WAVE EQUATION

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Contents

1. Cauchy Problem

1.1. Solution formula and existence.

1.1.1. *D'Alembert formula — 1D.* We will give the formal solution of Cauchy problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R} \\ u|_{t=0} &= g(x), \\ u_t|_{t=0} &= h(x). \end{aligned} \tag{1.1}$$

By factorizing the operator $\partial_{tt} - \partial_{xx} = (\partial_t + \partial_x)(\partial_t - \partial_x)$, we will solve the following two transport equations,

$$\begin{aligned} v_t + v_x &= 0, \\ v|_{t=0} &= h(x) - g'(x) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} u_t - u_x &= v, \\ u|_{t=0} &= g(x) \end{aligned} \tag{1.3}$$

By the solution of transport equation, (1.2) has solution

$$v(x,t) = h(x-t) + g'(x-t).$$

(1.3) has solution

$$u(x,t) = g(x+t) + \int_0^t v(x+(t-s),s)ds.$$

Thus the solution of (1.1) is

$$\begin{aligned} u(x,t) &= g(x+t) + \int_0^t h(x+t-2s) + g'(x+t-2s)ds \\ &= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t}(h(y) - g'(y))dy \\ &= g(x+t) - \frac{1}{2}g(x+t) + \frac{1}{2}g(x-t) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy \\ &= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy. \end{aligned}$$

D'Alembert formula refers to

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy. \tag{1.4}$$

which is the formal solution of (1.1).

From this formula, we are ready to get the existence of solution for smooth initial data,

Theorem 1.1. *If $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, then $u \in C^2(\mathbb{R} \times [0,+\infty))$ satisfies wave equation $u_{tt} - u_{xx} = 0$, and*

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0), \qquad \lim_{(x,t)\to(x_0,0)} u_t(x,t) = h(x_0).$$

Some properties of the solution. By using D'Alembert formula, there are some important sets in (x,t) space.

(1) We call $\{y \in \mathbb{R} | |y - x| \leq t\}$ the **domain of dependence** of point (x,t).
(2) $\{(x,t) \in \mathbb{R} \times [0,+\infty) | x \geq x_1 - t \text{ and } x \leq x_2 + t\}$ the **range of influence** of $[x_1, x_2]$.
(3) $\{(x,t) \in \mathbb{R} \times [0,+\infty) | x \geq x_1 + t \text{ and } x \leq x_2 - t\}$ the **determining region** of $[x_1, x_2]$.
(4) $x + t$ and $x - t$ are **characteristics** of the wave equation.

Due to characteristics, wave equation has the property of **Finite speed propagation of singularity**.

Nonhomogeneous problem By the same method we can also find the solution of nonhomogeneous problem

$$\begin{aligned} u_{tt} - u_{xx} = f(x,t), &\quad x \in \mathbb{R}, t > 0 \\ u|_{t=0} = g(x), &\quad u_t|_{t=0} = h(x) \end{aligned} \tag{1.5}$$

which is,

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy + \frac{1}{2}\int_0^t ds \int_{x-(t-s)}^{x+(t-s)} f(y,s)dy. \tag{1.6}$$

Theorem 1.2. *If $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, $f \in C^2(\mathbb{R} \times [0,+\infty))$, then u in (1.6), a function in $C^2(\mathbb{R} \times [0,+\infty))$, is a classical solution of (1.5).*

By using the solution formula, it is easy to check that

Corollary 1.1. *If g, h and f are odd (even, or periodic) in x, so is u.*

1.1.2. *Half-line problem.* By using extension, we will be able to give the solution formula of the following problem

$$
\begin{aligned}
u_{tt} - u_{xx} = 0, &\quad \text{in } \mathbb{R}^+ \times \mathbb{R}^+ \\
u|_{t=0} = g(x), &\quad u_t|_{t=0} = h(x) \\
u|_{x=0} = 0 &
\end{aligned}
\tag{1.7}
$$

For compatibility we need $h(0) = g(0) = 0$. Due to the homogeneous Dirichlet boundary condition at $x = 0$, we use odd extension. Let

$$
\tilde{g} = \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x < 0 \end{cases}
$$

The same extensions for $\tilde{u}(x,t)$ and $\tilde{h}(x)$. Then we have that $\tilde{u}$ satisfies

$$
\begin{aligned}
\tilde{u}_{tt} - \tilde{u}_{xx} = 0, &\quad \text{in } \mathbb{R}^+ \times \mathbb{R} \\
\tilde{u}|_{t=0} = \tilde{g}(x), & \\
\tilde{u}_t|_{t=0} = \tilde{h}(x). &
\end{aligned}
$$

By D'Alembert formula, $\tilde{u}$ has the representation

$$
\tilde{u}(x,t) = \frac{1}{2}(\tilde{g}(x+t) + \tilde{g}(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)dy.
$$

We need to get back to the domain $\{(x,t) : x > 0, t > 0\}$ and drop the tildes in the formula. There are two cases, in the case of $x \geq t$, our solution has the representation

$$
u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy.
$$

And in the case of $0 \leq x < t$, the solution is

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}(g(x+t) - g(t-x)) + \frac{1}{2}\Big(\int_0^{x+t} h(y)dy - \int_0^{t-x} h(-y)dy\Big). \\
&= \frac{1}{2}(g(x+t) - g(t-x)) + \frac{1}{2}\int_{t-x}^{x+t} h(y)dy.
\end{aligned}
$$

Remark 1.1. For nonhomogeneous boundary condition $u|_{x=0} = u_D(t)$, one can use new variable $v = u - u_D(t)$ to do the same discussion, where $v_{tt} - v_{xx} = -(u_D)_{tt}$.

Remark 1.2. It is an easy exercise to get half-line problem with homogeneous Neumann boundary condition $u_x|_{x=0} = 0$ by using even extension.

1.1.3. *Kirchhoff formula in 3D and Poisson formula in 2D.* We will reduce the multi-dimension problem into a half-line problem by using Spherical mean of the solution.

The spherical mean of a function $u(x,t)$ on $\partial B(x,r)$ is given by

$$
U(x;r,t) = ⨍_{\partial B(x,r)} u(y,t)dS_y \tag{1.8}
$$

Lemma 1.1. *If $u \in C^m([0,+\infty) \times \mathbb{R}^n)$ is a solution of*

$$
\begin{aligned}
u_{tt} - \Delta u = 0, &\quad \textit{in } (0,+\infty) \times \mathbb{R}^n \\
u|_{t=0} = g, &\quad u_t|_{t=0} = h.
\end{aligned}
\tag{1.9}
$$

Then the spherical mean of u, $U(x;r,t)\in C^m([0,+\infty)\times[0,+\infty))$ *satisfies*

$$\begin{aligned} U_{tt}-U_{rr}-\frac{n-1}{r}U_r=0, &\quad \textit{in } (0,+\infty)\times\mathbb{R}^+\\ U|_{t=0}=G, &\quad U_t|_{t=0}=H, \end{aligned}$$

which is called the **Euler-Poisson-Darboux equation**.

