value property, we have

$$\begin{aligned} u_\varepsilon(x) &= \int_{B(x,\varepsilon)} \frac{1}{\varepsilon^n} j(\frac{x-y}{\varepsilon}) u(y) dy \\ &= \frac{1}{\varepsilon^n} \int_0^\varepsilon \left[j(\frac{r}{\varepsilon}) \int_{\partial B(x,r)} u(y) dS_y \right] dr \\ &= \frac{1}{\varepsilon^n} u(x) \int_0^\varepsilon j(\frac{r}{\varepsilon}) n\alpha(n) r^{n-1} dr \\ &= u(x) \int_{B(0,\varepsilon)} j_\varepsilon(y) dy = u(x). \end{aligned}$$

Thus $u(x) = u_\varepsilon(x) \in C^\infty(\Omega_\varepsilon)$, $\forall \varepsilon > 0$. □

2.4. **Liouville theorem.**

Theorem 2.5. *If $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded, then u is a constant.*

Proof. *** The proof will use local regularity estimates for harmonic function which was not talked about in this course. $\forall x_0 \in R^n$, $r > 0$,

$$|Du(x_0)| \leq \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))} \leq \frac{C_1 \alpha(n)}{r} \|u\|_{L^\infty(\mathbb{R}^n)} \to 0, \text{ as } r \to \infty.$$

Then $Du \equiv 0$, which implies u is a constant. □

Corollary 2.3. *$f \in C_0^2(\mathbb{R}^n)$, $n \geq 3$, then any bounded solution of $-\Delta u = f$ in $\mathbb{R}^n$ has the form*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy + C.$$

Proof. First we know that $\int_{\mathbb{R}^n} \Phi(x-y) f(y) dy$ is a bounded solution since $\Phi(x) \to 0$ as $|x| \to \infty$. If there is another bounded solution $\tilde{u}$, then $w = u - \tilde{u}$ is harmonic, thus by Liouville's theorem, w is a constant. □

3. Green's Function

The main goal is to get the representation formula for the solution of boundary value problem

$$\begin{aligned} -\Delta u = f \quad \text{in } \Omega \\ u|_{\partial\Omega} = g \end{aligned} \tag{3.1}$$

The natural question to ask is that is it possible to have solution formula for this problem? Is our the fundamental solution useful?

Let's start from a formal calculation, $\forall x \in \Omega$,

$$\begin{aligned} u(x) &= \langle \delta(x-y), u(y)\rangle = \langle -\Delta_y \Phi(x,y), u(y)\rangle = -\int_\Omega \Delta_y \Phi(x,y) u(y) dy \\ &= \int_\Omega \Phi(x,y)(-\Delta_y u(y))dy - \int_{\partial\Omega} \nabla_y \Phi(x,y)\cdot \gamma u(y) dS_y + \int_{\partial\Omega} \Phi(x,y)\nabla_y u(y)\cdot \gamma dS_y. \end{aligned}$$

Then formally, if $u|_{\partial\Omega} = g$ and $-\Delta u = f$, we have

$$u(x) = \int_\Omega \Phi(x,y) f(y) dy - \int_{\partial\Omega} \nabla_y \Phi(x,y)\cdot \gamma g(y) dS_y + \int_{\partial\Omega} \Phi(x,y)\nabla_y u(y)\cdot \gamma dS_y.$$

where the last term is still unknown. We will try to consider another function $G(x,y)$ to replace the fundamental solution $\Phi(x,y)$. And this $G(x,y)$ satisfies

$$\begin{aligned} -\Delta_y G(x,y) = \delta(y-x) \\ G(x,y)|_{y\in\partial\Omega} = 0. \end{aligned}$$

A good candidate of $G(x,y)$ is $\Phi(x,y) + g(x,y)$ with $g(x,y)$ satisfies

$$\begin{aligned} -\Delta_y g(x,y) = 0 \\ g|_{\partial\Omega} = -\Phi(x,y)|_{\partial\Omega} \end{aligned}$$

Once we can solve the above problem for $g(x,y)$, we will have the solution representation of (3.1),

$$u(x) = \int_\Omega G(x,y) f(y) dy - \int_{\partial\Omega} \nabla_y G(x,y)\cdot \gamma g(y) dS_y.$$

We will give a proof of the above discussion after the definition.

Definition 3. (Green's function)

$$G(x,y) = \Phi(x,y) + g(x,y)$$

is called the **Green's function** of (3.1), where $g(x,y) \in C^2(\Omega\times\Omega)$ is a solution of

$$\begin{aligned} -\Delta_y g(x,y) = 0, \qquad \text{in } \Omega \\ g(x,y)|_{y\in\partial\Omega} = -\Phi(x,y) \end{aligned}$$

Theorem 3.1. *Ω is an open subset of $\mathbb{R}^n$, $\partial\Omega$ is piecewise smooth, $u \in C^2(\Omega)\cap C^1(\Omega)$, then $\forall x \in \Omega$,*

$$u(x) = \int_\Omega \Phi(x,y)(-\Delta_y u(y))dy - \int_{\partial\Omega} \nabla_y \Phi(x,y)\cdot \gamma u(y) dS_y + \int_{\partial\Omega} \Phi(x,y)\nabla_y u(y)\cdot \gamma dS_y. \tag{3.2}$$