Proof. By direct calculations, we have

$$\begin{aligned} U_r(x;r,t) &= \frac{\partial}{\partial r}⨍_{\partial B(x,r)}u(y,t)dS_y=\frac{\partial}{\partial r}⨍_{\partial B(0,1)}u(x+rz,t)dS_z\\ &= ⨍_{\partial B(0,1)}\nabla u(x+rz,t)\cdot z dS_z=⨍_{\partial B(x,r)}\nabla u(y,t)\cdot\frac{y-x}{r}dS_y\\ &= ⨍_{\partial B(x,r)}\nabla u\cdot\gamma dS_y=\frac{r}{n}⨍_{B(x,r)}\Delta u(y,t)dy. \end{aligned}$$

As a consequence,

$$\lim_{r\to 0+}U_r(x;r,t)=0.$$

If we take one derivative more,

$$\begin{aligned} U_{rr}(x;r,t) &= \frac{\partial}{\partial r}\Big(\frac{r}{n}⨍_{B(x,r)}\Delta u(y,t)dy\Big)=\frac{1}{n\alpha(n)}\frac{\partial}{\partial r}\Big(r^{1-n}\int_{B(x,r)}\Delta u(y,t)dy\Big)\\ &= \frac{1-n}{n}\frac{1}{\alpha(n)r^n}\int_{B(x,r)}\Delta u(y,t)dy+\frac{1}{n\alpha(n)r^{n-1}}\frac{\partial}{\partial r}\int_{B(x,r)}\Delta u(y,t)dy\\ &= (\frac{1}{n}-1)⨍_{B(x,r)}\Delta u dy+⨍_{\partial B(x,r)}\Delta u dS_y. \end{aligned}$$

and

$$\lim_{r\to 0+}U_{rr}(x;r,t)=\frac{1}{n}\Delta u(x,t).$$

Then by iteration, if $u\in C^m$, we have $U\in C^m$.

Back to the first order derivative, by using the wave equation $u_{tt}-\Delta u=0$, we have

$$U_r=\frac{r}{n}⨍_{B(x,r)}u_{tt}dy=\frac{1}{n\alpha(n)r^{n-1}}\int_{B(x,r)}u_{tt}dy.$$

Multiplication of it by r^{n-1} gives

$$r^{n-1}U_r=\frac{1}{n\alpha(n)}\int_{B(x,r)}u_{tt}dy.$$

The the desired equation follows from Taking one more derivative of it, i.e.

$$(r^{n-1}U_r)_r=\frac{1}{n\alpha(n)}\int_{\partial B(x,r)}u_{tt}dS_y=r^{n-1}⨍_{\partial B(x,r)}u_{tt}dS_y=r^{n-1}U_{tt}.$$

□

In the case $n=3$, we will get Kirchhoff's formula by using Euler-Poisson-Darboux equation. Let $\widetilde{U}=rU$, $\widetilde{G}=rG$ and $\widetilde{H}=rH$, we have

$$\widetilde{U}_r=U+rU_r,$$

and moreover,

$$\widetilde{U}_{tt}=rU_{tt}=rU_{rr}+2U_r=(U+rU_r)_r=\widetilde{U}_{rr}.$$

Now $\widetilde{U}$ solves the half-line problem

$$\begin{aligned}\widetilde{U}_{tt}-\widetilde{U}_{rr}=0, &\quad \text{in } \mathbb{R}^+\times\mathbb{R}^+\\ \widetilde{U}|_{t=0}=\widetilde{G}, \widetilde{U}_t|_{t=0}=\widetilde{H}&\\ \widetilde{U}|_{r=0}=0.&\end{aligned}$$

By solution representation in half-line problem, we have

$$\widetilde{U}(x;r,t)=\frac{1}{2}(\widetilde{G}(r+t)-\widetilde{G}(t-r))+\frac{1}{2}\int_{t-r}^{r+t}\widetilde{H}(y)dy, \quad \forall 0<r<t.$$

Since $u(x,t)$ is a continuous function, its value on (x,t) is exactly the limit of its spherical mean. Thus we have

$$\begin{aligned}u(x,t) &= \lim_{r\to 0^+}\frac{\widetilde{U}(x;r,t)}{r}=\lim_{r\to 0^+}U(x;r,t)\\ &= \lim_{r\to 0^+}\left[\frac{1}{2}\frac{\widetilde{G}(r+t)-\widetilde{G}(t-r)}{r}+\frac{1}{2r}\int_{t-r}^{r+t}\widetilde{H}(y)dy\right]\\ &= \widetilde{G}'(t)+\widetilde{H}(t)\end{aligned}$$

By definition of $\widetilde{G}$ and $\widetilde{H}$, going back to variables g and h, we arrive at

$$u(x,t)=\frac{\partial}{\partial t}\left[t\fint_{\partial B(x,t)} g dS_y\right]+t\fint_{\partial B(x,t)} h dS_y.$$

Then if we do calculation one step more,

$$\begin{aligned}\frac{\partial}{\partial t}\fint_{\partial B(x,t)} g(y)dS_y &= \frac{\partial}{\partial t}\fint_{\partial B(0,1)} g(x+tz)dS_z\\ = \fint_{\partial B(0,1)}\nabla g(x+tz)\cdot z dS_z &= \fint_{\partial B(x,t)}\nabla g(y)\cdot(y-x)dS_y,\end{aligned}$$

we will have the 3-D **Kirchhoff formula**,

$$u(x,t)=\fint_{\partial B(x,t)}[g(y)+\nabla g(y)\cdot(y-x)+th(y)]dS_y. \tag{1.10}$$

In the case $n=2$, we will get Poisson's formula by the method of descent.

If $u(x_1,x_2,t)$ is a solution in 2-D. Let $\bar{u}(x_1,x_2,x_3,t)=u(x_1,x_2,t)$, then $\bar{u}$ solves the wave equation in 3-D,

$$\begin{aligned}\bar{u}_{tt}-\Delta\bar{u}=0 &\quad \text{in } \mathbb{R}^3\times(0,\infty)\\ \bar{u}|_{t=0}=\bar{g}, &\quad \bar{u}_t|_{t=0}=\bar{h}\end{aligned}$$

where $\bar{g}(x_1,x_2,x_3)=g(x_1,x_2)$ and $\bar{h}(x_1,x_2,x_3)=h(x_1,x_2)$. Then by Kirchhoff's formula in 3-D, we have

$$u(x,t)=\frac{\partial}{\partial t}\left[t\fint_{\partial B(\bar{x},t)}\bar{g}(y)dS_y\right]+t\int_{\partial B(\bar{x},t)}\bar{h}(y)dS_y,$$

where $\bar{x} = (x_1, x_2, 0)$. Due to the fact that $\bar{g}(y_1, y_2, y_3) = g(y_1, y_2)$, we can simplify the integral on $\partial B(\bar{x}, t)$ by

$$\begin{aligned} ⨍_{\partial B(\bar{x},t)} \bar{g}(y) dS_y &= \frac{1}{4\pi t^2} \int_{\partial B(\bar{x},t)} \bar{g} dS_y \\ &= \frac{2}{4\pi t^2} \int_{B(x,t)} g(y)(1 + |D\gamma(y)|^2)^{\frac{1}{2}} dy, \end{aligned}$$

where $\gamma(y) = \sqrt{t^2 - (y-x)^2}$ and $(1 + |D\gamma(y)|^2)^{\frac{1}{2}} = t(t^2 - |y-x|^2)^{-\frac{1}{2}}$. Therefore,

$$\begin{aligned} ⨍_{\partial B(\bar{x},t)} \bar{g}(y) dS_y &= \frac{1}{2\pi t} \int_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} dy \\ &= \frac{t}{2} ⨍_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} dy. \end{aligned}$$