Proof. $\forall \varepsilon > 0$ small enough, we have

$$\begin{aligned} & \int_\Omega \Phi(x,y)(-\Delta_y u(y))dy = \lim_{\varepsilon\to 0^+}\int_{\Omega\setminus B_\varepsilon(x)} \Phi(x,y)(-\Delta_y u(y))dy \\ = & \lim_{\varepsilon\to 0^+}\int_{\Omega\setminus B_\varepsilon(x)} -\Delta_y\Phi(x,y)u(y)dy - \lim_{\varepsilon\to 0^+}\int_{\partial\Omega}(\Phi(x,y)\nabla u(y)\cdot\gamma - \nabla\Phi(x,y)\cdot\gamma u(y))dS_y \\ & -\lim_{\varepsilon\to 0^+}\int_{\partial B(x,\varepsilon)}(\Phi(x,y)\nabla u(y)\cdot\gamma - \nabla\Phi(x,y)\cdot\gamma u(y))dS_y \\ = & \ 0 - \lim_{\varepsilon\to 0^+}\int_{\partial\Omega}(\Phi(x,y)\nabla u(y)\cdot\gamma - \nabla\Phi(x,y)\cdot\gamma u(y))dS_y + u(x). \end{aligned}$$

where we have used facts

$$\begin{aligned} \left|\int_{\partial B(x,\varepsilon)} \Phi(x,y)\nabla u(y)\cdot\gamma dS_y\right| \le C\varepsilon\max_{\partial B(x,\varepsilon)}|\nabla u| \to 0, \\ \int_{\partial B(x,\varepsilon)} u(y)\nabla\Phi(x,y)\cdot\gamma dS_y = ⨍_{\partial B(x,\varepsilon)} u(y)dS_y \to u(x). \end{aligned}$$

□

Theorem 3.2. *(Green's function is symmetric with its two variables)*

$$G(x,y) = G(y,x).$$

We give the main idea of the prove here. The technical point is the same as the proof of the above theorem. $\forall\varepsilon > 0$ small enough such that $B(x,\varepsilon)\cup B(y,\varepsilon)\subset\Omega$, let $\Omega_\varepsilon = \Omega\setminus(B(x,\varepsilon)\cup B(y,\varepsilon))$. Notice that $G(x,z) = G(y,z) = 0$ on $z\in\partial\Omega$,

$$\begin{aligned} 0 &= \int_{\Omega_\varepsilon}(G(y,z)\Delta_z G(x,z) - G(x,z)\Delta_z G(y,z))dz \\ &= \int_{\partial\Omega_\varepsilon}(G(y,z)\nabla_z G(x,z)\cdot\gamma - G(x,z)\nabla_z G(y,z)\cdot\gamma)dS_z \\ &= \int_{\partial B(x,\varepsilon)\cup\partial B(y,\varepsilon)}(G(y,z)\nabla_z G(x,z)\cdot\gamma - G(x,z)\nabla_z G(y,z)\cdot\gamma)dS_z \end{aligned}$$

We just take $\partial B(y,\varepsilon)$ as an example, the same discussion for the term on $\partial B(x,\varepsilon)$,

$$\begin{aligned} \left|\int_{\partial B(y,\varepsilon)} G(y,z)\nabla_z G(x,z)\cdot\gamma dS_z\right| \le C(\varepsilon+\varepsilon^{n-1}) \to 0, \\ -\int_{\partial B(y,\varepsilon)} G(x,z)\nabla_z G(y,z)\cdot\gamma dS_z = ⨍_{\partial B(y,\varepsilon)} G(x,z)dS_z + o(\varepsilon^{n-1}) \to -G(x,y). \end{aligned}$$

3.1. **Half space problem.** The half space we study here is $\mathbb{R}^n_+ = \{x = (x_1,\cdots,x_n)\in\mathbb{R}^n | x_n > 0\}$.

$\forall x = (x_1,\cdots,x_{n-1},x_n)\in\mathbb{R}^n_+$, we call $\tilde{x} = (x_1,\cdots,x_{n-1},-x_n)$ is x's reflection in the plane $\{x_n = 0\}$.

We study the following boundary value problem

$$\begin{aligned} -\Delta u = f, \quad \text{in } \mathbb{R}^n_+. \\ u|_{\partial\mathbb{R}^n_+} = g, \end{aligned}$$

Our goal here is to find Green's function $G(x,y)$ of this problem and write the solution by using formula

$$u(x) = \int_\Omega G(x,y)f(y)dy - \int_{\partial\Omega} \nabla_y G(x,y)\cdot\gamma g(y)dS_y.$$

$\forall x\in\mathbb{R}^n_+$, the Green's function should be a solution of

$$-\Delta_y G = \delta(y-x) \quad y\in\mathbb{R}^n_+$$
$$G|_{y\in\partial\mathbb{R}^n_+} = 0.$$

The the Green's function of half space problem is easy to obtain, i.e.

$$G(x,y) = \Phi(x,y) - \Phi(\tilde{x},y), \quad x,y\in\mathbb{R}^n_+, x\neq y.$$

Then

$$\frac{\partial G}{\partial y_n}(x,y) = \frac{\partial\Phi}{\partial y_n}(y-x) - \frac{\partial\Phi}{\partial y_n}(y-\tilde{x}) = \frac{-1}{n\alpha(n)}\Big(\frac{y_n-x_n}{|y-x|^n} - \frac{y_n+x_n}{|y-\tilde{x}|^n}\Big).$$

Therefore, $\forall y\in\partial\mathbb{R}^n_+$,

$$\frac{\partial G}{\partial\gamma}(x,y) = -\frac{\partial G}{\partial y_n}(x,y) = -\frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n}.$$

Then the solution of boundary value problem can be represented by

$$u(x) = \frac{2x_n}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+}\frac{g(y)}{|x-y|^n}dy, \quad \forall x\in\mathbb{R}^n_+.$$

which is called the **Poisson formula** of half space problem.

The function

$$K(x,y) := \frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n}, \quad x\in R^n_+, y\in\partial\mathbb{R}^n_+$$

is called the **Poisson kernel** for $\mathbb{R}^n_+$.

Theorem 3.3. *Assume $g\in C(\mathbb{R}^{n-1})\cap L^\infty(\mathbb{R}^{n-1})$, u is defined by the Poisson formula. Then $u\in C^\infty(\mathbb{R}^n_+)\cap L^\infty(\mathbb{R}^n_+)$, $-\Delta u = 0$ in $\mathbb{R}^n_+$ and $\forall x^0\in\partial\mathbb{R}^n_+$,*

$$\lim_{x\in\mathbb{R}^n_+, x\to x^0} u(x) = g(x^0).$$

Proof. $-\Delta u = 0$ is easy to check. Notice that $\forall x\in\mathbb{R}^n_+$,

$$\int_{\partial\mathbb{R}^n_+} K(x,y)dy = 1.$$

Since $\forall x\neq y$, $K(x,y)$ is a smooth function in x, we know directly that $u\in C^\infty(\mathbb{R}^n_+)$ and

$$\Delta u(x) = \int_{\partial\mathbb{R}^n_+}\Delta_x K(x,y)g(y)dy = 0, \quad \forall x\in\mathbb{R}^n_+.$$

For boundary condition, $\forall x_0\in\partial\mathbb{R}^n_+$, $\forall\varepsilon>0$, choose $\delta>0$ small enough such that $\forall y\in\partial\mathbb{R}^n_+$ and $|y-x^0|<\delta$, we have

$$|g(y)-g(x^0)|<\varepsilon.$$

Then $\forall x \in \mathbb{R}^n_+$ and $|x - x^0| < \delta/2$, we have

$$|u(x) - g(x_0)| = \Big| \int_{\partial\mathbb{R}^n_+} K(x,y)(g(y) - g(x^0)) \Big|$$

$$\begin{aligned}
&\le \int_{\partial\mathbb{R}^n_+ \cap B(x^0,\delta)} K(x,y)|g(y) - g(x^0)|dy + \int_{\partial\mathbb{R}^n_+ \setminus B(x^0,\delta)} K(x,y)|g(y) - g(x^0)|dy \\
&\le \varepsilon + 2\|g\|_{L^\infty} \int_{\partial\mathbb{R}^n_+ \setminus B(x^0,\delta)} K(x,y)dy \\
&\le \frac{2^{n+2}\|g\|_{L^\infty} x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \setminus B(x^0,\delta)} \frac{1}{|y - x^0|^n} dy \to 0, \qquad \text{as } x_n \to 0+ .
\end{aligned}$$

□

3.2. **problem in a ball.** We will give an exact formula for the Green's function in a ball.

$\forall x \in B^n(0,1)$. We need that $G(x,y) = 0$, $\forall y \in \partial B^n(0,1)$. Let $\tilde{x}$ be the inversion of x, i.e. $\tilde{x} = \frac{x}{|x|^2}$, thus

$$|\tilde{x} - y| \cdot |x| = |x - y|, \quad \forall y \in \partial B^n(0,1)$$

and

$$G(x,y) = \Phi(|x - y|) - \Phi(|y - x|) = \Phi(|y - x|) - \Phi(|x| \cdot |y - \tilde{x}|), \quad \forall y \in \partial B^n(0,1)$$

Since Φ is the fundamental solution,

$$-\Delta_y \Phi(|x| \cdot |y - \tilde{x}|) = 0, \quad \forall y \neq \tilde{x}.$$

As a consequence,

$$G(x,y) = \Phi(|y - x|) - \Phi(|x| \cdot |y - \tilde{x}|), \quad \forall y \in B^n(0,1),$$

is called the Green's function on $B^n(0,1)$.