Then taking derivative with respect to t, we have

$$\begin{aligned} \frac{\partial}{\partial t}\left[t^2 ⨍_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} dy \right] &= \frac{\partial}{\partial t}\left[t ⨍_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{\frac{1}{2}}} dz \right] \\ = ⨍_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{\frac{1}{2}}} dz &+ t ⨍_{B(0,1)} \frac{\nabla g(x+tz) \cdot z}{(1-|z|^2)^{\frac{1}{2}}} dz \\ = t ⨍_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} dy &+ t ⨍_{B(x,t)} \frac{\nabla g(y) \cdot (y-x)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} dy \end{aligned}$$

Thus the 2-D **Poisson's formula** is

$$u(x,t) = \frac{1}{2} ⨍_{B(x,t)} \frac{tg(y) + t^2 h(y) + t\nabla g(y) \cdot (y-x)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} dy. \tag{1.11}$$

By Kirchhoff's and Poisson's formula in 3-D and 2-D, we have the following existence result.

Theorem 1.3. $n = 2, 3$. *If* $g \in C^3(\mathbb{R}^n)$, $h \in C^2(\mathbb{R}^n)$ *with* $Q = \mathbb{R}^n \times (0, +\infty)$, *then* $u \in C^2(\bar{Q})$ *is a classical solution of* (1.9).

Difference of Solution behavior between 3-D and 2-D If we have a closed look at the Kirchhoff formula (1.10) and the Poisson formula (1.11), we can easily find the main difference is the integral. Integral over the sphere (which is the boundary of a domain) in Kirchhoff formula and integral over the ball in Poisson formula.

Let's assume that the initial data has compact support Ω, where Ω is connected and regular enough. $\forall x_0 \notin \Omega$ and $d_1 = \text{dist}(x_0, \Omega) > 0$, $d_2 = \max\{\text{dist}(x_0, x) : x \in \Omega\}$, then the possible nonzero point of $u(x_0, t)$ can only be interval $[d_1, d_2]$. While in 2-D, the possible nonzero point of $u(x_0, t)$ must be the half line $[d_1, +\infty)$. That's explained why one can hear the others' voice in 3-D, and the water wave diffuse the whole space in 2-D. A picture is needed here.

1.2. **Uniqueness — Energy method.** With existence theory at hand, uniqueness is a natural question to ask. Is the existed classical solution unique? Furthermore, is it stable? in which sense?

We first introduce a useful lemma.

Lemma 1.2. *(Gronwall's inequality) Assume* $G(\tau) \geq 0$, $G'(\tau) \in C[0,T]$, $G(0) = 0$ *and* $\forall \tau \in [0,T]$, *the following inequality holds*

$$\frac{dG(\tau)}{d\tau} \leq CG(\tau) + F(\tau)$$

where C is a constant, $F(\tau) \geq 0$ nondecreasing in τ. Then

$$\frac{dG(\tau)}{d\tau} \leq e^{C\tau}F(\tau),$$

and

$$G(\tau) \leq C^{-1}(e^{C\tau}-1)F(\tau).$$

Proof. Multiplying the given inequality by $e^{-C\tau}$ and integrate it on $[0,\tau]$, we have

$$e^{-C\tau}G(\tau) \leq \int_0^\tau e^{-Ct}F(t)dt \leq F(\tau)C^{-1}(1-e^{-C\tau}).$$

□

The Cauchy problem we considered is revisited here

$$\begin{aligned} u_{tt} - u_{xx} &= f, \quad \text{in } \mathbb{R}^+ \times \mathbb{R} := Q \\ u|_{t=0} &= g(x), \\ u_t|_{t=0} &= h(x). \end{aligned} \tag{1.12}$$

The energy inequality of 1-D Cauchy problem (1.12) is

Theorem 1.4. *If $u \in C^1(\mathbb{R}\times[0,+\infty)) \cap C^2(\mathbb{R}\times(0,+\infty))$ is a solution of (1.12), then $\forall(x_0,t_0) \in \mathbb{R}\times(0,+\infty)$, we have*

$$\int_{\Omega_\tau}[u_t^2(x,\tau)+u_x^2(x,\tau)]dx \leq M\Big(\int_{\Omega_0}(h^2+g_x^2)dx + \int\int_{K_\tau} f^2(x,t)dxdt\Big),$$

$$\int\int_{K_\tau}[u_t^2(x,t)+u_x^2(x,t)]dxdt \leq M\Big(\int_{\Omega_0}(h^2+g_x^2)dx + \int\int_{K_\tau} f^2(x,t)dxdt\Big).$$

where $K = \{(x,t) \in \mathbb{R}\times[0,+\infty) : |x-x_0| < t_0 - t\}$, $K_\tau = K \cap \{0 \leq t \leq \tau\}$, $\Omega_\tau = K \cap \{t = \tau\}$, $M = e^{t_0}$.

Proof. Multiply the equation by u_t and integrated on K_τ, we have

$$\int\int_{K_\tau}(u_tu_{tt} - u_tu_{xx})dxdt = \int\int_{K_\tau} u_t f dxdt$$

Notice that the boundary of K_τ is $\partial K_\tau = \Omega_0 \cup \Omega_\tau \cup \Gamma_\tau^1 \cup \Gamma_\tau^2$, we can calculate the left hand side by using divergence theorem,

$$\begin{aligned} & \int\int_{K_\tau}\frac{1}{2}(u_t^2+u_x^2)_t dxdt - \int\int_{K_\tau}(u_tu_x)_x dxdt \\ = & \int_{\partial K_\tau}(\frac{1}{2}(u_t^2+u_x^2), -u_tu_x)^T \cdot \gamma dl \\ = & \int_{\Omega_\tau}\frac{1}{2}(u_t^2+u_x^2)dx - \int_{\Omega_0}\frac{1}{2}(u_t^2+u_x^2)dx \\ & + \int_{\Gamma_\tau^1}\frac{1}{\sqrt{2}}(\frac{1}{2}(u_t^2+u_x^2)+u_tu_x)dl + \int_{\Gamma_\tau^2}\frac{1}{\sqrt{2}}(\frac{1}{2}(u_t^2+u_x^2)-u_tu_x)dl \\ \geq & \int_{\Omega_\tau}\frac{1}{2}(u_t^2+u_x^2)dx - \int_{\Omega_0}\frac{1}{2}(h^2+g_x^2)dx. \end{aligned}$$

where γ is the exterior unit normal vector of ∂K_τ, has values $\gamma = (-1,0)$ on Ω_0, $\gamma = (1,0)$ on Ω_τ, $\gamma = \frac{1}{\sqrt{2}}(1,-1)$ on Γ_τ^1 and $\gamma = \frac{1}{\sqrt{2}}(1,1)$ on Γ_τ^2.