Now we will give the Poisson's formula for $B^n(0,r)$.

$$\begin{aligned}
-\Delta u = 0, &\quad \text{in } B^n(0,1) \\
u|_{\partial B(0,1)} = h.&
\end{aligned}$$

By Green's formula we have the solution is

$$u(x) = -\int_{\partial B(0,1)} h(y) \nabla G(x,y) \cdot \gamma dS_y.$$

We will explicitly calculate this formula.

$$\begin{aligned}
\nabla_y \Phi(y - x) &= -\frac{1}{n\alpha(n)} \frac{y - x}{|x - y|^n} \\
\nabla_y \Phi(|x|(y - \frac{x}{|x|^2})) &= -\frac{1}{n\alpha(n)} \nabla_y \frac{1}{|x|^{n-2}|y - \frac{x}{|x|^2}|^{n-2}} \\
&= \frac{-1}{n\alpha(n)} \frac{1}{|x|^{n-2}} \frac{y - \frac{x}{|x|^2}}{|y - \frac{x}{|x|^2}|^n} = \frac{-1}{n\alpha(n)} \frac{y|x|^2 - x}{[|x|(y - \frac{x}{|x|^2})]^n} \\
&= \frac{-1}{n\alpha(n)} \frac{y|x|^2 - x}{|x - y|^n},
\end{aligned}$$

Where we have used the fact $y \in \partial B(0,1)$, $|x| \cdot |y - \frac{x}{|x|^2}| = |x-y|$.

$$\begin{aligned} \nabla_y G(x,y) \cdot \gamma|_{\partial B(0,1)} &= \frac{-1}{n\alpha(n)} \Big(\frac{y-x}{|x-y|^n} - \frac{y|x|^2 - x}{|x-y|^n} \Big) \cdot y \Big|_{y \in \partial B(0,1)} \\ &= \frac{-1}{n\alpha(n)} \frac{|y|^2 - x \cdot y - |y|^2|x|^2 + x \cdot y}{|x-y|^n} \Big|_{|y|=1} \\ &= \frac{-1}{n\alpha(n)} \frac{|y|^2(1-|x|^2)}{|x-y|^n} \Big|_{|y|=1} = \frac{-1}{n\alpha(n)} \frac{1-|x|^2}{|x-y|^n} \end{aligned}$$

Thus the solution formula is

$$u(x) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{h(y)}{|x-y|^n} dS_y.$$

For problems on $B(0,r)$, by doing scaling $\tilde{u}(x) = u(rx)$, $\tilde{h}(x) = h(rx)$, we will have the **Poisson's formula**

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{h(y)}{|x-y|^n} dS_y, \quad \forall x \in B(0,r). \tag{3.3}$$

We call

$$\frac{r^2 - |x|^2}{n\alpha(n)r} \frac{1}{|x-y|^n}$$

the **Poisson's kernel** for $B(0,r)$.

Theorem 3.4. *If $h \in C(\partial B)$, then $u \in C^\infty(B)$, $-\Delta u = 0$ and $\lim_{x \to x^0} u(x) = h(x^0)$, $\forall x^0 \in \partial B$.*

4. Maximum Principle

For more general equations. Let Ω be a bounded open subset of $\mathbb{R}^n$.

$$Lu = -\Delta u + c(x)u = f, \text{ in } \Omega$$

Theorem 4.1. *(Weak maximum principle) Let $0 \le c(x) \le \bar{c}$ in Ω, if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $Lu \le 0$ in Ω, then*

$$\sup_{\Omega} u(x) \le \sup_{\partial\Omega} u^+(x),$$

where $u^+(x) = \max\{u(x), 0\}$.

Proof. Assume $Lu < 0$ in Ω. If $\exists x_0 \in \Omega$ such that

$$0 \le u(x_0) = \max_{\Omega} u,$$

then

$$-\Delta u|_{x_0} + c(x_0)u(x_0) \ge 0,$$

which is a contradiction.

If $Lu \le 0$ in Ω, we introduce an auxiliary function

$$w(x) = u(x) + \varepsilon e^{ax_1}$$

where a is to be determined later, then we can choose a such that $-a^2 + \bar{c} < 0$, and

$$Lw = Lu + \varepsilon e^{ax_1}(-a^2 + c(x)) < 0.$$

Our above discussion applies $\sup_{\Omega} w \le \sup_{\partial\Omega} w^+$, then the results hold by taking $\varepsilon \to 0$.

□

Remark 4.1. If $c \equiv 0$, then $\sup_{\partial\Omega} u^+$ in the theorem can be replaced by $\sup_{\partial\Omega} u$.

Remark 4.2. If $Lu \geq 0$, then $\inf_{\Omega} u \geq \inf_{\partial\Omega}(-u^-)$.

We will consider the problem

$$\begin{aligned} -\Delta u &= f, \qquad \text{in } \Omega \\ u &= \varphi \qquad \text{on } \partial\Omega \end{aligned} \tag{4.1}$$

Theorem 4.2. *If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a solution of* (4.1)*, then*

$$\max_{\Omega} |u| \leq \Phi + CF,$$

where $\Phi = \max_{\partial\Omega} |\varphi|$, $F = \sup_{\Omega} |f|$, $C \sim n, \text{diam}\Omega$.

Proof. Without loss of generality, let $x = 0 \in \Omega$, let

$$w(x) = \pm u + \frac{F}{2n}(d^2 - |x|^2) + \Phi,$$

then

$$-\Delta w = \pm f + F \geq 0, \quad w|_{\partial\Omega} \geq \Phi \pm \varphi \geq 0.$$

By comparison principle, we have $w \geq 0$ in $\bar{\Omega}$, which implies

$$\max_{\Omega} |u| \leq \Phi + \frac{F}{2n} d^2.$$

□

5. Variational Problem

We show in this part that the boundary value problem of Poisson equation is equivalent to a variational problem. Namely

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{5.1}$$

is equivalent to the following problem in some sense,

$$\begin{aligned} J(u) &= \min_{v \in M_g} J(v) \\ J(v) &= \frac{1}{2}\int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} fv dx \\ M_g &= \{v \in C^1(\bar{\Omega}) | v = g \text{ on } \partial\Omega\}. \end{aligned} \tag{5.2}$$

5.1. Dirichlet principle.

Theorem 5.1. *(Dirichlet principle) Assume $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, then u is a solution of* (5.1) *if and only if u is a solution of* (5.2)*.*

Proof. "⇒". $\forall v \in M_g$, we choose $u - v$ as test function in (5.1),

$$\int_{\Omega} -\Delta u (u - v) = \int_{\Omega} f(u - v).$$

Integral by parts with boundary condition $u - v = 0$ on $\partial\Omega$ shows

$$\int_{\Omega} \nabla u \cdot \nabla (u - v) = \int_{\Omega} f(u - v).$$

Equivalently,

$$\int_\Omega |\nabla u|^2 - \int_\Omega fu = \int_\Omega \nabla u\cdot\nabla v - \int_\Omega fv \le \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega fv.$$

Then we have

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu \le \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega fv$$

which implies directly that

$$I(u) \le I(v), \quad \forall v \in M_g.$$

"$\Leftarrow$" $\forall v \in M_0$, we have $u+\varepsilon v \in M_g$. Let $j(\varepsilon) = J(u+\varepsilon v)$, since u is a solution of (5.2), we know that $j'(\varepsilon)|_{\varepsilon=0} = 0$, more precisely,

$$\begin{aligned} &\frac{d}{d\varepsilon}\left[\int_\Omega \frac{1}{2}|\nabla(u+\varepsilon v)|^2 - \int_\Omega f(u+\varepsilon v)\right]_{\varepsilon=0} \\ = \;& \int_\Omega \nabla(u+\varepsilon v)|_{\varepsilon=0}\cdot\nabla v - \int_\Omega fv = \int_\Omega \nabla u\cdot\nabla v - \int_\Omega fv = \int_\Omega(-\Delta u - f)v. \end{aligned}$$

These holds true for any $v \in C_0^1(\bar\Omega)$. Thus u is a solution of (5.2).

□

$-\Delta u = f$ in Ω is called the **Euler-Lagrange equation** of variational problem (5.2).

In the 19th century, it is thought that variational problem always has a solution. But Weierstrass said sometimes the infimum couldn't be achieved by a function in the function set. Here is an example,

Example 1. *(Weierstrass) Variational problem. Let* $M = \{\varphi(x) \in C[0,1] | \varphi'(x)$ *is continuous except finite discontinuity point of the first kind, and* $\varphi(0) = 1, \varphi(1) = 0\}$*. The functional is*

$$F(\varphi) = \int_0^1 [1+(\varphi')^2]^{\frac{1}{4}}\,dx.$$

It is obvious that $\min\limits_{\varphi\in M}(\varphi) = 1$*. In fact, we only need to prove* $\forall\delta > 0$, $\exists\varphi_\delta \in M$ *such that*

$$I(\varphi_\delta) \le 1+\delta,$$

where we can choose

$$\varphi_\delta = \begin{cases} \frac{1}{\delta^2}(\delta^2 - x) & 0 \le x \le \delta^2 \\ 0 & \delta^2 < x \le 1 \end{cases}$$

On the other hand, we couldn't find any $\varphi \in M$ *such that* $I(\varphi) = 1$*. Otherwise,* $\varphi' = 0$ *a.e., then* $\varphi \equiv C$*, which contradicts with* $\varphi(0) = 1$, $\varphi(1) = 0$.

Another fact is that even the boundary value problem (5.1) has a solution in $C^2(\Omega)\cap C(\bar\Omega)$, it may not be obtained by solving the variational problem (5.2). Here is an example by Hadamard,

Example 2. $\Omega = B(0,1)$, $f \equiv 0$, $\varphi(\theta) = \sum\limits_{n=1}^{\infty}\dfrac{\sin n^4\theta}{n^2} \in C(\partial\Omega)$, $0 \le \theta \le 2\pi$.