And the right hand side can be estimated by

$$\int\int_{K_\tau} u_t f \le \frac{1}{2}\int\int_{K_\tau} u_t^2 + \frac{1}{2}\int\int_{K_\tau} f^2.$$

Combined the above discussions together, we have

$$\int_{\Omega_\tau} (u_t^2 + u_x^2) \le \int_{\Omega_0} (h^2 + g_x^2) + \int\int_{K_\tau} u_t^2 + \int\int_{K_\tau} f^2.$$

Now let

$$\begin{aligned} G(\tau) &= \int\int_{K_\tau} (u_t^2 + u_x^2)dxdt = \int_0^\tau \int_{x_0-(t_0-t)}^{x_0+(t_0-t)} (u_t^2 + u_x^2)dxdt, \\ F(\tau) &= \int_{\Omega_0} (h^2 + g_x^2)dx + \int\int_{K_\tau} f^2 dxdt. \end{aligned}$$

Our above estimates is equivalently

$$\frac{dG(\tau)}{d\tau} \le G(\tau) + F(\tau)$$

where $F(\tau)$ is increasing in τ. Then Gronwall's inequality implies that

$$G(\tau) \le (e^\tau - 1)F(\tau) \le e^{t_0}F(\tau).$$

□

We can also get the L^2 estimate from the energy estimate.

Theorem 1.5. *If $u \in C^1(\mathbb{R} \times [0, +\infty)) \cap C^2(\mathbb{R} \times (0, +\infty))$ is a solution of (1.12), then $\forall (x_0, t_0) \in \mathbb{R} \times (0, +\infty)$,*

$$\begin{aligned} \int_{\Omega_\tau} u^2(x,\tau)dx &\le M_1\Big(\int_{\Omega_0} (g^2 + h^2 + g_x^2)dx + \int\int_{K_\tau} f^2 dxdt\Big) \\ \int\int_{K_\tau} u^2(x,t)dxdt &\le M_1\Big(\int_{\Omega_0} (g^2 + h^2 + g_x^2)dx + \int\int_{K_\tau} f^2 dxdt\Big) \end{aligned}$$

where $M_1 = e^{t_0}(e^{t_0} + 1)$, $\tau \in [0, t_0]$ and the definition of domains K_τ, Ω_τ and Ω_0 are the same as before.

Proof. We only need to prove that $\|u\|_{L^2(\Omega_\tau)}$ and $\|u\|_{L^2(K_\tau)}$ can be controlled by $\|u_t\|_{L^2(K_\tau)}$. In fact,

$$\int_{\Omega_\tau} (u^2(x,\tau) - u^2(x,0))dx = \int_{\Omega_\tau}\int_0^\tau \partial_t u^2(x,t)dtdx \le \int\int_{K_\tau} (u^2 + u_t^2)dxdt.$$

By Gronwall's inequality, we have

$$\int_{\Omega_\tau} u^2(x,\tau)dx \le e^{t_0}\Big(\int_{\Omega_0} g^2(x)dx + \int\int_{K_\tau} u_t^2 dxdt\Big).$$

$$\int\int_{K_\tau} u^2(x,t)dxdt \le e^{t_0}\Big(\int_{\Omega_0} g^2(x)dx + \int\int_{K_\tau} u_t^2 dxdt\Big).$$

Thus the L^2 estimate is a direct consequence by energy estimates. □

Uniqueness is a direct corollary of energy estimates. Let $Q = \mathbb{R} \times (0, +\infty)$.

Corollary 1.2. *If u_1 and u_2 are two $C^2(Q) \cap C^1(\bar{Q})$ solutions of the Cauchy problem (1.12), then $u_1 = u_2$ in Q.*

Proof. Let $w = u_1 - u_2$, then $w_{tt} - w_{xx} = 0$ in Q and $w|_{t=0} = w_t|_{t=0} = 0$. Then energy estimates for w gives that $\forall (x_0, t_0) \in Q$, $0 < \tau < t_0$,

$$\int_{\Omega_\tau} (w^2 + w_t^2 + w_x^2) dx \leq 0,$$

which implies $w = 0$ in Ω_τ. Since (x_0, t_0) is arbitrary, the uniqueness is proved. □

Stability in the sense of H^1 norm, a H^1 norm of a function is defined by $\|u\|_{H^1} = \|u\|_{L^2} + \|\nabla u\|_{L^2}$.

Corollary 1.3. *If u_1, u_2 are $C^2(Q) \cap C^1(\bar{Q})$ solutions of the Cauchy problem* (1.12) *with different data f_1, g_1, h_1 and f_2, g_2, h_2. Then*

$$\|u_1 - u_2\|_{H^1(K_\tau)} \leq M \left(\|g_1 - g_2\|_{H^1(\Omega_0)} + \|h_1 - h_2\|_{L^2(\Omega_0)} + \|f_1 - f_2\|_{L^2(K_\tau)} \right),$$

where

$$\begin{aligned}
\|u\|^2_{H^1(K_\tau)} &= \int\int_{K_\tau} [u^2(x,\tau) + u_t^2(x,\tau) + u_x^2(x,\tau)] dx dt \\
\|g\|^2_{H^1(\Omega_0)} &= \int_{\Omega_0} [g^2(x) + g_x^2(x)] dx \\
\|h\|^2_{L^2(\Omega_0)} &= \int_{\Omega_0} h^2(x) dx \\
\|f\|^2_{L^2(K_\tau)} &= \int\int_{K_\tau} f^2(x,t) dx dt.
\end{aligned}$$

Proof. Just notice that $w = u_1 - u_2$ is a solution of

$$\begin{aligned}
w_{tt} - w_{xx} &= f_1 - f_2 \quad \text{in } Qm \\
w|_{t=0} = g_1 - g_2, &\quad w_t|_{t=0} = h_1 - h_2,
\end{aligned}$$

then using the energy estimates and L^2 estimates. □

2. Initial boundary value problem in 1D

We will consider the initial boundary value problem 1-D

$$\begin{aligned}
u_{tt} - u_{xx} &= 0 \qquad x \in (0,1), t > 0 \\
u|_{t=0} &= g(x), \qquad u_t|_{t=0} = h(x) \\
u|_{x=0} = u|_{x=1} &= 0.
\end{aligned} \tag{2.1}$$

Here we give the Dirichlet boundary condition. One can also give Neumann boundary condition and Robin boundary condition.

For derivation of the equation and physical meaning of boundary conditions, check Salsa's book [1] Page. 227-228. Page. 230.

The problem (2.1) can be solved by separation of variables. We first give a formal calculation.