We know that (5.1) *has a unique solution* $u_0 \in C(\bar\Omega)\cap C^2(\Omega)$ *with expression*

$$u_0(\rho,\theta) = \sum_{n=1}^{\infty}\frac{\sin n^4\theta}{n^2}\rho^{n^4}.$$

On the other hand we can prove that

$$J(u_0) = +\infty.$$

In fact,

$$\begin{aligned} J(u_0) &= \lim_{r\to 1-}\int\int_{\rho\le r}|\nabla u_0|^2dxdy = \lim_{r\to 1-}\int\int_{\rho\le r}\Big[\Big(\frac{\partial u_0}{\partial\rho}\Big)^2+\frac{1}{\rho^2}\Big(\frac{\partial u_0}{\partial\theta}\Big)^2\Big]\rho d\rho d\theta \\ &= \lim_{r\to 1-}2\pi\int_0^r\sum_{n=1}^\infty n^4\rho^{2n^4-1}d\rho = \lim_{r\to 1-}\pi\sum_{n=1}^\infty r^{2n^4} = +\infty. \end{aligned}$$

We call $H^1(\Omega)$ the Sobolev spaces such that

$$H^1(\Omega) = \{u|u, Du\in L^2(\Omega)\}$$

with norm and inner product

$$\|u\|_{H^1} = \|u\|_{L^2}+\|\nabla u\|_{L^2}, \quad \langle u,v\rangle_{H^1} = \int_\Omega uv + \int_\Omega \nabla u\cdot\nabla v.$$

H^1 is a Hilbert space. $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$, the completion of $C_0^\infty(\Omega)$ with H^1 norm.

For bounded Ω with uniform cone condition, $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.

$(-\Delta)^{-1}$ with homogenous Dirichlet boundary condition is a compact operator in $L^2(\Omega)$, since

$$(-\Delta)^{-1}: L^2(\Omega)\to H^1(\Omega)\hookrightarrow\hookrightarrow L^2(\Omega).$$

Definition 4. If $\exists u\in H_0^1(\Omega)$ such that

$$J(u) = \min_{v\in H_0^1}\Big(\frac{1}{2}\int_\Omega|\nabla v|^2-\int_\Omega fv\Big)$$

we call u is a solution of (5.2).

Definition 5. If $\exists u\in H_0^1$ such that $\forall v\in H_0^1(\Omega)$,

$$\int_\Omega\nabla u\cdot\nabla v = \int_\Omega fv,$$

then we call u a weak solution of (5.1)

Theorem 5.2. *If $u\in H_0^1(\Omega)$, then u is a weak solution of* (5.1) *if and only if u is a solution of* (5.2).

The proof of this theorem is left to readers.

5.2. **Lax-Milgram.** We first list the Lax-Milgram theorem from functional analysis, then prove the existence of weak solution of (5.1).

Theorem 5.3 (Lax-Milgram theorem)**.** *H is a Hilbert space, assume $a(u,v)$ is a bi-linear mapping from H to $\mathbb{R}$, satisfies*

- *Bounded. $\exists M\ge 0$ such that $|a(u,v)|\le M\|u\|\cdot\|v\|$, $\forall u,v\in H$.*
- *Coercive. $\exists\delta>0$ such that $a(u,u)\ge\delta\|u\|^2$, $\forall u\in H$.*

Then for any bounded linear functional $F(v)$ on H, there exists a unique $u\in H$ such that

$$F(v) = a(u,v), \quad \forall v\in H.$$

and

$$\|u\|\le\frac{1}{\delta}\|F\|.$$

Proof. For any fixed $u \in H$, Riesz representation theorem implies that $\exists Au \in H$ such that

$$a(u,v) = (Au, v), \quad \forall v \in H.$$

The linearity of Au in u is obvious due to the fact that $a(u,v)$ is linear in u. Furthermore,

$$(Au, v) \le M\|u\| \cdot \|v\|, \quad \Rightarrow \quad \|Au\| \le M\|u\|.$$

Coercivity gives that $\forall u \in H$,

$$\delta\|u\|^2 \le a(u,u) = (Au, u) \le \|Au\| \cdot \|u\|, \quad \Rightarrow \quad \|Au\| \ge \delta\|u\|.$$

Thus A^{-1} exists. We claim that $R(A) = H$.

First $R(A)$ is closed. In fact, choose any Cauchy sequence $\{Au_k\}$ in $R(A)$, then $\lim_{k\to\infty} Au_k = v$. By coercivity, we have

$$\delta\|u_k - u_l\| \le \|Au_k - Au_l\|,$$

which means $\{u_k\}$ is also a Cauchy sequence in H. $\exists u \in H$ such that

$$\lim_{k\to\infty} u_k = u.$$

Thus

$$Au = \lim_{k\to\infty} Au_k = v.$$

If $R(A) \ne H$, $\exists w \ne 0$ in H such that

$$(Au, w) = 0, \quad \forall u \in H,$$

which contradicts with coercivity if we choose $w = u$. Thus $R(A) = H$.

For any linear functional $F(v)$ on H, by Riesz representation theorem, we have a unique $w \in H$ s.t.

$$F(v) = (w, v).$$

Let $u = A^{-1}w$, we have

$$\|u\| \le \|A^{-1}\| \cdot \|w\| \le \frac{1}{\delta}\|F\|$$

and

$$F(v) = (Au, v).$$

□

Theorem 5.4. *For $f \in L^2(\Omega)$, there exists a solution $u \in H_0^1(\Omega)$ of* (5.1).

Proof. Let the bilinear functional defined by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v.$$

Then it is coercive

$$a(u,u) \ge \|\nabla u\|_{L^2}^2 \ge C\|u\|_{H^1}^2.$$

Lax-Milgram theorem implies that $\forall f \in L^2(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \langle f, v\rangle, \quad \forall v \in H_0^1(\Omega).$$

□

5.3. **Solvability of variational problem.** *** Our goal in this subsection is to prove the unique solvability of variational problem (5.2).

Theorem 5.5. *Solution of* (5.2) *in* $H_0^1(\Omega)$ *is unique.*

Proof. Let $u_1, u_2 \in H_0^1(\Omega)$ are two solutions of (5.2), i.e.

$$J(u_1) = J(u_2) = m = \inf_{v \in H_0^1(\Omega)} J(v),$$

then

$$0 = \frac{1}{2}\int_\Omega |\nabla u_1|^2 - \frac{1}{2}\int_\Omega |\nabla u_2|^2 - \int_\Omega (u_1 - u_2) f.$$

Notice that fact

$$\left|\frac{\nabla(u_1 - u_2)}{2}\right|^2 + \left|\frac{\nabla(u_1 + u_2)}{2}\right|^2 = \frac{1}{2}|\nabla u_1|^2 + \frac{1}{2}|\nabla u_2|^2,$$

we have

$$\begin{aligned}
\int_\Omega \left|\frac{\nabla(u_1 - u_2)}{2}\right|^2 &= \int_\Omega \frac{1}{2}|\nabla u_1|^2 + \int_\Omega \frac{1}{2}|\nabla u_2|^2 - \int_\Omega \left|\frac{\nabla(u_1 + u_2)}{2}\right|^2 \\
&\quad - \int_\Omega u_1 f - \int_\Omega u_2 f + 2\int_\Omega \frac{u_1 + u_2}{2} f \\
&= J(u_1) + J(u_2) - 2J(\frac{u_1 + u_2}{2}) \le 0
\end{aligned}$$

which implies that

$$\|\nabla(u_1 - u_2)\|_{L^2} = 0.$$

Poincare inequality gives

$$\|u_1 - u_2\|_{L^2} = 0 \quad \Rightarrow \quad u_1 = u_2 \text{ a.e. in } \Omega.$$

□

Lemma 5.1. *(Friedrich inequality for $H_0^1(\Omega)$)*

$$\|u\|_{L^2(\Omega)} \le 2d\|\nabla u\|_{L^2(\Omega)},$$

where $d = diam\Omega$.

Proof. Let $u \in C_0^1(\Omega)$, without loss of generality assume

$$\Omega \subset \{x | 0 \le x_i \le 2d, 1 \le i \le n\} = \bar{Q}.$$

Let $\tilde{u} = \begin{cases} u & x \in \bar{\Omega} \\ 0 & x \in \bar{Q}\backslash\bar{\Omega} \end{cases}$. It is obvious that $\tilde{u} \in C_*^1(\bar{Q})$, piecewise C^1 function, and

$$\tilde{u}|_{\partial\bar{Q}} = 0.$$

By Newton-Leibnitz formula

$$\tilde{u}(x_1, x_2, \cdots, x_n) = \int_0^{x_1} \frac{\partial \tilde{u}}{\partial x_1} dx_1,$$

then

$$\tilde{u}^2 = \Big(\int_0^{x_1} \frac{\partial \tilde{u}}{\partial x_1} dx_1\Big)^2 \le x_1 \int_0^{x_1} \Big(\frac{\partial \tilde{u}}{\partial x_1}\Big)^2 dx_1 \le 2d \int_0^{2d} \left|\frac{\partial \tilde{u}}{\partial x_1}\right|^2 dx_1.$$

Integration in Q gives

$$\int_Q \tilde{u}^2 dx \le 2d \int_Q \int_0^{2d} \left| \frac{\partial \tilde{u}}{\partial x_1} \right|^2 dx_1 dx \le 4d^2 \int_Q |\nabla \tilde{u}|^2 dx.$$

Thus we arrive at

$$\|u\|_{L^2(\Omega)} \le 2d \|\nabla u\|_{L^2(\Omega)}.$$

If $u \in H_0^1(\Omega)$, we can choose $\{u_m\}_{m=1}^\infty \subset C_0^1(\Omega)$ such that

$$\|u_m - u\|_{H^1} \to 0, \quad m \to \infty,$$

and

$$\|u_m\|_{L^2} \le 2d \|\nabla u_m\|_{L^2},$$

our result can be obtained by taking $m \to \infty$. □

Theorem 5.6. *(Existence) $f \in L^2(\Omega)$, then* (5.2) *has a solution $u \in H_0^1(\Omega)$.*

Proof. First we prove that $J(u)$ has a lower bound. In fact, by Hölder and Friedrich inequality,

$$J(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 - \int_\Omega fv \ge \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{1}{4}\|\nabla v\|_{L^2}^2 - C\|f\|_{L^2}^2 \ge -C(d)\|f\|_{L^2}^2.$$