Suppose that our solution has a factorized form $u(x,t) = X(x)T(t)$, once we put it into the equation we will get immediately

$$XT'' - X''T = 0 \quad \Rightarrow \quad \frac{X''}{X} = \frac{T''}{T}$$

Since both side of this equation are functions of different variables x and t, once they are equal, they must be constant independent of x and t. We denote the constant by $-\lambda$. Taking account of the boundary condition we have

$$\begin{aligned} X'' + \lambda X &= 0, \qquad X(0) = X(1) = 0, \\ T'' + \lambda T &= 0. \end{aligned}$$

Then the solutions with some undetermined constants A, B, C and D are (Later on, we will prove that $\lambda \geq 0$ so that one can take square root of it)

$$\begin{aligned} X(x) &= C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x \\ T(t) &= A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t \end{aligned}$$

Boundary conditoin for X shows that

$$C = 0, \quad D\sin\sqrt{\lambda} = 0, \quad \Rightarrow \quad \lambda = (n\pi)^2, n = 1, 2, 3, \cdots$$

Thus for any fixed n we can have a solution with undetermined constants A_n and B_n,

$$\begin{aligned} u_n(x,t) &= (A_n\cos\sqrt{\lambda_n}t + B_n\sin\sqrt{\lambda_n}t)\sin\sqrt{\lambda_n}x \\ &= (A_n\cos n\pi t + B_n\sin n\pi t)\sin n\pi x. \end{aligned}$$

By superposition principle, we know the finite summation of solutions is still a solution. For any fixed N, we denote

$$u_N(x,t) = \sum_{n=1}^{N} u_n(x,t) = \sum_{n=1}^{N}(A_n\cos n\pi t + B_n\sin n\pi t)\sin n\pi x.$$

Now if initial data $g(x)$ and $h(x)$ has the same form, say

$$g(x) = \sum_{n=1}^{N} g_n\sin n\pi x, \quad h(x) = \sum_{n=1}^{N} h_n\sin n\pi x,$$

then u_N must be a solution with this initial data. One could naturally ask how about the case with general initial data, what is $u(x,t)$ then? Can we use ∞ to replace N? The problem went directly to the theory of Fourier Series. We first write initial data g and h into Fourier Series, which can be done for L^2 functions, then the corresponding series $\lim_{N\to\infty} u_N(x,t)$ can be expected to be a solution for smooth enough initial data. We will prove this seriously later. Before that, we introduce the so called Sturm-Liouville problem, which is the theoretical basis of the method of separation of variables.

2.1. **Solution formula by separation of variables.**

2.1.1. *Eigenvalue problem.* We will study the following eigenvalue problem

$$\begin{aligned} X'' + \lambda X &= 0, \quad 0 < x < 1 \\ -\alpha_1 X'(0) + \beta_1 X(0) &= 0 \quad \alpha_i, \beta_i \geq 0 \\ \alpha_2 X'(1) + \beta_2 X(1) &= 0 \quad \alpha_i + \beta_i > 0. \end{aligned} \tag{2.2}$$

Theorem 2.1. *(Sturm-Liouville theorem)*

1. *All eigenvalues of (2.2) are nonnegative. In addition, if $\beta_1 + \beta_2 > 0$, then all eigenvalues are positive.*

(2) *Eigenvalues are countable and increasing to infinity, i.e.*

$$0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lim_{n\to\infty} \lambda_n = \infty.$$

(3) *Eigenfunctions of different eigenvalues are orthoganal in the following sense*

$$\int_0^1 X_\lambda X_\mu dx = 0, \quad \textit{for } \lambda \ne \mu,$$

(4) $\forall f \in L^2(0,1)$, *it holds that*

$$f(x) = \sum_{n=1}^{\infty} C_n X_n(x), \ \textit{in the sense of } L^2(0,1), \quad C_n = \frac{\int_0^1 f(x) X_n(x) dx}{\int_0^1 X_n^2 dx},$$

or

$$\lim_{n\to\infty} \|f(x) - f_n(x)\|_{L^2} = 0$$

where $f_n(x) = \sum_{i=1}^{n} C_i X_i(x)$ *is called the generalized Fourier Series.*

Proof. We will only proof the first three statements. The last one can be found in functional analysis in the part of compact self-adjoint operators.

(1) Multiply the equation by X_λ and integrate it on $(0,1)$, we have

$$X_\lambda X'_\lambda \Big|_0^1 - \int_0^1 (X'_\lambda)^2 dx + \lambda \int_0^1 X_\lambda^2 dx = 0.$$

Boundary conditions show that

$$\begin{aligned} -\alpha_1 X'_\lambda(0) X_\lambda(0) + \beta_1 X_\lambda^2(0) = 0, \qquad & -\alpha_1 (X'_\lambda(0))^2 + \beta_1 X_\lambda(0) X'_\lambda(0) = 0 \\ \alpha_2 X'_\lambda(1) X_\lambda(1) + \beta_2 X_\lambda^2(1) = 0, \qquad & \alpha_2 (X'_\lambda(1))^2 + \beta_2 X_\lambda(1) X'_\lambda(1) = 0. \end{aligned}$$

From these, we get

$$\begin{aligned} X'_\lambda(0) X_\lambda(0) &= \frac{1}{\alpha_1 + \beta_1} (\alpha_1 (X'_\lambda(0))^2 + \beta_1 X_\lambda^2(0)) \\ X'_\lambda(1) X_\lambda(1) &= \frac{-1}{\alpha_2 + \beta_2} (\beta_2 X_\lambda^2(1) + \alpha_2 (X'_\lambda(1))^2). \end{aligned}$$

As a consequence we know the nonnegativity of

$$\lambda \int_0^1 X_\lambda^2 dx = \int_0^1 (X'_\lambda)^2 dx - X'_\lambda(1) X_\lambda(1) + X'_\lambda(0) X_\lambda(0) \ge 0.$$

Thus we have $\lambda \ge 0$ and and furthermore

$$\lambda = 0 \text{ if and only if } X'_\lambda \equiv 0 \text{ and } \frac{\beta_1}{\alpha_1 + \beta_1} X_\lambda^2(0) + \frac{\beta_2}{\alpha_2 + \beta_2} X_\lambda^2(1) = 0,$$

or equivalently

$$X_\lambda \equiv C \text{ and } \Big(\frac{\beta_1}{\alpha_1 + \beta_1} + \frac{\beta_2}{\alpha_2 + \beta_2} \Big) C^2 = 0.$$

We can see from this expression that if $\beta_1 + \beta_2 > 0$, then $X_\lambda \equiv 0$. In the end, we get that in the case of $\beta_1 + \beta_2 > 0$, λ must be positive.

(2) We have already $\lambda \geq 0$ and $\lambda = 0$ iff $\beta_1 = \beta_2 = 0$. From $X'' + \lambda X = 0$, we know that there exist A and B such that

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x,$$
$$X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x.$$

We will study it in the following three cases:

(a) Dirichlet boundary condition. $\alpha_1 = \alpha_2 = 0$, we have $X(0) = X(1) = 0$. So we have $A = 0$ and $B\sin\sqrt{\lambda} = 0$. As a consequence,

$$\lambda_n = (n\pi)^2, \quad X_n(x) = \sin n\pi x, \quad n = 1, 2, \cdots.$$

Obviously, in this case, λ_n is monotone and increasing to ∞.

(b) Neumann boundary condition. $\beta_1 = \beta_2 = 0$, we have $X'(0) = X'(1) = 0$, then $B = 0$, $A\sin\sqrt{\lambda} = 0$. Thus we have

$$\lambda_n = (n\pi)^2, \quad X_n(x) = \cos n\pi x, \quad n = 0, 1, 2, \cdots.$$

(c) Robin boundary condition. $\alpha_1\beta_2 + \alpha_2\beta_1 > 0$, from the boundary condition we have

$$\beta_1 A - \alpha_1 B\sqrt{\lambda} = 0,$$
$$\beta_2(A\cos\sqrt{\lambda} + B\sin\sqrt{\lambda}) - \alpha_2\sqrt{\lambda}(A\sin\sqrt{\lambda} - B\cos\sqrt{\lambda}) = 0.$$

By calculations, $\frac{A}{B} = \frac{\alpha_1\sqrt{\lambda}}{\beta_1}$, and

$$\beta_2\alpha_1\sqrt{\lambda}\frac{1}{\tan\sqrt{\lambda}} + \beta_2\beta_1 - \alpha_1\alpha_2\lambda + \beta_1\alpha_2\sqrt{\lambda}\frac{1}{\tan\sqrt{\lambda}} = 0.$$

Let $\xi = \sqrt{\lambda}$, then

$$\tan\xi = \frac{(\beta_2\alpha_1 + \beta_1\alpha_2)\xi}{\alpha_1\alpha_2\xi^2 - \beta_1\beta_2}.$$

From this formulation, we know that λ_n is increasing to ∞.