Let

$$m = \inf_{v \in H_0^1(\Omega)} J(v).$$

Let $\{v_k\}_{k=1}^\infty \subset H_0^1(\Omega)$ be a minimizing sequence such that

$$J(v_k) \le m + \frac{1}{k}.$$

We want to prove that $\{v_k\}$ is a Cauchy sequence in $H^1(\Omega)$, by using similar discussions to the uniqueness proof, for $k, l \to \infty$,

$$\left\| \nabla \frac{(v_k - v_l)}{2} \right\|_{L^2}^2 = J(v_k) + J(v_l) - 2J(\frac{v_k + v_l}{2}) \le m + \frac{1}{k} + m + \frac{1}{l} - 2m \le \frac{1}{k} + \frac{1}{l} \to 0.$$

Then there must $\exists u \in H_0^1(\Omega)$ such that

$$v_k \to u \quad \text{in } H^1(\Omega).$$

Taking limit in the energy, we have $J(v_k) \to J(u)$ and $J(u) = m$.

□

6. Energy Estimate

Energy methods for Poisson equation is easy. I will not talk about it here. But leave it as an exercise. The energy estimate also shows that $-\Delta u = f$ in Ω and $u = h$ on $\partial\Omega$ has at most one solution in $C^2(\Omega) \cap C^1(\bar{\Omega})$.

7. Problems

(1) Try to derive energy estimates for Dirichlet problem of Possion equation.

(2) Modify the proof of the mean value formulas to show for $n \geq 3$ that

$$u(0) = \fint_{\partial B(0,r)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f dx,$$

provided $\begin{cases} -\triangle u = f & x \in B(0,r) \\ u = g & x \in \partial B(0,r) \end{cases}$.

(3) We say $v \in C^2(\bar{\Omega})$ is subharmonic if $-\triangle v \leq 0$ in Ω.

(a) Prove for subharmonic v that

$$v(x) \leq \fint_{B(x,r)} v dy, \quad \text{for all } B(x,r) \subset \Omega.$$

(b) Prove that therefore $\max_{\bar{\Omega}} v = \max_{\partial\Omega} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

(d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

(4) Let $B^+(R) = \{(x,y) : x^2 + y^2 < R^2, y > 0\}$, try to find the Green's function of the following problem

$$\begin{cases} -\triangle u = f(x,y), & (x,y) \in B^+(R), \\ u|_{\partial B^+(R) \cap \{y>0\}} = \varphi(x,y), & \\ u_y|_{y=0} = \psi(x,0), & -R \leq x \leq R. \end{cases}$$

Furthermore, give the representation formula of solution.

(5) Ω is a bounded open subset of $\mathbb{R}^n$, $u(x)$ is a classical solution of

$$\begin{cases} -\triangle u + c(x)u = f(x), & x \in \Omega, \\ (\nabla u \cdot \gamma + \alpha(x)u)|_{\Gamma_1} = \varphi_1, & u|_{\Gamma_2} = \varphi_2 \end{cases}$$

where $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_2 \neq \emptyset$.

If $c(x) \geq 0$, $\alpha(x) \geq \alpha_0 > 0$, try to prove the following estimate,

$$\max_{\Omega} |u(x)| \leq C(\alpha_0, \text{diam}\Omega) \Big[\sup_{\Omega} |f| + \sup_{\Gamma_1} |\varphi_1| + \sup_{\Gamma_2} |\varphi_2| \Big].$$

(6) Try to get the Euler-Lagrange equation of the following variational problem

$$J(u) = \min_{v \in M_0} J(v), \text{ with } M_0 = \{u \in C^2(\Omega) \cap C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\},$$

(a) $J(v) = \displaystyle\int_\Omega (\frac{1}{p}|\nabla v|^p - fv) dx$, $p > 1$

(b) $J(v) = \displaystyle\int_\Omega (\frac{1}{2m}|\nabla v^m|^2 - fv) dx$, $m > 0$

(c) $j(v) = \displaystyle\int_\Omega (\sqrt{1 + |\nabla v|^2} dx + v^p) dx$, $p > 1$

(7) If $u \in H_0^1(\Omega)$ is a weak solution of

$$-\triangle u + u = f,$$

prove that u is a solution of variational problem

$$J(u) = \min_{v \in H_0^1(\Omega)} J(v),$$

where $J(v) = \dfrac{1}{2}\int_\Omega |\nabla v|^2 dx + \dfrac{1}{2}\int_\Omega v^2 dx - \int_\Omega fv dx$.

(8) Assume $f \in L^2(\Omega)$, $\varphi \in H^1(\Omega)$, $c(x) \geq 0$ and $c(x) \in C(\bar{\Omega})$, prove that variational problem

$$J(u) = \min_{v \in M_\varphi} J(v)$$

has a unique solution in $M_\varphi = \{u \in H^1(\Omega) : u - \varphi \in H_0^1(\Omega)\}$, where

$$J(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 + c(x)v^2 - fv)dx.$$

Furthermore, show that the solution of variational problem is a weak solution of

$$-\triangle u + c(x)u = f \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega.$$

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