(3) Multiply the equation by X_μ and X_λ separately and integrate on $(0, 1)$, we have

$$X_\mu X'_\lambda\Big|_0^1 - \int_0^1 X'_\mu X'_\lambda + \lambda\int_0^1 X_\lambda X_\mu = 0$$
$$X_\lambda X'_\mu\Big|_0^1 - \int_0^1 X'_\mu X'_\lambda + \mu\int_0^1 X_\lambda X_\mu = 0$$

The difference between this two equations shows that

$$(\lambda - \mu)\int_0^1 X_\lambda X_\mu = -X_\mu X'_\lambda\Big|_0^1 + X_\lambda X'_\mu\Big|_0^1.$$

Now the boundary condition for X_λ and X_μ are

$$-\alpha_1 X'_\lambda(0) + \beta_1 X_\lambda(0) = 0, \qquad \alpha_2 X'_\lambda(1) + \beta_2 X_\lambda(1) = 0$$
$$-\alpha_1 X'_\mu(0) + \beta_1 X_\mu(0) = 0, \qquad \alpha_2 X'_\mu(1) + \beta_2 X_\mu(1) = 0$$

These algebraic systems have non zero solutions, thus the coefficient determinents are 0, i.e.

$$\begin{vmatrix} X'_\lambda(0) & X_\lambda(0) \\ X'_\mu(0) & X_\mu(0) \end{vmatrix} = 0, \qquad \begin{vmatrix} X'_\lambda(1) & X_\lambda(1) \\ X'_\mu(1) & X_\mu(1) \end{vmatrix} = 0.$$

Thus we have

$$(\lambda-\mu)\int_0^1 X_\mu X_\lambda dx = 0.$$

Since $\lambda \neq \mu$, we know X_λ and X_μ are authogonal.

□

Remark 2.1. When $\beta_1 = \beta_2 = 0$, it is Neumann boundary condition. In this case, $\lambda = 0$ is an eigenvalue, its eigenfunction is $X_0 = 1$.

Remark 2.2. $\{X_n(x)\}$ is a complete authogonal basis of $L^2(0,1)$, after normalization it is

$$X_n^*(x) = \frac{X_n(x)}{\|X_n(x)\|_{L^2}}.$$

Then $\forall f \in L^2(0,1)$, the fourier coefficient C_n^* is

$$C_n^* = \frac{\int_0^1 f(x)X_n(x)dx}{\|x_n(x)\|_{L^2}}$$

which is the inner product of $f(x)$ and $X_n^*(x)$.

2.1.2. *Separation of variable.* Formally the solution of (2.1) is

$$u(x,t) = \sum_{n=1}^{\infty}(A_n \cos n\pi t + B_n \sin n\pi t)\sin n\pi x. \tag{2.3}$$

Next we will give the method on determining coefficients A_n and B_n by using initial data.

Take $t = 0$ in (2.3),

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin n\pi x, \quad \text{and} \quad u_t(x,0) = \sum_{n=1}^{\infty} n\pi B_n \sin n\pi x.$$

Assume initial data g and h has the following Fourier expansion by sine functions

$$g(x) = \sum_{n=1}^{\infty} g_n \sin n\pi x, \qquad g_n = 2\int_0^1 g(x)\sin n\pi x dx,$$

$$h(x) = \sum_{n=1}^{\infty} h_n \sin n\pi x, \qquad h_n = 2\int_0^1 h(x)\sin n\pi x dx.$$

Then a natural choice of the coefficients are

$$A_n = g_n, \quad B_n = \frac{1}{n\pi}h_n.$$

Thus the solution expression is

$$u(x,t) = \sum_{n=1}^{\infty}(g_n \cos n\pi t + \frac{h_n}{n\pi}\sin n\pi t)\sin n\pi x. \tag{2.4}$$

Summary. There are three main steps in separation of variables

(1) Separation of variable formally and set up the eigenvalue problem,
(2) Solve eigenvalue problem, and solve the ODE for $T(t)$,
(3) Summation, fixed the coefficients from initial data.

Questions remained.

(1) How about the solution for other boundary conditions? Neumann and Robin?
(2) Inhomogeneous boundary condition? Homogenization.

(3) Inhomogeneous equation $u_{tt} - u_{xx} = f$?
(4) Under what condition is $u(x,t)$ a solution?

For Neumann and Robin boundary conditions, the related eigenvalue problem is already studied in Sturm-Liouville theorem. We will answer the remaining three questions in the following three subsections.

2.1.3. *Inhomogeneous equation.* We briefly explain how to deal with the non homogeneous equations. Here we use $(0,l)$ instead of $(0,1)$.

$$\begin{aligned} u_{tt} - u_{xx} &= f \qquad x \in (0,l), t > 0 \\ u|_{x=0} = u|_{x=l} &= 0 \\ u|_{t=0} = g(x) &\qquad u_t|_{t=0} = h(x). \end{aligned} \tag{2.5}$$

Firstly we know that the eigenfunctions are $\sin\frac{n\pi x}{l}$, $n = 1, 2, \cdots$. Then assume that

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi}{l}x \\ f(x,t) &= \sum_{n=1}^{\infty} f_n(t) \sin\frac{n\pi}{l}x \\ g(x) &= \sum_{n=1}^{\infty} g_n \sin\frac{n\pi}{l}x \\ h(x) &= \sum_{n=1}^{\infty} h_n \sin\frac{n\pi}{l}x. \end{aligned}$$

Then solve the ODE for $T_n(t)$,

$$\begin{aligned} &T_n''(t) + (\frac{n\pi}{l})^2 T_n(t) = f_n(t) \\ &T_n(0) = g_n, \quad T_n'(0) = h_n. \end{aligned}$$

One can get that the solution is

$$T_n(t) = g_n \cos\frac{n\pi}{l}t + \frac{l}{n\pi}h_n \sin\frac{n\pi}{l}t + \frac{l}{n\pi}\int_0^t f_n(\tau)\sin\frac{n\pi}{l}(t-\tau)d\tau.$$

Then by replacing $T_n(t)$ in the solution $u(x,t)$ by this, we get the solution for inhomogeneous equation.

2.1.4. *Inhomogeneous boundary conditions.* The problem with inhomogeneous boundary condition is

$$\begin{aligned} u_{tt} - u_{xx} &= f \qquad x \in (0,l), t > 0 \\ u|_{x=0} = u_0(t) &\qquad u|_{x=l} = u_1(t) \\ u|_{t=0} = g(x) &\qquad u_t|_{t=0} = h(x). \end{aligned} \tag{2.6}$$

We will use homogenization technic. Introduce a new function $v(x,t)$ such that the homogeneous boundary conditions are true for $v(x,t)$, more precisely, let

$$u(x,t) = v(x,t) + \frac{x}{l}u_1(t) + \frac{l-x}{l}u_0(t),$$

then $v(x,t)$ solves

$$\begin{aligned}
&v_{tt} - v_{xx} = f(x,t) - \frac{x}{l}u_1'' - \frac{l-x}{l}u_0'' \\
&v|_{x=0} = v|_{x=l} = 0 \\
&v|_{t=0} = g(x) - \frac{x}{l}u_1(0) - \frac{l-x}{l}u_0(0) \\
&v_t|_{t=0} = h(x) - \frac{x}{l}u_1'(0) - \frac{l-x}{l}u_0'(0)
\end{aligned}$$

By the method of dealing with inhomogeneous equations, we can get a formula for $v(x,t)$, which gives the solution formula for $u(x,t)$.

2.2. **Existence of solution for** (2.1)**.** Now we have the solution formula (2.4). Under what conditions is u a classical solution of (2.1)? We need that u is at least twice differentiable in both x and t. According to the theory on function Series, we need that

$$\sum_{n=1}^{\infty} u_n, \quad \sum_{n=1}^{\infty} Du_n, \quad \sum_{n=1}^{\infty} D^2 u_n$$

are uniformly convergent in $(0,1)\times(0,T)$.

To have classical solution of (2.1), we also need some compatibility conditions,

$$g(0) = g(1) = 0, \quad h(0) = h(1) = 0, \quad g''(0) = g''(1) = 0. \tag{2.7}$$

Theorem 2.2. *$g \in C^3[0,1]$, $h \in C^2[0,1]$ and they satisfies the compatibility condition* (2.7)*, then $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) \in C^2(\bar{Q})$ is a solution of* (2.1)*.*

Proof. Integral by parts on the coefficients of g and h by using compatibility conditions,

$$\begin{aligned}
\frac{h_n}{n\pi} &= \frac{2}{n\pi}\int_0^1 h(x)\sin n\pi x dx = -\frac{2}{(n\pi)^3}\int_0^1 h''(x)\sin n\pi x dx := -\frac{2}{(n\pi)^3}a_n \\
g_n &= 2\int_0^1 g(x)\sin n\pi x dx = \frac{2}{(n\pi)^3}\int_0^1 g'''(x)\cos n\pi x dx := \frac{2}{(n\pi)^3}b_n.
\end{aligned}$$

Then we have

$$u(x,t) = \sum_{n=1}^{\infty}\Big(\frac{2}{(n\pi)^3}b_n\cos n\pi t - \frac{2}{(n\pi)^3}a_n\sin n\pi t\Big)\sin n\pi x.$$

Moreover the following estimates holds

$$\begin{aligned}
&|u_n| \le \frac{C}{n^3}, \quad |Du_n| \le \frac{C}{n^2} \\
&|D^2 u_n| \le \frac{C}{n}(|a_n| + |b_n|) \le C(\frac{1}{n^2} + |a_n|^2 + |b_n|^2),
\end{aligned}$$

where the right hand side of the last inequality can be bounded by Bessel inequality,

$$\sum_{n=1}^{\infty}|a_n|^2 \le 2\int_0^1 |h''|^2 dx, \quad \sum_{n=1}^{\infty}|b_n|^2 \le 2\int_0^1 |g'''|^2 dx.$$

□

2.3. **Uniqueness and Stability — Energy estimates.** Let $Q_\tau = (0,1)\times(0,\tau)$, we have the following energy estimates for initial boundary value problem of wave equation (2.1) in Q_τ. Then uniqueness and stability can be obtained from that.

Theorem 2.3. *Assume $u \in C^2(Q_\tau)\cap C^1(\bar{Q}_\tau)$, then*

$$\int_0^1 (u^2+u_t^2+u_x^2)dx \le M\Big(\int_0^1 (h^2+g^2+g_x^2)dx + \int_{Q_\tau} f^2 dxdt\Big).$$

Proof. Multiply the wave equation by u_t and integrate it on Q_τ,

$$\int_{Q_\tau} \frac{\partial}{\partial t}(u_t^2+u_x^2) \le \int_{Q_\tau} f^2 + \int_{Q_\tau} u_t^2.$$

Notice that $\int_{Q_\tau} = \int_0^\tau \int_0^1$, we have

$$\int_0^1 (u_t^2+u_x^2)|_{t=\tau} \le \int_0^1 (h^2+g_x^2) + \int_{Q_\tau} f^2 + \int_{Q_\tau} u_t^2.$$

By Gronwall's inequality,

$$\int_0^1 (u_t^2+u_x^2)|_{t=\tau} \le M\Big(\int_0^1 (h^2+g_x^2) + \int_{Q_\tau} f^2\Big).$$

Similar to the discussion in Cauchy problem, we have the L^2 estimates.

□

2.4. **Resonance.** Consider initial boundary value problem

$$\begin{aligned} u_{tt}-u_{xx} &= A(x)\sin\omega t, \quad x\in(0,1), t>0\\ u|_{x=0,1} &= 0,\\ u|_{t_0} = u_t|_{t=0} &= 0. \end{aligned}$$

Compatibility conditions $A(0)=A(1)=0$ is needed for existence of classical solution. We assume $A\in C^1$. The solution formula from separation of variable is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^\infty \frac{1}{n\pi}\int_0^t f_n(\tau)\sin n\pi(t-\tau)d\tau\cdot \sin n\pi x\\ &= \sum_{n=1}^\infty \frac{a_n}{n\pi}\sin n\pi x\int_0^t \sin\omega\tau\cdot\sin n\pi(t-\tau)d\tau, \end{aligned}$$

where

$$a_n = 2\int_0^1 A(x)\sin n\pi x dx.$$

If we calculate further, we will see

$$
\begin{aligned}
& \int_0^t \sin\omega\tau\cdot\sin n\pi(t-\tau)d\tau \\
= & \int_0^t -\frac{1}{2}(\cos(n\pi t+(\omega-n\pi)\tau)-\cos(-(\omega+n\pi)\tau-n\pi t))d\tau \\
= & \frac{1}{2}\int_0^t \cos((\omega+n\pi)\tau-n\pi t)d\tau-\frac{1}{2}\int_0^t \cos((\omega-n\pi)\tau+n\pi t)d\tau \\
\overset{\omega\neq n\pi}{=} & \frac{1}{2(\omega+n\pi)}\sin((\omega+n\pi)\tau-n\pi t)\Big|_0^t-\frac{1}{2(\omega-n\pi)}\sin((\omega-n\pi)\tau+n\pi t)\Big|_0^t \\
\overset{\omega\neq n\pi}{=} & \frac{1}{2(\omega+n\pi)}(\sin\omega t+\sin n\pi t)-\frac{1}{2(\omega-n\pi)}(\sin\omega t-\sin n\pi t).
\end{aligned}
$$

If $\omega = k\pi$ for some k, then

$$
\begin{aligned}
u_k(x,t) &= \frac{a_k}{k\pi}\Big(\frac{2\sin k\pi t}{k\pi+k\pi}-\frac{1}{2}\int_0^t \cos k\pi t d\tau\Big)\sin k\pi x \\
&= \Big(\frac{a_k}{(k\pi)^2}\sin k\pi t-\frac{a_k}{2k\pi}t\cdot\cos k\pi t\Big)\sin k\pi x.
\end{aligned}
$$

Thus in the case of $\omega = k\pi$, we have

$$
\begin{aligned}
u(x,t) = & \sum_{n\neq k}\Big(\frac{1}{2(\omega+n\pi)}(\sin\omega t+\sin n\pi t)-\frac{1}{2(\omega-n\pi)}(\sin\omega t-\sin n\pi t)\Big)\sin n\pi x \\
& +\Big(\frac{a_k}{(k\pi)^2}\sin k\pi t-\frac{a_k}{2k\pi}t\cdot\cos k\pi t\Big)\sin n\pi x
\end{aligned}
$$

As $t\to\infty$, we have $u_k(x,t)$ must blow up at some point.

3. Appendix-On Fourier Series

$\forall f\in L^1(-l,l)$, it can be written into a Series by using sines and cosines functions

$$
f(x)\sim\frac{A_0}{2}+\sum_{n=1}^{\infty}(A_n\cos\frac{n\pi x}{l}+B_n\sin\frac{n\pi x}{l}).
$$

where

$$
\begin{aligned}
A_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx, \quad n=0,1,2,\cdots \\
B_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx, \quad n=1,2,\cdots
\end{aligned}
$$

are called Fourier coefficients of f.

If $f(x)$ is an odd function, then $B_n = 0$, and

$$
f(x)\sim\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos\frac{n\pi x}{l}, \quad A_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx
$$

If $f(x)$ is an even function, then $A_n = 0$, and

$$
f(x)\sim\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{l}, \quad B_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx.
$$

For any fixed $N \geq 1$, $(S_N f)(x) = \frac{A_0}{2} + \sum_{n=1}^{N}(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l})$ is called Trigonometric polynomials.

$\sin\frac{n\pi x}{l}$, $\cos\frac{m\pi x}{l}$, $n, m = 1, 2, \cdots$ are orthogonal in the sense that

$$\frac{1}{l}\int_{-l}^{l}\cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}dx = \delta_{mn}$$

$$\frac{1}{l}\int_{-l}^{l}\sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}dx = \delta_{mn}$$

$$\frac{1}{l}\int_{-l}^{l}\sin\frac{m\pi x}{l}\cos\frac{n\pi x}{l}dx = 0$$

Moreover, $\{1, \sqrt{2}\cos\frac{n\pi x}{l}, \sqrt{2}\sin\frac{n\pi x}{l}\}_{n=1}^{\infty}$ is an orthonormal basis in $L^2(-l, l)$, where the inner product in $L^2(-l, l)$ is defined by $\frac{1}{2l}\int_{-l}^{l} f(x)\bar{g}(x)dx$.

Theorem 3.1. *(Convergence in L^2 norm)*

$$\lim_{N\to\infty}\|f(x) - (S_N f)(x)\|_{L^2} = 0, \qquad \text{for } f \in L^2(-l, l).$$

Theorem 3.2. *(Bessel inequality) For $f \in L^2(-l, l)$, it holds*

$$\frac{A_0^2}{2} + \sum_{n=1}^{\infty}(A_n^2 + B_n^2) \leq \frac{1}{l}\int_{-l}^{l} f^2 dx.$$

Theorem 3.3. *(Parseval's equality) For $f \in L^2(-l, l)$, it holds*

$$\frac{A_0^2}{2} + \sum_{n=1}^{\infty}(A_n^2 + B_n^2) = \frac{1}{l}\int_{-l}^{l} f^2 dx.$$

4. Problems

(1) Verify that $u(x, t) = \frac{F(x - at) + G(x + at)}{h - x}$ is a solution of

$$\Big(1 - \frac{x}{h}\Big)^2\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial}{\partial x}\Big[\Big(1 - \frac{x}{h}\Big)^2\frac{\partial u}{\partial x}\Big]$$

where $h > 0$, $a > 0$ are constants, F, G are any function in C^2.

(2) (a) Show the general solution of the PDE $u_{xy} = 0$ is $u(x, t) = F(x) + G(y)$ for arbitrary function F, G.
(b) Using the change of variables $\xi = x + t$, $\eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.
(c) Use the above two facts to derive d'Alembert's formula.

(3) Give energy estimates for half-line problem and the Cauchy problem in Multi-D case.

(4) (Equal partition of energy) Suppose that $u \in C^2(\mathbb{R} \times [0, \infty))$ is a solution of the following Cauchy problem

$$u_{tt} - u_{xx} = 0 \qquad (x, t) \in \mathbb{R} \times (0, \infty)$$

$$u|_{t=0} = g, \quad u_t|_{t=0} = h \qquad x \in \mathbb{R}.$$

where g, h have compact support. Let kinetic energy be $k(t) = \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} u_t^2(x,t)dx$, potential energy be $p(t) = \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} u_x^2(x,t)dx$. Try to prove

(a) $k(t) + p(t)$ is a constant independent of t.
(b) $k(t) = p(t)$ for large enough t.

(5)

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \qquad x \in (0,+\infty), t \in (0,+\infty) \\ u|_{x=0} &= \cos \omega t \\ u|_{t=0} &= Ae^{-x^2}, \qquad u_t|_{t=0} = 0. \end{aligned}$$

Find the condition for A and ω such that solution $u \in C^2(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+)$, and give this solution formula.

(6) If u is a classical solution of

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \qquad x \in (0,1), t \in (0,+\infty) \\ u|_{x=0} = u|_{x=1} &= 0 \\ u|_{t=0} &= 0, \qquad u_t|_{t=0} = x^2(1-x). \end{aligned}$$

what is the limit

$$\lim_{t\to+\infty} \int_0^1 (u_t^2 + u_x^2)dx.$$

(7) Solve eigenvalue problem

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \quad x \in (0,l) \\ X(0) = X'(l) &= 0. \end{aligned}$$

(8)

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \quad x \in (0,1) \\ X'(0) + X(0) &= 0, \quad X(1) = 0. \end{aligned}$$

(a) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.
(b) Find an equation for the positive eigenvalues $\lambda = \beta^2$.
(c) Show graphically from part (8b) that there are an infinite number of positive eigenvalues.
(d) Is there a negative eigenvalue?

(9) Apply separation of variables to get formal solution of

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \qquad (x,t) \in (0,1) \times (0,\infty) \\ u_x|_{x=0} = A \sin \omega t, u|_{x=1} &= 0 \quad t \geq 0 \\ u|_{t=0} = 1, \quad u_t|_{t=0} &= 0 \qquad x \in [0,1]. \end{aligned}$$

(10)

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \qquad x \in (0,1), t \in (0,+\infty) \\ u|_{x=0} = u|_{x=1} &= 0 \\ u|_{t=0} = \alpha x^4 + \beta x^3 + \sin x, &\qquad u_t|_{t=0} = \gamma \cos x. \end{aligned}$$

Solve the problem and give the conditions on α, β and γ such that the solution you gave is a classical one.

(11) Find the solution of initial boundary values for heat equation by separation of variables.

$$\begin{aligned} u_t - u_{xx} &= \sin x\pi, \qquad x \in (0,1), t \in (0,+\infty) \\ u|_{x=0} = u|_{x=1} &= 0 \\ u|_{t=0} &= 0. \end{aligned}$$

(12) **Discussions** One can get solution formula of (2.1) by D'Alembert and Fourier series, are they the same?

References

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